

$$N = \sum_k \frac{1}{z^{-1} e^{\beta E_k} + 1}$$

$$E = \sum_k \frac{E_k}{z^{-1} e^{\beta E_k} + 1}$$

$$\ln \Xi = \sum_k \ln (1 + z e^{-\beta E_k})$$

$$E_k = \frac{\hbar^2 k^2}{2m} \text{ in 3D:}$$

$$\sum_k = \frac{gV}{(2\pi)^3} \int d^3 \vec{k} \quad \begin{array}{l} g: \text{spin degeneracies} \\ g = 2s+1 \end{array}$$

$$N = \frac{gV}{(2\pi)^3} \int d^3 \vec{k} \frac{1}{z^{-1} e^{\beta \frac{\hbar^2 k^2}{2m}} + 1}$$

$$= \frac{2gV}{\pi^{\frac{1}{2}}} \frac{1}{\lambda^3} \int_0^\infty \frac{x^{\frac{1}{2}}}{z^{-1} e^x + 1} dx$$

$$\text{Define } f_m(z) = \frac{1}{\Gamma(m)} \int_0^\infty dx \frac{x^{m-1}}{z^{-1} e^x + 1}$$

$$N = \frac{gV}{\lambda^3} f_{3/2}(z), \quad E = \frac{3}{2} \frac{gV}{\lambda^3} f_{5/2}(z), \quad \ln \Xi = \frac{gV}{\lambda^3} f_{3/2}(z), \quad E = \frac{3}{2} pV$$

* Non-degenerate limit $z \ll 1$, $z = e^{\beta \mu}$

Expand $f_m(z)$ in powers of z :

$$f_m(z) = \frac{1}{\Gamma(m)} \int_0^\infty dx x^{m-1} z e^{-x} \frac{1}{1 + z e^{-x}}$$

$$= \frac{1}{\Gamma(m)} \int_0^\infty dx x^{m-1} \sum_{l=1}^\infty (-1)^{l+1} (z e^{-x})^l$$

$$= \sum_{l=1}^\infty (-1)^{l+1} \frac{z^l}{l^m} = -\text{Li}_m(-z)$$

$$\text{Equation of state } \frac{p}{k_B T} = n + B_2 n^2 + B_3 n^3 + \dots$$

$$\textcircled{1} \quad \frac{n \lambda^3}{g} = f_{3/2}(z) = z - \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} - \frac{z^4}{4^{3/2}} + \dots$$

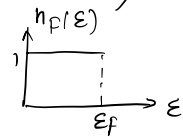
$$\textcircled{2} \quad \frac{p}{k_B T} = \frac{g}{\lambda^3} f_{5/2}(z) \quad \frac{p}{k_B T} = n + \frac{1}{2^{5/2}} \frac{\lambda^3}{g} n^2 + \left(\frac{1}{8} - \frac{2}{3^{5/2}} \right) \left(\frac{\lambda^3}{g} \right)^2 n^3 + \dots$$

For boson, $z \in (0,1)$; for fermions: no such restriction

$$\boxed{T=0} \quad n_F = \frac{1}{e^{\beta(\epsilon-\mu)} + 1} \quad \begin{cases} 1 & \epsilon < \mu \\ 0 & \epsilon > \mu \end{cases}$$

Fermions tend to occupy each energy level until μ successively

μ : last filled energy level $\mu(T=0) = \epsilon_F$ Fermi energy

$$n_F(\epsilon) = \theta(\epsilon_F - \epsilon)$$


$$N = \frac{gV}{(2\pi)^3} \times (\text{Volume in } k\text{-space with } |\vec{k}| \leq k_F)$$

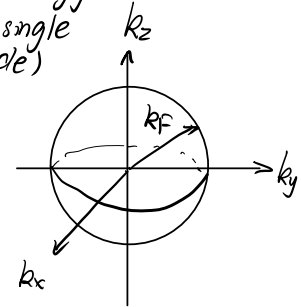
$$= \frac{gV}{(2\pi)^3} \cdot \frac{4}{3}\pi k_F^3 = \frac{gV}{6\pi^2} k_F^3, \quad \frac{\hbar^2 k_F^2}{2m} = \epsilon_F$$

the energy level
(of a single
particle)

$$k_F = \left(\frac{6\pi^2 n}{g}\right)^{\frac{1}{3}} \quad \epsilon_F = \frac{\hbar^2}{2m} \left(\frac{6\pi^2 n}{g}\right)^{\frac{2}{3}} = \mu(T=0)$$

locus of points with $|\vec{k}| = k_F$ Fermi surface

$|\vec{k}| \leq k_F$: Fermi Sea



In general, Fermi surfaces are not spherical (non-isotropical)

$$E = \frac{gV}{(2\pi)^3} \int d^3k \epsilon(k) n_F(\epsilon_k) = \frac{gV}{(2\pi)^3} 4\pi \int_0^{k_F} dk k^2 \frac{\hbar^2 k^2}{2m}$$

$$= \frac{gV\hbar^2}{20\pi^2 m} k_F^5 = \frac{3}{5} \frac{gV}{6\pi^2} k_F^3 \frac{\hbar^2 k_F^2}{2m} = \frac{3}{5} N \epsilon_F$$

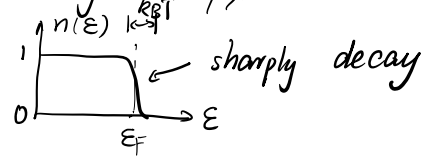
$$pV = \frac{2}{3} E = \frac{2}{3} \times \frac{3}{5} N \epsilon_F = \frac{2}{5} N \epsilon_F \quad p \neq 0 \text{ at } T=0!$$

(Recall bosons $p \sim T^{\frac{5}{2}} \xrightarrow{T \rightarrow 0} 0$) degeneracy pressure
White dwarf

* Electrons in metals $\epsilon_F \sim 1\text{eV} \sim 10^4\text{K} \gg \text{room temperature}$

\Rightarrow Must be visualized as degenerate Fermi gas.

$T \neq 0$ degenerate Fermi gas $\beta\mu \gg 1$

$$n_{\epsilon} = \frac{1}{e^{\beta(\epsilon - \mu)} + 1}$$


$n(\epsilon)$ differs significantly from $T=0$ only within a window

* fermions deep within the Fermi sea is hard to be

$|\epsilon - \mu| \leq k_B T$ thermally excited (Pauli Exclusion) Only particles near Fermi surface can be thermally excited

* $\mu(T) = ?$

$$\frac{n\lambda^3}{g} = f_{3/2}(z), \quad E = \frac{3}{2} \frac{gV}{\lambda^3} k_B T f_{5/2}(z), \quad pV = \frac{2}{3} E$$

(Sommerfield Expansion)

$$\text{Asymptotic Expansion of } f_m(z) = \frac{(\ln z)^m}{m!} \left[1 + \frac{\pi^2}{6} \frac{m(m-1)}{(\ln z)^2} + O\left(\frac{1}{(\ln z)^4}\right) \right]$$

$$\frac{n\lambda^3}{g} = f_{3/2}(z) \simeq \frac{(\ln z)^{3/2}}{(\frac{3}{2})!} \left[1 + \frac{\pi^2}{8} \frac{1}{(\ln z)^2} + \dots \right] = \frac{(\ln z)^{3/2}}{(\frac{3}{2})!}$$

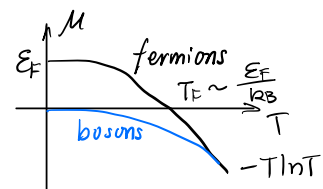
$$\ln z = \left(\frac{(\frac{3}{2})! n \lambda^3}{g} \right)^{2/3} = \left(\frac{3\sqrt{\pi} n \lambda^3}{4g} \right)^{2/3} = \left(\frac{3\sqrt{\pi} n}{4g} \right)^{2/3} \frac{\hbar^2 \beta}{2m} 4\pi = \beta \epsilon_F$$

o First order correction

$$\begin{aligned} \ln z &= \left(\frac{(\frac{3}{2})! n \lambda^3}{g} \right)^{2/3} \left[1 + \frac{\pi^2}{8} \frac{1}{(\ln z)^2} \right]^{-2/3} \\ &= \left(\frac{(\frac{3}{2})! n \lambda^3}{g} \right)^{2/3} \left[1 + \frac{\pi^2}{8} \left(\frac{k_B T}{\epsilon_F} \right)^2 \right]^{-2/3} \simeq \beta \epsilon_F \left[1 - \frac{\pi^2}{12} \left(\frac{k_B T}{\epsilon_F} \right)^2 \right] \end{aligned}$$

$\rightarrow \sim 10^{-4}$

$$\mu(T) = \epsilon_F \left[1 - \frac{\pi^2}{12} \left(\frac{k_B T}{\epsilon_F} \right)^2 \right] \quad \text{shift in } \mu \quad O\left(\left(\frac{k_B T}{\epsilon_F}\right)^2\right)$$



$$p = \frac{g}{\lambda^3} k_B T f_{5/2}(z) \simeq \frac{g}{\lambda^3} k_B T \frac{(\ln z)^{5/2}}{(\frac{5}{2})!} \left[1 + \frac{5\pi^2}{8} \frac{1}{(\ln z)^2} + \dots \right]$$

$$\text{Use } \ln z \simeq \beta \epsilon_F \left[1 - \frac{\pi^2}{12} \left(\frac{k_B T}{\epsilon_F} \right)^2 \right]$$

$$p \simeq \frac{g}{\lambda^3} k_B T \frac{8}{15\sqrt{\pi}} (\beta \epsilon_F)^{5/2} \left[1 - \frac{\pi^2}{12} \left(\frac{k_B T}{\epsilon_F} \right)^2 \right]^{5/2} \left[1 + \frac{5\pi^2}{8} \frac{1}{(\beta \epsilon_F)^2} \right]$$

$$= p_F \left[1 - \frac{5}{2} \times \frac{\pi^2}{12} \left(\frac{k_B T}{\epsilon_F} \right)^2 + \frac{5\pi^2}{8} \left(\frac{k_B T}{\epsilon_F} \right)^2 \right] = p_F \left[1 + \frac{5\pi^2}{12} \left(\frac{k_B T}{\epsilon_F} \right)^2 \right]$$

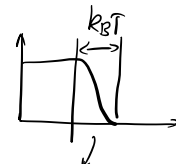
$$E = \frac{3}{2} pV = \frac{3}{5} N \epsilon_F \left[1 + \frac{5\pi^2}{12} \left(\frac{k_B T}{\epsilon_F} \right)^2 \right]$$

heat capacity

$$C_V = \frac{dE}{dT} = \frac{3}{5} N \epsilon_F \times \frac{5\pi^2}{12} \times 2 \frac{k_B^2 T}{\epsilon_F^2} = \frac{\pi^2}{2} N k_B \left(\frac{k_B T}{\epsilon_F} \right) \propto T$$

$$\propto g(\epsilon_F) k_B^2 T = g(\epsilon_F) k_B T \cdot k_B$$

↑ density of state at Fermi surface



only this part of particles contributes to C_V (can be thermally excited)

Recall for bosons $C_V \sim T^{\frac{d}{2}}$

$$\text{Metals} = \underset{\substack{\downarrow \\ \text{phonon}}}{\alpha T^3} + \underset{\substack{\downarrow \\ \text{electron}}}{\gamma T}$$

* Sommerfeld Expansion

$$f_m(z) = \frac{1}{\Gamma(m)} \int_0^\infty dx \frac{x^{m-1}}{z^{-1}e^x + 1} = \frac{1}{m!} \int_0^\infty dx x^m \frac{d}{dx} \left(\frac{-1}{z^{-1}e^x + 1} \right)$$

peaked at $x \simeq \ln z$

Let $x = \ln z + t$, so we can use the $-\infty$ as the lower limit

$$\begin{aligned} f_m(z) &= \frac{1}{m!} \int_{-\infty}^{+\infty} dt (\ln z + t)^m \frac{d}{dt} \left(\frac{-1}{e^t + 1} \right) \\ &\simeq \frac{1}{m!} \int_{-\infty}^{+\infty} dt (\ln z)^m \left(1 + \frac{t}{\ln z} \right)^m \frac{d}{dt} \left(\frac{-1}{e^t + 1} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{m!} \int_{-\infty}^{+\infty} dt (\ln z)^m \sum_{l=0}^{\infty} \frac{m!}{l!(m-l)!} \left(\frac{t}{\ln z}\right)^l \frac{d}{dt} \left(\frac{-1}{e^t+1}\right) \\
&= \sum_{l=0}^{\infty} (\ln z)^{m-l} \frac{1}{l!(m-l)!} \int_{-\infty}^{+\infty} \frac{t^l}{(e^t+1)(e^{-t}-1)} dt
\end{aligned}$$

if l is odd, then the integral would be 0

$$l=0: \int_{-\infty}^{+\infty} d\left(\frac{-1}{e^t+1}\right) = 1$$

$$\begin{aligned}
l \text{ is even } \int_{-\infty}^{+\infty} t^l d\left(\frac{-1}{e^t+1}\right) &= 2 \int_0^{+\infty} t^l d\left(\frac{-1}{e^t+1}\right) = 2l \int_0^{+\infty} \frac{t^{l-1}}{e^t+1} dt \\
&= 2l \Gamma(l) f_l(1) = 2l! f_l(1)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow f_m(z) &= \frac{(\ln z)^m}{m!} \left[1 + 2 \sum_{l=2,4,6,\dots}^{\infty} \frac{m!}{(m-l)!} \frac{f_l(1)}{(\ln z)^l} \right] \\
&= 1 + \left[\frac{\pi^2}{6} \frac{m(m-1)}{(\ln z)^2} + \frac{7\pi^4}{360} \frac{m(m-1)(m-2)(m-3)}{(\ln z)^4} + O(\ln z)^{-6} \right]
\end{aligned}$$

* Electrons in a magnetic field: Pauli Paramagnetism

electron spin, or, magnetic moment couples to external \vec{B}

$$E_{\text{spin}} = \mu_B \cdot B \sigma \rightarrow \begin{cases} +1 & \text{spin up} \\ -1 & \text{spin down} \end{cases}$$

$$\downarrow$$

$$\mu_B = \frac{e\hbar}{2mc} \text{ Bohr magneton}$$

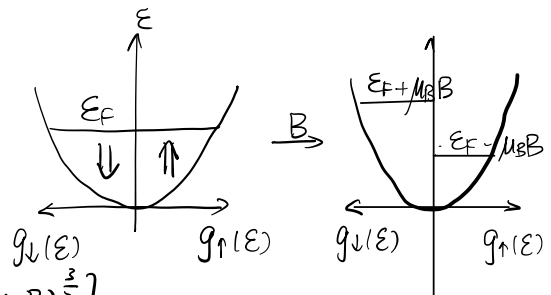
$$H = \frac{p^2}{2m} + \mu_B \cdot B \sigma$$

$$g_{\uparrow}(\epsilon) = g_{\downarrow}(\epsilon) = \frac{2\pi V}{h^3} (2m)^{\frac{3}{2}} \epsilon^{\frac{1}{2}}$$

$$N_{\downarrow} - N_{\uparrow} = \frac{4\pi V}{3h^3} (2m)^{\frac{3}{2}} \left[(\epsilon_F + \mu_B B)^{\frac{3}{2}} - (\epsilon_F - \mu_B B)^{\frac{3}{2}} \right]$$

$$\left(\frac{4\pi V}{3h^3} (2m)^{\frac{3}{2}} \left[(\tilde{\epsilon}_F + \mu_B B)^{\frac{3}{2}} - (\tilde{\epsilon}_F - \mu_B B)^{\frac{3}{2}} \right] \right) = \frac{4\pi V}{3h^3} (2m)^{\frac{3}{2}} (\epsilon_F^{\frac{3}{2}} + \epsilon_F^{\frac{3}{2}})$$

$$\delta \epsilon_F \sim \left(\frac{\mu_B B}{\epsilon_F} \right) \quad \mu_B \sim 10^{-5} \text{ eV} \cdot \text{T}^{-1}$$



$$= \frac{4\pi V}{3h^3} (2m)^{\frac{3}{2}} \epsilon_F^{\frac{3}{2}} \cdot \frac{3}{2} \cdot 2 \cdot \frac{\mu_B B}{\epsilon_F}$$

$$= \frac{4\pi V}{h^3} (2m)^{\frac{3}{2}} \epsilon_F^{\frac{1}{2}} \mu_B B$$

$$= g(\epsilon_F) \mu_B B$$

magnetization $M = \mu_B (N_{\downarrow} - N_{\uparrow}) = g(\epsilon_F) \mu_B^2 B$

susceptibility $\chi = \frac{\partial M}{\partial B} \Big|_{B \rightarrow 0} = g(\epsilon_F) \mu_B^2$

① $\chi_{\text{Pauli}} > 0$ paramagnetism

② $T \rightarrow 0$ χ_{Pauli} saturates to a finite value

③ high temp $\chi = \frac{\mu_B^2 N}{k_B T}$ "Curie's Law"

$T \neq 0$, $N = \frac{gV}{\lambda^3} f_{\frac{3}{2}}(z)$ $e^{\beta(\epsilon - \mu + \mu_B B \sigma)}$ $z^{-1} \rightarrow z^{-1} e^{\beta \mu_B B \sigma}$

$$N_{\downarrow} - N_{\uparrow} = \frac{V}{\lambda^3} [f_{\frac{3}{2}}(ze^{\beta \mu_B B}) - f_{\frac{3}{2}}(ze^{-\beta \mu_B B})]$$

when T is large, $e^{\beta \mu_B B} \approx 1 + \beta \mu_B B$

$$\chi(T) \simeq \chi(0) \left[1 - \frac{\pi^2}{12} \left(\frac{k_B T}{\epsilon_F} \right)^2 \right]$$

Landau diamagnetism

e^- orbital motion couples to B

QM $E_n = (n + \frac{1}{2}) \hbar \omega_c + \frac{\hbar^2 k_z^2}{2m}$ $\omega_c = \frac{eB}{m}$ cyclotron frequency

Landau level

susceptibility $\chi = -\frac{1}{3} \chi_{\text{Pauli}} = -\frac{1}{3} g(\epsilon_F) \mu_B^2 < 0$
 \uparrow
 $(\frac{m}{m_*})^2$

In an exterior magnetic field $\vec{B} = B\hat{k}$, an electron moves along a spiral, its energy would be

$$\varepsilon = \frac{e\hbar B}{mc} (j + \frac{1}{2}) + \frac{p_z^2}{2m} \quad j = 0, 1, 2, \dots$$

All those states of which the values of the quantity $\frac{p_x^2 + p_y^2}{2m}$ lay between $\frac{e\hbar B}{mc} \cdot j$ and $\frac{e\hbar B}{mc} (j+1)$ would now "coalesce together" into a single state characterized by the quantum number j . The number of these states is given by

$$g_j = \frac{1}{h^2} \int dx dy dp_x dp_y = \frac{L_x L_y}{h^2} \pi \cdot 2m \cdot \frac{e\hbar B}{mc} [(j+1) - j] = L_x L_y \frac{eB}{hc}$$

Then the Grand Canonical Partition Function would be given by

$$\begin{aligned} \ln \Xi &= \sum_{\varepsilon} \ln (1 + z e^{-\beta \varepsilon}) \\ &= \frac{g_j}{h} \sum_{j=0}^{\infty} \int dz dp_z \ln \left[1 + z \exp \left(-\beta \left[\frac{e\hbar B}{mc} (j + \frac{1}{2}) + \frac{p_z^2}{2m} \right] \right) \right] \\ &= \frac{eBV}{h^2 c} \sum_{j=0}^{\infty} \int dp_z \ln \left(1 + z \exp \left[-\beta \frac{e\hbar B}{mc} (j + \frac{1}{2}) \right] \cdot e^{-\beta \frac{p_z^2}{2m}} \right) \end{aligned}$$

The magnetization $m = \frac{M}{V} = \frac{N}{V} \langle \mu \rangle = \frac{k_B T}{V} \left\langle \frac{\partial \ln \Xi}{\partial B} \right\rangle$, then the susceptibility

$$\chi = \frac{\partial m}{\partial B} = \frac{1}{V} \frac{\partial^2}{\partial B^2} \langle k_B T \ln \Xi \rangle$$

In the high temp limit, $z \ll 1$. Expand $\ln (1 + z e^{-\beta \varepsilon})$ in the series of z .

$$\begin{aligned} \ln \Xi &= \frac{eBV}{h^2 c} \sum_{j=0}^{\infty} \int_0^{\infty} dp_z z \exp \left[-\beta \frac{e\hbar B}{mc} (j + \frac{1}{2}) \right] \cdot e^{-\beta \frac{p_z^2}{2m}} \\ &= V \frac{eB}{hc} \sqrt{\frac{2\pi m}{\beta}} z \sum_{j=0}^{\infty} \exp \left[-\beta \frac{e\hbar B}{mc} (j + \frac{1}{2}) \right] \end{aligned}$$

$$= V \frac{eB}{\hbar c} \sqrt{\frac{2\pi m}{\beta}} z \cdot \frac{e^{-\beta \frac{e\hbar B}{2mc}}}{1 - e^{-\beta \frac{e\hbar B}{2mc}}} = V \frac{eB}{\hbar \lambda} z (z \sinh \beta \frac{e\hbar B}{2mc})^{-1}$$

The equilibrium number of particles \bar{N} and the magnetic moment M would be: (denote $\beta \frac{e\hbar B}{2mc} = \beta \mu_{\text{eff}} B = x$, where $\mu_{\text{eff}} = \frac{e\hbar}{2mc}$) $\ln \Xi = \frac{Vz}{\lambda^3} \frac{x}{\sinh x}$

$$\bar{N} = z \frac{\partial}{\partial z} \ln \Xi = \frac{zV}{\lambda^3} \frac{x}{\sinh x}$$

$$M = \frac{1}{\beta} \frac{\partial \ln \Xi}{\partial B} = \mu_{\text{eff}} \frac{\partial \ln \Xi}{\partial x} = \mu_{\text{eff}} \frac{zV}{\lambda^3} \left(\frac{1}{\sinh x} - \frac{x \cosh x}{\sinh^2 x} \right)$$

Combining these two equations, we get:

$$M = -\bar{N} \mu_{\text{eff}} L(x), \quad L(x) = \coth x - \frac{1}{x}$$

If the field intensity B and temp T satisfy $\mu_{\text{eff}} B \ll k_B T$ (weak field and high temp, $x \ll 1$), then \bar{N} and M reduce to

$$\bar{N} \simeq \frac{zV}{\lambda^3}, \quad M \simeq \mu_{\text{eff}} \frac{zV}{\lambda^3} \left[\frac{1}{x} - \frac{1 + \frac{x^2}{2}}{x(1 + \frac{x^2}{6})} \right] \simeq -\mu_{\text{eff}} \frac{zV}{\lambda^3} \frac{x}{3} = -\bar{N} \mu_{\text{eff}}^2 \frac{B}{3k_B T}$$

so the counterpart to the Curie's Law is:

$$\chi_{\text{dia}} = \frac{1}{V} \frac{\partial \bar{N}}{\partial B} = -\bar{n} \mu_{\text{eff}}^2 / 3 k_B T = -\frac{1}{3} \chi_{\text{para}}$$

For the free electron gas, the net susceptibility

$$\chi = \chi_{\text{para}} - \chi_{\text{dia}} = \frac{n}{k_B T} (\mu_B^2 - \frac{1}{3} \mu_{\text{eff}}^2)$$

In a finite temp T (but still $\mu_{\text{eff}} B \ll k_B T$), the summation can be handled by the Euler summation formula:

$$\sum_{j=0}^{\infty} f(j + \frac{1}{2}) \simeq \int_0^{\infty} f(x) dx + \frac{1}{24} f'(0)$$

$$\begin{aligned}
\ln \Xi &= \frac{eBV}{h^2 c} \sum_{j=0}^{\infty} \int dp_z \ln \left(1 + z \exp \left[-\beta \frac{e\hbar B}{mc} \left(j + \frac{1}{2} \right) \right] \cdot e^{-\beta \frac{p_z^2}{2m}} \right) \\
&= \frac{VeB}{h^2 c} \left[\int_0^{\infty} dx \int_{-\infty}^{+\infty} dp_z \ln \left[1 + z \exp \left(-\beta \mu_{\text{eff}} B x \right) \cdot e^{-\beta \frac{p_z^2}{2m}} \right] \right. \\
&\quad \left. - \frac{1}{12} \beta \mu_{\text{eff}} B \int_{-\infty}^{+\infty} dp_z \frac{1}{z^{-1} e^{\beta \frac{p_z^2}{2m}} + 1} \right] \\
x' = Bx &\Rightarrow \frac{2Ve}{h^2 c} \int_0^{\infty} dx' \int_0^{+\infty} dp_z \ln \left(1 + z \exp \left[-\beta \left(2\mu_{\text{eff}} x' + \frac{p_z^2}{2m} \right) \right] \right) \\
&\quad - \frac{VeB^2}{6h^2 c} \beta \mu_{\text{eff}} \int_0^{+\infty} dp_z \frac{1}{z^{-1} e^{\beta \frac{p_z^2}{2m}} + 1}
\end{aligned}$$

the first term is independent of B , and the second term:

$$\begin{aligned}
& - \frac{VeB^2}{6h^2 c} \beta \mu_{\text{eff}} \int_0^{+\infty} dp_z \frac{1}{z^{-1} e^{\beta \frac{p_z^2}{2m}} + 1} = - \frac{VeB^2}{6h^2 c} \beta \mu_{\text{eff}} \sqrt{\frac{m}{2\beta}} \int_0^{+\infty} dy \frac{y^{-\frac{1}{2}}}{z^{-1} e^y + 1} \\
& = - \frac{VeB^2}{6h^2 c} \beta^{\frac{1}{2}} \mu_{\text{eff}} \sqrt{\frac{m}{2}} \sqrt{\pi} f_{\frac{1}{2}}'(z) = - \frac{VB^2}{6h^3} (2\pi m)^{\frac{3}{2}} \beta^{\frac{1}{2}} \mu_{\text{eff}}^2 f_{\frac{1}{2}}'(z)
\end{aligned}$$

The weak-field susceptibility of the gas:

$$\begin{aligned}
\chi &= \frac{1}{V} \frac{\partial M}{\partial B} = \frac{1}{\beta V} \frac{\partial \ln \Xi}{\partial B^2} \\
&= - \frac{1}{3h^3 \beta^{\frac{1}{2}}} (2\pi m)^{\frac{3}{2}} \mu_{\text{eff}}^2 f_{\frac{1}{2}}'(z)
\end{aligned}$$

so this effect is diamagnetic in character.

When $z \ll 1$, $f_{\frac{1}{2}}(z) \simeq z \simeq n\lambda^3$, so at the high temp limit

$$\chi_{\infty} = - \frac{n \mu_{\text{eff}}^2}{3 k_B T} = - \frac{1}{3} \chi_{\infty, \text{para}}$$

; when $z \gg 1$, $f_{\frac{1}{2}}(z) = \frac{2}{\sqrt{\pi}} (\ln z)^{\frac{1}{2}} \simeq 2 \sqrt{\frac{\beta \epsilon_F}{\pi}}$, at $T=0$:

$$\chi_0 = - \frac{n \mu_{\text{eff}}^2}{2 \epsilon_F} = - \frac{1}{3} \chi_{0, \text{para}}$$