$$N=\sum_k n_k = \sum_k \frac{1}{e^{\beta(\varepsilon_k-\mu)}-1} = \sum_k \frac{1}{\frac{e^{\beta\varepsilon_k}}{z}-1}$$

$$E=\sum_k \varepsilon_k n_k = \sum_k \frac{\varepsilon_k}{z^{-1}e^{\beta\varepsilon_k}-1}$$

$$\ln\Xi = -\sum_k \ln\left(1-ze^{-\beta\varepsilon_k}\right) = \frac{pV}{k_BT}$$

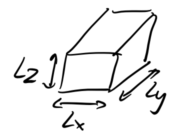
translational invariaty

(in 3-dim)

Non-relativistic free particle $\varepsilon_k=\frac{\hbar^2k^2}{2m}$ dispersion relation

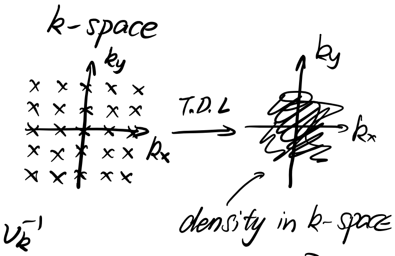
wavefunction: $\phi_k(x)=\frac{e^{i\vec{k}\cdot\vec{x}}}{\sqrt{V}}$ Periodic B.C. $\psi_k(x_i+L)=\psi_k(x_i)$

"In the limit of thermodynamics, the volume is overwhelmingly large, so the B.C. would not influence the bulk."




$$e^{ik(x+L)}=e^{ikx} \Rightarrow k=\frac{2\pi}{L}n \quad n\in\mathbb{Z}$$

$\vec{k}$ is quantized to $\vec{k}=\left(\frac{2\pi}{L_x}n_x,\ \frac{2\pi}{L_y}n_y,\ \frac{2\pi}{L_z}n_z\right)$




$\sum_{\vec{k}} = \int d^3\vec{k}\cdot$(density of state in $\vec{k}$-space) $\rho_k = v_k^{-1}$

$v_k$ is the volume per $\vec{k}$ state in $\vec{k}$-space $=\frac{(2\pi)^3}{V}$

$$=\frac{V}{(2\pi)^3}\int d^3\vec{k}$$

semi-classical check $\vec{p}=\hbar\vec{k}$

$$\sum_k = \frac{V}{(2\pi)^3}\int d^3\vec{k} = \frac{V}{h^3}\int d^3p = \int\frac{d^3p\,d^3q}{h^3}$$

in general in d-dim: $\frac{L^d}{(2\pi)^d}\int d^d\vec{k}$

$$\int\frac{d^3q\,d^3p}{h^3} \overset{\varepsilon=\frac{p^2}{2m}}{=} \int_0^\infty d\varepsilon\, \underbrace{\frac{2\pi V}{h^3}(2m)^{\frac{3}{2}}\varepsilon^{\frac{1}{2}}}_{g(\varepsilon)}$$

density of state

# of states in $[\varepsilon, \varepsilon+d\varepsilon]$

$$N=\frac{V}{(2\pi)^3}\int d^3\vec{k}\,\frac{1}{z^{-1}e^{\beta\frac{\hbar^2k^2}{2m}}-1}$$

$$=\frac{V}{2\pi^2}\int_0^\infty \frac{k^2\,dk}{z^{-1}e^{\beta\frac{\hbar^2k^2}{2m}}-1}$$

Define dimensionless quantity

denote $x\equiv\beta\frac{\hbar^2k^2}{2m}$, $k^2=\frac{2m}{\beta\hbar^2}x$, $dk=\sqrt{\frac{m}{2\beta\hbar^2}}\frac{dx}{\sqrt{x}}=\frac{\pi^{\frac{1}{2}}}{\lambda}\frac{dx}{\sqrt{x}}$

notice that $\hbar=\frac{h}{2\pi}$!

$$N=\frac{V}{2\pi^2}\frac{\pi^{\frac{1}{2}}}{\lambda}\int_0^\infty\frac{dx\,x^{-\frac{1}{2}}\,\frac{4\pi}{\lambda^2}x}{z^{-1}e^x-1}$$

$$=\frac{2V}{\pi^{\frac{1}{2}}}\frac{1}{\lambda^3}\int_0^\infty\frac{\sqrt{x}}{z^{-1}e^x-1}\,dx$$

Define $g_m(z) = \frac{1}{\Gamma(m)} \int_0^\infty dx \frac{x^{m-1}}{z^{-1}e^x - 1}$ Polylogarithm $Li_m(z)$

$$N = \frac{2V}{\pi^{\frac{1}{2}}} \frac{1}{\lambda^3} \cdot \Gamma\left(\frac{3}{2}\right) g_{\frac{3}{2}}(z) = \frac{V}{\lambda^3} g_{\frac{3}{2}}(z)$$

$$E = \frac{V}{(2\pi)^3} \int d^3\vec{k} \frac{\frac{\hbar^2 k^2}{2m}}{z^{-1} e^{\beta \frac{\hbar^2 k^2}{2m}} - 1}$$

$$= \frac{V}{2\pi^2} \int_0^\infty dk \frac{k^2 \frac{\hbar^2 k^2}{2m}}{z^{-1} e^{\beta \frac{\hbar^2 k^2}{2m}} - 1} = \frac{2V}{\pi^{\frac{1}{2}}} \frac{1}{\lambda^3} \frac{1}{\beta} \int_0^\infty \frac{x^{\frac{3}{2}}}{z^{-1}e^x - 1} dx$$

$$= \frac{3}{2} \frac{V}{\lambda^3} \frac{1}{\beta} g_{\frac{5}{2}}(z)$$

$$\ln \Xi = -\frac{V}{(2\pi)^3} \int d^3\vec{k} \ln\left(1 - z e^{-\beta \frac{\hbar^2 k^2}{2m}}\right)$$

$$= -\frac{2V}{\pi^{1/2}} \frac{1}{\lambda^3} \int_0^\infty dx\, x^{\frac{1}{2}} \ln(1 - z e^{-x})$$

$$= \frac{2V}{\pi^{1/2}} \frac{1}{\lambda^3} \cdot \frac{2}{3} \int_0^\infty dx \frac{x^{\frac{3}{2}}}{z^{-1}e^x - 1}$$

$$= \frac{V}{\lambda^3} g_{\frac{5}{2}}(z) = pV\beta$$

$$\Rightarrow pV = \frac{2}{3}E, \quad E = \frac{3}{2}pV$$

$$\begin{cases} \frac{p}{k_B T} = \frac{1}{\lambda^3} g_{\frac{5}{2}}(z) \\ n = \frac{1}{\lambda^3} g_{\frac{3}{2}}(z) \end{cases}$$

* Non-degenerate Limit $(z \ll 1)$

$$g_m(z) = \frac{1}{\Gamma(m)} \int_0^\infty \frac{x^{m-1}}{z^{-1}e^x - 1} dx$$

$$= \frac{1}{\Gamma(m)} \int_0^\infty \frac{x^{m-1} z e^{-x}}{1 - z e^{-x}} dx$$

$$= \frac{1}{\Gamma(m)} \int_0^\infty x^{m-1} \sum_{l=1}^\infty (z e^{-x})^l$$

$$= \frac{1}{\Gamma(m)} \sum_{l=1}^\infty z^l \int_0^\infty dx\, x^{m-1} e^{-lx}$$

$$= \frac{1}{\Gamma(m)} \sum_{l=1}^{\infty} \frac{z^l}{l^m} \boxed{\int_0^\infty dt\, t^{m-1} e^{-t}} = \Gamma(m)$$

$$= \sum_{l=1}^{\infty} \frac{z^l}{l^m}$$

$$\boxed{g_m(z) = \sum_{l=1}^{\infty} \frac{z^l}{l^m}} \quad (z \in [0,1],\ m<1)$$

$$\begin{cases} \frac{P}{k_BT} = \frac{1}{\lambda^3}\left(z + \frac{z^2}{2^{\frac{5}{2}}} + \frac{z^3}{3^{\frac{5}{2}}} + \cdots\right) \\ n = \frac{1}{\lambda^3}\left(z + \frac{z^2}{2^{\frac{3}{2}}} + \frac{z^3}{3^{\frac{3}{2}}} + \cdots\right) \end{cases}$$

$$\frac{P}{k_BT} = n\sum_{l=1}^{\infty} a_l (n\lambda_T^3)^{l-1} = \sum_{l=1}^{\infty} \frac{1}{\lambda^3}\frac{z^l}{l^{\frac{5}{2}}}$$

$$\Rightarrow \sum_{l=1}^{\infty} a_l (n\lambda_T^3)^l = \sum_{l=1}^{\infty} \frac{z^l}{l^{\frac{5}{2}}}$$

$$\Rightarrow \sum_{l=1}^{\infty} \frac{z^l}{l^{5/2}} = \sum_{l=1}^{\infty} a_l \left(\sum_{m=1}^{\infty} \frac{z^m}{m^{3/2}}\right)^l$$

$$\frac{P}{nk_BT} = 1 - \underbrace{\frac{1}{2^{5/2}}}_{-0.17678} \lambda_T^3 n - \underbrace{\left(\frac{2}{3^{5/2}} - \frac{1}{8}\right)}_{-0.0030}(\lambda_T^3 n)^2 - \cdots$$

Virial Expansion: Bose statistics, no interaction

"Reduction in pressure" $\Rightarrow$ "Effective Attraction"

when $n\lambda_T^3 \ll 1$ (the average distance are much bigger than thermal wavelength, particles almost don't interact with each other)

$$\Rightarrow \frac{P}{k_BT} = n$$

* The degenerate Bose gas $\Rightarrow z \lesssim 1$

$$z = e^{\beta\mu} > 0$$

$$Q = \prod_k Q_k = \prod_k \sum_{n_k=0}^{\infty} e^{-\beta(\varepsilon_k - \mu)n_k}$$ convergent $\Rightarrow \varepsilon_k - \mu > 0\ \forall k \Rightarrow \mu < \min\{\varepsilon_k\}$

$\mu < 0 \Rightarrow$ for ideal Bose gas w/ $\varepsilon_k \sim k^2 \Rightarrow 0 < z < 1$

$N = \frac{V}{\lambda^3} g_{3/2}(z)$ $\quad g_m(z)$ is monotonically $\nearrow$ for $z \in (0,1)$

◦ Min $g_m(0) = 0$; ◦ Max $g_m(1) = \zeta(m)$ ($m > 1$, or $g_m(1)$ diverges)

- high temp: $T \nearrow \to \frac{V}{\lambda^3} \nearrow$, $g_{3/2}(z \to 0) \to 0$ $N$ fixed
- low temp: $T \searrow \to \frac{V}{\lambda^3} \searrow$, $g_{3/2}(z)$ is bounded from above by a finite value $g_{3/2}(z) \leq g_{3/2}(1) = \zeta(\frac{3}{2}) = 2.612$

$\Rightarrow$ Impossible to keep $N$ fixed as temp is lowered.

hit $z=1$ when $n = \frac{1}{\lambda^3}\zeta(\frac{3}{2}) \Rightarrow n\lambda_{T_c}^3 = \zeta(\frac{3}{2})$

$$\Rightarrow T_c = \frac{h^2}{2\pi m k_B}\left(\frac{n}{\zeta(\frac{3}{2})}\right)^{\frac{2}{3}}$$

$T < T_c$ $\quad n^* = \frac{1}{\lambda^3_{T<T_c}}\zeta(\frac{3}{2}) < n \Rightarrow$

$z \lesssim 1$ $\quad \langle n_{k=0} \rangle = \frac{1}{z^{-1}-1} \to$ diverges as $z \to 1$

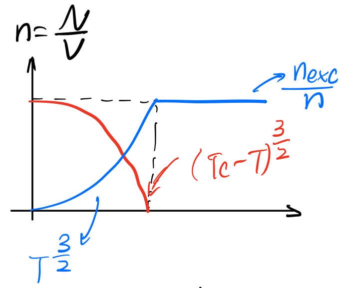
macroscopically large $\propto V$

a huge # of particles populate the lowest energy state

$n^* V \to$ total # of particles occupying $k \neq 0$

$N = N_{exc} + N_0$ particles at ground state

$$\begin{cases} n\lambda_{T_c}^3 = \zeta(\frac{3}{2}) \\ n_{exc}\lambda^3_{T<T_c} = \zeta(\frac{3}{2}) \end{cases} \qquad \frac{n_{exc}}{n} = \left(\frac{T}{T_c}\right)^{\frac{3}{2}} \qquad \frac{n_0}{n} = 1 - \left(\frac{T}{T_c}\right)^{\frac{3}{2}}$$




Macroscopic # of particles condense into a single quantum state only in $\vec{k}$-space

$\Rightarrow$ Bose - Einstein Condensation (Einstein, 1925) not in $\vec{x}$-space

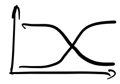
Why? $\sum_k \to \frac{V}{(2\pi)^3}\int d^3k = \frac{V}{(2\pi)^3}4\pi\int dk\, k^2$ $\boxed{k=0}$!

$$N = \underbrace{\frac{V}{\lambda^3}g_{\frac{3}{2}}(z)}_{N_{exc}} + \underbrace{\frac{1}{z^{-1}-1}}_{N_0}$$

$z \ll 1$ $\quad N_0 \sim O(1)$

$T \searrow$, $N_0 \nearrow (z \nearrow)$

$T=0$, $N_{exc}=0$, $N = \frac{z}{1-z}$ $\Rightarrow z_{max} = \frac{N}{1+N} \sim 1 - \frac{1}{N}$

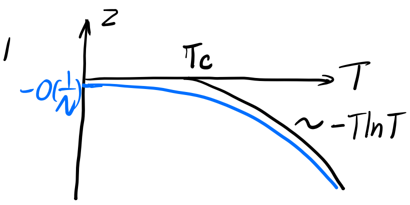
if $N$ is not big enough  * Finite $N$, $z$ never reach 1

Thermodynamic limit ($N \to \infty$, $V \to \infty$, $n$ is finite)

$$n = \underbrace{\frac{1}{\lambda^3} g_{3/2}(z)}_{n_{exc}} + \underbrace{\frac{1}{V}\frac{z}{1-z}}_{n_0}$$

$n_0$ nonzero, $\frac{z}{1-z} \sim O(N)$

$z \sim 1 - O(\frac{1}{N}) \to 1$




## Thermodynamic Properties

$$\ln \Xi = \underbrace{-\frac{V}{(2\pi)^3}\int d^3\vec{k}\, \ln(1-z^{-1}e^{\beta\varepsilon_k})}_{\text{Excited states}} \underbrace{-\ln(1-z)}_{k=0 \text{ contribution}} \sim O(\ln N)$$

$$= \frac{V}{\lambda^3} g_{5/2}(z)$$

$$\Rightarrow p = \frac{k_B T}{\lambda^3} g_{5/2}(z)$$

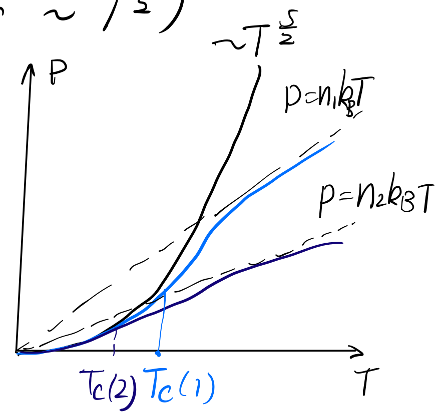
$T < T_c \quad z = 1$

$$p = \frac{k_B T}{\lambda^3}\zeta\left(\frac{5}{2}\right) \approx 1.341 \frac{k_B T}{\lambda^3} \sim T^{\frac{5}{2}}$$ independent of density $n$

( the particles at ground states make no contribution to pressure. the density of excited states $\sim T^{\frac{3}{2}}$)

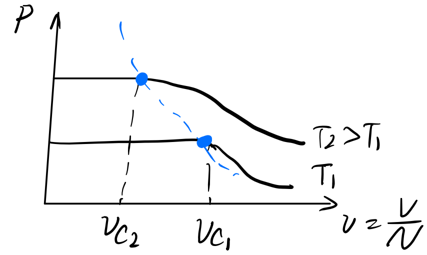
$\langle n_{k=0} \rangle \sim O(N) \sim V$

$$\langle n_{k=k_1} \rangle \sim \frac{1}{e^{\beta\frac{\hbar^2 k_1^2}{2m}} - 1} \sim \frac{1}{e^{\beta\frac{h^2}{2mL^2}} - 1} \sim L^2 \sim V^{\frac{2}{3}}$$



$n\lambda_T^3 = \zeta(\frac{3}{2})$ alternatively, fix $T$, reduce volume, $n \nearrow$

$\Rightarrow$ critical density for BEC: $n_c(T)$, $v_c(T)$




along the coexistence line

$$\frac{N}{V} = \underbrace{\frac{N_{exc}}{V}}_{n_{exc}(T)} + \frac{N_0}{V} \qquad N_{exc}\downarrow \Rightarrow N_0 \uparrow$$

$$\begin{cases} p = \frac{k_B T}{\lambda_T^3}\zeta(\frac{5}{2}) & p \sim T^{\frac{5}{2}} \\ v_c = \frac{\lambda_T^3}{\zeta(\frac{3}{2})} & v_c \sim T^{-\frac{3}{2}} \end{cases} \Rightarrow p \sim v_c^{-\frac{5}{3}}$$

Clausius Clapeyron Eqn.

$$\frac{dp}{dT} = \frac{S_2 - S_1}{v_2 - v_1} = \frac{L}{T(v_2 - v_1)} = -\frac{L}{T v_c(T)}$$

1 excited
2 condensed

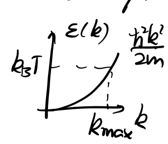
$$p = \frac{k_B T}{\lambda^3}\zeta(\frac{5}{2}) \qquad L = -\frac{5}{2}\frac{k_B T}{\lambda^3} v_c(T)\zeta(\frac{5}{2})$$

$$\frac{dp}{dT} = \frac{5}{2}\frac{k_B}{\lambda^3}\zeta(\frac{5}{2}) \qquad = -\frac{5}{2}k_B T\frac{\zeta(\frac{5}{2})}{\zeta(\frac{3}{2})}$$

* Energy & heat capacity

$$E = \begin{cases} \frac{3}{2}k_B T\frac{V}{\lambda^3}g_{5/2}(z) & T > T_c \\ \frac{3}{2}k_B T\frac{V}{\lambda^3}g_{5/2}(1) & T \le T_c \end{cases}$$

a quick argument for $C_v$.




the temp $T$ is able to excite all the states with energy lower than $k_B T$. so $k_{max} \sim T^{1/2}$. the # of states $(|\vec{k}| \le k_{max}) \sim T^{\frac{3}{2}}$, the contribution of a single state $\sim k_B$ so $C_v \sim T^{\frac{3}{2}}$

$T \le T_c$

$$C_v = \left(\frac{\partial E}{\partial T}\right)_v = \frac{5}{2}\cdot\frac{3}{2}k_B\frac{V}{\lambda^3}g_{5/2}(1) = \frac{15}{4}k_B\frac{V}{\lambda^3}\zeta(\frac{5}{2})$$

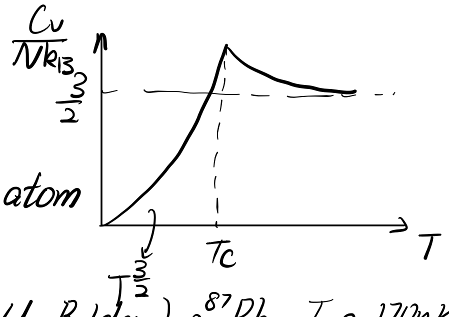
$T > T_c$

$$C_v = \left(\frac{\partial E}{\partial T}\right)_v = \frac{15}{4}k_B\frac{V}{\lambda^3}\zeta(\frac{5}{2}) + \frac{3}{2}k_B T\frac{V}{\lambda^3}g_{5/2}(z)\frac{dz}{dT}$$

$$g_m(z) = \sum_{l=1}^{\infty}\frac{z^l}{l^m}, \qquad \frac{dg_m(z)}{dz} = \frac{1}{z}\sum_{l=1}^{\infty}\frac{z^l}{l^{m-1}} = \frac{1}{z}g_{m-1}(z)$$

$$n\lambda^3 = g_{3/2}(z) \Rightarrow \left(\frac{\partial z}{\partial T}\right)_v =$$

$$\Rightarrow C_v = \frac{15}{4} N k_B \frac{g_{5/2}(z)}{g_{3/2}(z)} - \frac{9}{4} N k_B \frac{g_{3/2}(z)}{g_{1/2}(z)}$$




Experimental Observation of BEC

in ultra-cold atomic gases $\to$ alkali atom

1995 Eric Cormell & Carl Wieman (CU Bolder) $\to$ $^{87}Rb$ $T_c \sim 170nK$

Wolfgang Ketterle (MIT) $\to$ $^{23}Na \sim 2\mu K$

$^4He$

(2001 Nobel prize) Weakly interacting Boson! $\Rightarrow$ superfluid

Superfluidity in $^4He$ (boson, abundant)

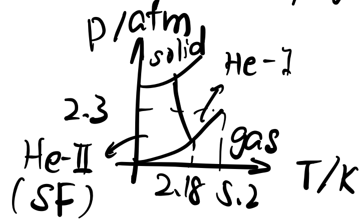
1908 Onnes Liquified at 4.2K

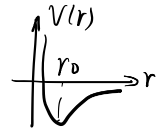
1911 Hg Superconductivity at 4.1K

1925 theoretical production of BEC

1927 $\lambda$ transition be discovered at 2.2K

1938 Kapitza Superfluidity (no viscosity)



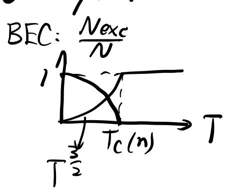

the zero point energy is big enough to escape from the energy trap

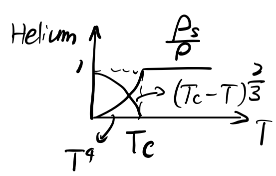
Differences between BEC & SF Helium

1) Helium is an incompressible liquid

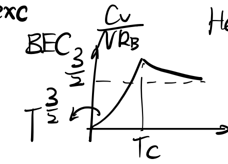
2) $\rho_s$ superfluid, $\rho_N$ normal

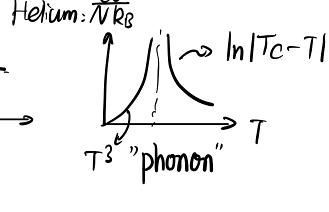
$\Updownarrow$ $n_0$ $\qquad$ $\Updownarrow$ $n_{exc}$




3) heat capacity



8.7, 8.8, 8. 10. 9.9. 9.10. 9.11

Black-body Radiation

Objects emit EM waves when heated

e.g. Steel : dark red → red → yellow → bright ⇒ : $T \nearrow$

black-body : perfect absorber of light

··· radiation : EM fields at a finite $T$ in thermal equilibrium

Quantized EM field ⇒ harmonic oscillator $\varepsilon_{n_{\vec{k},s}} = (n_{\vec{k},s} + \frac{1}{2})\hbar\omega_{\vec{k},s}$

$\vec{k}$ : wave vector ; $s$ : polarization

(i) a collection of harmonic oscillator (distinguishable) with energies $\varepsilon_{n_{\vec{k},s}}$

(ii) a gas of photon with energy $\hbar\omega_{\vec{k},s}$

(i) $E = \sum_{\vec{k},s} \varepsilon_{n_{\vec{k},s}}$

$$Z = \sum_{\{n_{\vec{k},s}\}} e^{-\beta E(\{n_{\vec{k},s}\})}$$

$$= \sum_{\{n_{\vec{k},s}\}} e^{-\beta \sum_{\vec{k},s} (n_{\vec{k},s} + \frac{1}{2})\hbar\omega_{\vec{k},s}}$$

if we don't sum but ingrate here, we shall get Wien's formula

$$= \prod_{\vec{k},s} \sum_{n_{\vec{k},s}=0}^{\infty} e^{-\beta(n_{\vec{k},s} + \frac{1}{2})\hbar\omega_{\vec{k},s}} = \prod_{\vec{k},s} \frac{e^{-\beta\frac{1}{2}\hbar\omega_{\vec{k},s}}}{1 - e^{-\beta\hbar\omega_{\vec{k},s}}}$$

$$\ln Z = \sum_{\vec{k},s} \left(-\frac{1}{2}\beta\hbar\omega_{\vec{k},s}\right) - \sum_{\vec{k},s} \ln\left(1 - e^{-\beta\hbar\omega_{\vec{k},s}}\right)$$

$$E = -\frac{\partial \ln Z}{\partial \beta} = \sum_{\vec{k},s} \hbar\omega_{\vec{k},s}\left(\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega_{\vec{k},s}} - 1}\right)$$

$E_0$ : independent of $\beta$ : constant

$$E - E_0 = \sum_{\vec{k},s} \frac{\hbar\omega_{\vec{k},s}}{e^{\beta\hbar\omega_{\vec{k},s}} - 1}$$

(ii) photon # is not conserved $\mu = 0$

Bose - Einstein distribution with $\mu = 0$

$$E - E_0 = \sum_{\vec{k},s} \varepsilon_{\vec{k},s}\, n_{\vec{k},s} = \sum_{\vec{k},s} \frac{\hbar\omega_{\vec{k},s}}{e^{\beta\hbar\omega_{\vec{k},s}} - 1}$$

(i) H.O. $(\vec{k},s)$ at energy $\varepsilon_{\vec{k},s}$ $\Longleftrightarrow$ (ii) # of photons with energy $\hbar\omega_{\vec{k},s}$ is $n_{\vec{k},s}$

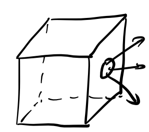
photon dispersion $\omega_{\vec{k},s} = c|\vec{k}|$, $\sum_{\vec{k}} \to \frac{V}{(2\pi)^3}\int d^3\vec{k}$, $\sum_s \to \times 2$

$$E = \frac{2V}{(2\pi)^3}\int d^3\vec{k}\, \frac{\hbar c k}{e^{\beta\hbar c k} - 1} = \frac{2V}{(2\pi)^3}\cdot 4\pi \int_0^\infty \frac{\hbar c k^3}{e^{\beta\hbar c k} - 1}\, dk \qquad x = \beta\hbar c k$$

$$= \frac{V}{\pi^2}\frac{\hbar c}{(\beta\hbar c)^4}\int_0^\infty \frac{x^3}{e^x - 1}\, dx = \frac{V}{\pi^2(\hbar c)^3}(k_B T)^4 \underline{\int_0^\infty dx\, \frac{x^3}{e^x - 1}} \qquad \Gamma(4)\zeta(4) = \frac{\pi^4}{15}$$

$$= \frac{V\pi^2 k_B^4}{15(\hbar c)^3} T^4 \qquad \frac{E}{V} = \frac{\pi^2}{15}\frac{k_B^4}{(\hbar c)^3} T^4 \sim T^4$$

* Energy flux — rate of energy flow per unit area



$$J = \frac{E}{V}\langle c_\perp \rangle \qquad c_\perp = \frac{1}{4\pi}\int d\Omega\, c\cos\theta$$

$$= \frac{c}{4\pi}\, 2\pi \int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\, d\theta = \frac{c}{4}$$

$$\Rightarrow J = \frac{\pi^2}{60}\frac{(k_B T)^4}{\hbar^3 c^2} = \sigma T^4$$ Stefan - Boltzmann Constant

* Radiation Pressure

$$pV = k_B T \ln Z = -k_B T \sum_{\vec{k},s} \ln\left(1 - e^{-\beta\hbar\omega_{\vec{k},s}}\right)$$

$$= -\frac{2V}{(2\pi)^3} k_B T \int d^3k\, \ln\left(1 - e^{-\beta\hbar c k}\right)$$

$$= -\frac{2V}{(2\pi)^3} k_B T\cdot 4\pi \cdot \int dk\, k^2 \ln\left(1 - e^{-\beta\hbar c k}\right)$$

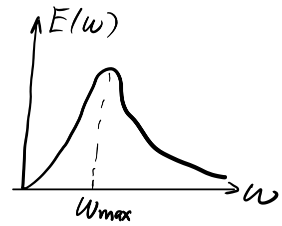
$$= \frac{V}{3\pi^2} k_B T \int dk\, \frac{\beta\hbar c\, k^3}{e^{\beta\hbar c k} - 1}$$

$$= \frac{1}{3}\frac{V}{\pi^2}\int dk \, \frac{k^3 \hbar c}{e^{\beta\hbar c k}-1}$$

$$= \frac{1}{3}E$$

$$\tilde{E}(\omega,T) = \frac{E}{V} = \frac{1}{\pi^2}\int dk \, \frac{k^3\hbar c}{e^{\beta\hbar ck}-1} \overset{\omega=ck}{=\!=} \frac{1}{\pi^2}\int \frac{d\omega}{c}\,\frac{(\frac{\omega}{c})^3\hbar c}{e^{\beta\hbar\omega}-1}$$

$$= \frac{\hbar}{\pi^2c^3}\int_0^\infty d\omega \, \frac{\omega^3}{e^{\beta\hbar\omega}-1} = \int_0^\infty d\omega \; u(\omega,T)$$




$$u(\omega,T)\,d\omega = \frac{\hbar}{\pi^2c^3}\,\frac{\omega^3}{e^{\beta\hbar\omega}-1}$$ Planck distribution

$\frac{du(\omega)}{d\omega} = 0 \Rightarrow \omega_{max} \propto k_BT$  Wien displacement

① $\beta\hbar\omega \gg 1$ high frequency limit

$$u(\omega,T)\,d\omega \approx \frac{\hbar}{\pi^2c^3}\,\omega^3 e^{-\beta\hbar\omega}\,d\omega \quad \text{(Wien, 1896)}$$

② $\beta\hbar\omega \ll 1$ classical limit

Equipartition Theorem

$$u(\omega,T)\,d\omega \approx \frac{1}{\pi^2c^3}\,k_BT\,\omega^2 d\omega \quad \text{(Rayleigh - Jeans, 1900)}$$

$$= k_BT\left(\frac{V}{\pi^2c^3}\,\omega^2 d\omega\right) \to g(\omega)\,d\omega$$ # of modes in $[\omega, \omega+d\omega]$

CMB $T \sim 2.7K$

* Heat capacity of solids

Solid: atoms form a periodic lattice, vibrating around its potential minimum

Kinetic Energy $K = \frac{1}{2}m\sum_{i=1}^{3N}\dot{u}_i^2$  $u_i = x_i - x_i^0 \leftarrow$ equilibrium position

Potential $\Phi \simeq \Phi(\{x_i^0\}) + \frac{1}{2}\sum_{ij}\frac{\partial^2\Phi}{\partial x_i\partial x_j}\Big|_{\{x_i^0\}} u_i u_j + \cdots$

$$H = \frac{1}{2} m \sum_{i=1}^{3N} \dot{u}_i^2 + \frac{1}{2} \sum_{ij} k_{ij} u_i u_j$$

normal mode $H = \frac{1}{2} m \sum_{i=1}^{3N} (\dot{q}_i^2 + \omega_i^2 q_i^2)$ 3N. H.O.

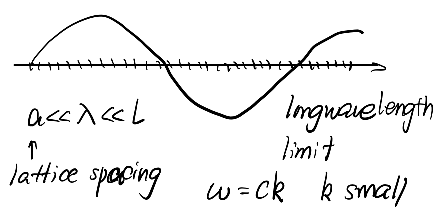
Equipartition $C_V = k_B \cdot 3N$ Dulong - Petit Law

$\Rightarrow$ Quantized

Einstein model $\omega_i = \omega_E$ $C_V = 3N k_B \left(\frac{\Theta_E}{T}\right)^2 \frac{e^{-\frac{\Theta_E}{T}}}{(e^{-\frac{\Theta_E}{T}} - 1)^2}$ $\Theta_E = \frac{\hbar \omega_E}{k_B}$

$C_V$ decays too fast! Excitations are "gapped"

In fact, there are "cheaper" excitations.




Quantized sound wave $\rightarrow$ $\varepsilon_{n_{\vec{k},s}} = (n_{\vec{k},s} + \frac{1}{2}) \hbar \omega_{\vec{k},s}$ phonon

Ideal phonon gas:

$$E = \sum_{\vec{k},s} \frac{\hbar \omega_{\vec{k},s}}{e^{\beta \hbar \omega_{\vec{k},s}} - 1} = \int d\omega\, g(\omega) \frac{\hbar \omega}{e^{\beta \hbar \omega} - 1}$$ $\sum_s$: $t(\times 2) + l(\times 1)$

$$g(\omega) = \frac{3V}{2\pi^2 c^3} \omega^2 = \frac{V\omega^2}{2\pi^2} \left(\frac{1}{c_l^3} + \frac{2}{c_t^3}\right)$$

$$E = \frac{3V}{2\pi^2 c^3} \int_0^\infty d\omega \frac{\hbar \omega^3}{e^{\beta \hbar \omega} - 1}$$ $\omega$ needs a cut-off, or $\lambda \to 0$ and "wave" doesn't make sense!

$$\int_0^{\omega_D} d\omega\, g(\omega) = 3N \quad \boxed{\frac{V}{2\pi^2 c^3} \omega_D^3 = 3N}$$ Debye

denote $x = \beta \hbar \omega$ $E = \frac{3V}{2\pi^2 c^3} \int_0^{x_D} \frac{dx}{\beta \hbar} \frac{\hbar \left(\frac{x}{\beta \hbar}\right)^3}{e^x - 1} = \frac{3V}{2\pi^2 c^3 \hbar^3} (k_B T)^4 \int_0^{x_D} dx \frac{x^3}{e^x - 1}$

① $\beta \hbar \omega_D \gg 1$ low temp limit

$$E = \frac{V\pi^2}{10 (\hbar c)^3} (k_B T)^4, \quad C_V = \frac{dE}{dT} = \frac{2\pi^2 V k_B^4}{5 \hbar^3 c^3} T^3 \sim T^3$$

② $\beta \hbar \omega_D \ll 1$ high temp limit $E = \frac{V}{2\pi^2 c^3 \hbar^3} (k_B T)^4 x_D^3 = \frac{V}{2\pi^2 c^3} (k_B T) \omega_D^3$

$$C_V = \frac{dE}{dT} = \frac{V}{2\pi^2 c^3} k_B \omega_D^3 = 3Nk_B$$

1D atom chain

$$H = \frac{1}{2m}\sum_{i=1}^{N} p_i^2 + \frac{\alpha}{2}\sum_{i=1}^{N}(u_i - u_{i+1})^2$$ assuming PBC

$$\begin{cases} \dot p_i = -\dfrac{\partial H}{\partial u_i} = -\alpha(u_i - u_{i+1}) + \alpha(u_{i-1} - u_i) \\ \dot u_i = \dfrac{\partial H}{\partial p_i} = \dfrac{p_i}{m} \end{cases}$$

$$\Rightarrow \ddot u_i = \frac{\alpha}{m}(u_{i-1} + u_{i+1} - 2u_i)$$

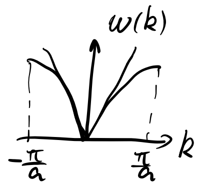
momentum space

Fourier Transform: $u_j(t) = \frac{1}{\sqrt{N}} \sum_{k,\omega_k} \tilde u_k e^{-ikja} e^{-i\omega_k t}$

$$\Rightarrow (-i\omega_k)^2 = \frac{\alpha}{m}(e^{-ika} + e^{ika} - 2)$$ $\cos ka - 1$

$$= -\frac{4\alpha}{m}\sin^2\frac{ka}{2}$$

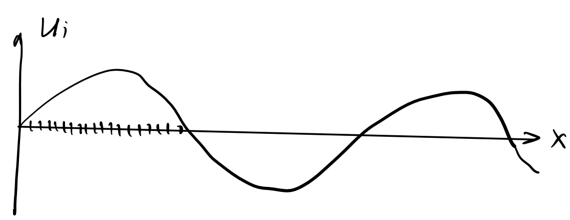
$$\omega_k = 2\sqrt{\frac{\alpha}{m}}\left|\sin\frac{ka}{2}\right|$$



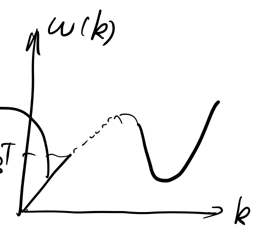

when $k \to 0$ $\omega_k \to k$

Even if there's other coupling $\frac{\alpha_1}{2}(u_i - u_{i+1})^2 + \frac{\alpha_2}{2}(u_i - u_{i+2})^2 + \cdots$

$$\omega(k) = \sqrt{\frac{\alpha_1(1-\cos ka) + \alpha_2(1-\cos 2ka) + \cdots}{m}} \xrightarrow{k\to 0} ck$$ Universality!




$\omega(k) \propto ck$ universal




low $k \to$ long $\lambda$

displacement field $u(x)$

$$H = \int_a dx\left[\frac{\alpha}{2}\left(\frac{du}{dx}\right)^2 + \cdots\right]$$

$\to \lambda$ is far bigger than a

$\to$ "coarse graining" $\to$ average field

cutoff (microscopical structure can't be ignored) UV (short $\lambda$)

in solids , continous translational invariance was replaced by

discrete translational invariance (only by a lattice spacing)

"gaplers phonon" Goldstone mode