

Colloquial: an abrupt, discontinuous change of the properties of a system

\* Non-analyticity in thermodynamic functions

\* Occurs only in the thermodynamics limit

### Critical Exponents

Quantities display power-law behaviour sufficiently close to the critical point

e.g. Liquid-gas transition

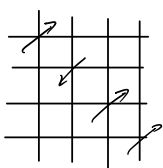
① order parameter as  $T \rightarrow T_c^-$   $v_g - v_l \sim (T_c - T)^\beta$

② critical isotherm ( $T = T_c$ )  $p - p_c \sim (v - v_c)^\delta$  or  $(v - v_c) \sim (p - p_c)^{\frac{1}{\delta}}$

③

\* The Ising model

◦ Describes spontaneous magnetization of a uniaxial magnet



d.o.f: a "spin", or magnet moment on

each site. 2 states per site:  $\sigma_i = \pm 1$

"Lattice model"

$$H = -J \sum \sigma_i \sigma_j - h \sum \sigma_i$$

$\langle ij \rangle \uparrow$  — magnetic field  
 summing over nearest  
 neighbour sites

- $J > 0$  : favours " $\uparrow\uparrow$ " or " $\downarrow\downarrow$ ", ferromagnetic
- $J < 0$  : favours " $\uparrow\downarrow$ " or " $\downarrow\uparrow$ " anti-ferromagnetic

"exchange interaction"  $\Rightarrow$   $\left\{ \begin{array}{l} \text{Coulomb interaction} \\ \text{anti-symmetric} \\ \text{wavefunction} \end{array} \right.$

Partition function

$$Z = \sum_{\{\sigma_i\}} e^{\beta J \sum_{\langle ij \rangle} \sigma_i \sigma_j + \beta h \sum_i \sigma_i} \neq Z_1^N \text{ (interacting system)}$$

In general, exact solution not possible

$d=1 \checkmark$ ;  $d=2$  ( $h=0$ ): square lattice (Onsager 1944)

\* Mean Field Theory

$$\langle \sigma_i \rangle = m$$

approximation: assume a uniform average magnetization

$$\sigma_i = m + (\sigma_i - m)$$

$$\sigma_i \sigma_j = [m + (\sigma_i - m)] [m + (\sigma_j - m)] \quad \text{"small"}$$

$$= m^2 + m(\sigma_i - m) + m(\sigma_j - m) + \cancel{(\sigma_i - m)(\sigma_j - m)}^0$$

$$= -m^2 + m(\sigma_i + \sigma_j)$$

$$H_{mf} = -J \sum_{\langle ij \rangle} [-m^2 + m(\sigma_i + \sigma_j)] - h \sum_i \sigma_i$$

$$= \frac{1}{2} J m^2 N q - (J m q + h) \sum_i \sigma_i \xrightarrow{\text{heff}}$$

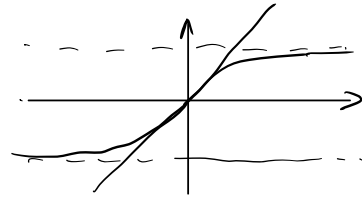
$q$  is the # of nearest-neighbor site

$$\begin{aligned} Z_{MF} &= e^{-\frac{1}{2} \beta N J q m^2} \left( \sum_{\sigma_i = \pm 1} e^{-\beta h_{\text{eff}} \sigma_i} \right)^N \\ &= e^{-\frac{1}{2} \beta N J q m^2} [2 \cosh(\beta h_{\text{eff}})]^N \end{aligned}$$

$$\langle \sigma_i \rangle = \frac{1}{N \beta} \frac{\partial \ln Z_{MF}}{\partial h} = \tanh(\beta h_{\text{eff}}) = \tanh(\beta J q m + \beta h)$$

Self-consistent equation

$$m = \tanh[\beta(J q m + h)]$$



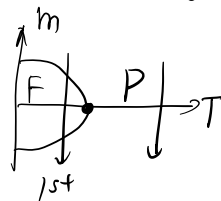
$$* h=0 \quad m = \tanh(\beta J q m)$$

$$\text{Let } x = \beta J q m \Rightarrow \frac{x}{\beta J q} = \tanh x$$

$$\textcircled{1} \beta J q < 1, \quad x = 0$$

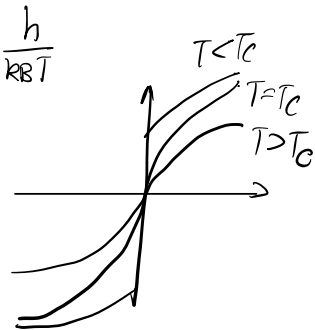
$$\textcircled{2} \beta J q > 1 \quad x = 0, \quad x = \pm x_0 \quad (\neq 0) \quad \text{spontaneous magnetization}$$

$$\textcircled{3} \beta J q = 1 \quad k_B T_c = J q$$



$$h \neq 0 \quad \text{As } T \rightarrow \infty (\beta \rightarrow 0) \quad m \approx \beta(h + J q m) \approx \frac{h}{k_B T}$$

Isotherm



Free energy within MFT

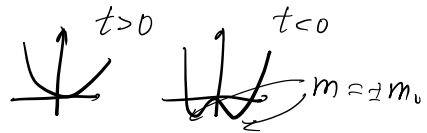
$$f_{MF} = \frac{F_{MF}}{N} = \frac{1}{2} J q^2 m^2 - k_B T \ln [2 \cosh(\beta J q m + \beta h)]$$

$h=0$ : Close to the critical point, Expand  $f$  in power of  $m$

$$f \simeq \frac{1}{2} Jq m - \frac{1}{2} \beta (Jq)^2 m^2 + \frac{1}{12} \beta^3 (Jq)^4 m^4 + O(m^6)$$

$$= \frac{1}{2} \frac{Jq(1-\beta Jq)}{t} m^2 + \frac{1}{12} \beta^3 (Jq)^4 m^4 + O(m^6)$$

$$f(m) = \frac{1}{2} t m^2 + u m^4 + O(m^6)$$



Thermal equil: saddle point:  $m^*$  minimizes  $f(m)$   $\frac{\partial f}{\partial m}|_{m^*} = 0$

$$t = Jq(1-\beta Jq) = \frac{1}{\beta_c}(1-\frac{\beta}{\beta_c}) \simeq T_c(1-\frac{T}{T_c}) \simeq T-T_c$$

Spontaneous Symmetry Breaking (SSB)

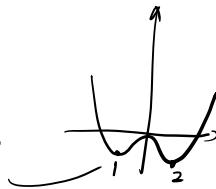
$$h=0, H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j$$

$\sigma_i \rightarrow -\sigma_i$  for all  $i$   $H$  invariant  
symmetry

$\mathbb{Z}_2$  symmetry

$$\langle \sigma_i \rangle \neq 0 \quad \text{SSB}$$

$$\langle \sigma_i \rangle = \frac{\sum_{\{\sigma_i\}} \sigma_i e^{\beta J \sum_{\langle ij \rangle} \sigma_i \sigma_j}}{Z} = 0?$$

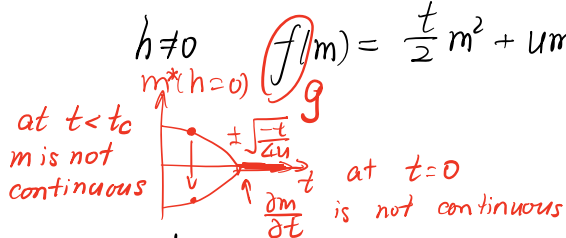


average of 1 and 2  
but the timescale from 1 to 2 is  $e^N$ !

$$\Rightarrow \langle \sigma_i \rangle = \lim_{h \rightarrow 0} \lim_{N \rightarrow \infty} \frac{1}{N\beta} \frac{\partial \ln Z(h)}{\partial h}$$

Always take TDL first!  
(order of limits matters)

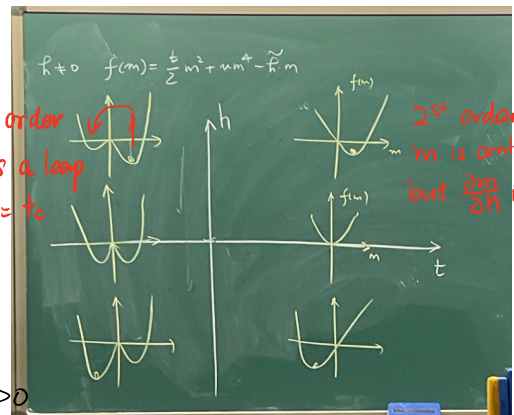
$$h \neq 0 \quad f(m) = \frac{t}{2} m^2 + u m^4 - \tilde{h} \cdot m$$



Critical exponents within MFT

$$0 \quad m \sim (t)^{\beta} \quad \text{at } h=0 \quad \nu_g - \nu_u \sim (T_c - T)^{\beta}$$

$$\frac{\partial f}{\partial m}|_{m^*} = t m + 4u m^3 = 0 \Rightarrow m^* = \begin{cases} 0 & t > 0 \\ \pm \sqrt{-t/4u} & t < 0 \end{cases}$$



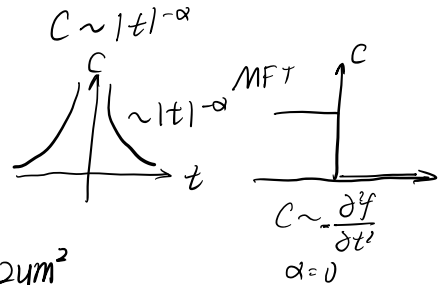
1st order  $m$  has a loop at  $t=t_c$   
2nd order  $m$  is continuous but  $\frac{\partial m}{\partial h}$  is not

$$\beta = \frac{1}{2} \quad \begin{array}{c} (t)^{\frac{1}{2}} \\ \uparrow \\ m \\ \text{---} t \end{array}$$

②  $m \sim h^{\frac{1}{\delta}}$  at  $t=0$   $v-v_c \sim (p-p_c)^{\frac{1}{\delta}}$  ④ heat capacity

$$\frac{\partial f}{\partial m} = 4nm^3 - h = 0 \quad m^* \sim \left(\frac{h}{4u}\right)^{\frac{1}{3}} \quad \delta = 3$$

③  $\chi = \left. \frac{\partial m}{\partial h} \right|_{h \rightarrow 0} \sim |t|^{-\gamma} \quad K_T \sim (T-T_c)^{-\gamma}$



$$\left. \frac{\partial f}{\partial m} \right|_{m^*} = 4um^3 - h = 0 \quad \frac{\partial h}{\partial m} = t + 12um^2$$

$$\chi = \frac{1}{t + 12um^{*2}} = \begin{cases} \frac{1}{t} & t > 0 \\ -\frac{1}{2t} & t < 0 \end{cases}$$

Exact solution of Ising model

$d=1$  MFT fails completely

$d=2$  MFT qualitatively correct, exponents are wrong

•  $d=1$  PBC ( $\sigma_i = \sigma_{i+N}$ )

$$H = -J \sum_i \sigma_i \sigma_{i+1} - \frac{h}{2} \sum_i (\sigma_i + \sigma_{i+1}) \quad \nearrow h \sum_i \sigma_i$$

$$Z = \sum_{\{\sigma_i\}} e^{\beta J \sum_i \sigma_i \sigma_{i+1} + \beta \frac{h}{2} \sum_i (\sigma_i + \sigma_{i+1})}$$

$$= \sum_{\{\sigma_i\}} \prod_{i=1}^N e^{\beta J \sigma_i \sigma_{i+1} + \beta \frac{h}{2} (\sigma_i + \sigma_{i+1})} \quad \rightarrow T[i]_{\sigma_i, \sigma_{i+1}}$$

$$T = \begin{pmatrix} e^{\beta J + \beta h} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J - \beta h} \end{pmatrix} \begin{matrix} \sigma_{i+1} = 1 \\ \sigma_{i+1} = -1 \end{matrix}$$

"transfer matrix"

$$= \sum_{\{\sigma_i = \pm 1\}} T_{\sigma_1, \sigma_2} T_{\sigma_2, \sigma_3} \dots T_{\sigma_{N-1}, \sigma_N} T_{\sigma_N, \sigma_1}$$

$$= \text{tr}(T^N) = \sum_i \lambda_i^N \quad \lambda_{\pm} = e^{\beta J} \cosh(\beta h) \pm \sqrt{e^{2\beta J} \cosh^2(\beta J) - 2 \sinh(\beta J)}$$

$$= \lambda_+^N + \cancel{\lambda_-^N} \xrightarrow{N \rightarrow \infty} \lambda_+^N$$

$$|h=0| \quad \lambda_+ = 2 \cosh(\beta J) \Rightarrow \ln Z = N \ln[2 \cosh(\beta J)] \quad \text{paramagnet}$$

$\Rightarrow$  No phase transition

Simple Argument

$$\begin{array}{c} \text{domain wall} \\ +++ \cdots +++ \Rightarrow ++++|--- \end{array}$$

$$\Delta E \sim +2J \quad \Delta S \sim k_B \ln \frac{L}{a} \quad \rightarrow \text{entropy always wins for } T \neq 0$$

$$\Delta F = \Delta E - T \Delta S = 2J - k_B T \ln \frac{L}{a} \xrightarrow{L \rightarrow \infty} \Delta F < 0$$



$$\begin{aligned} \Delta E &= 2Jl \\ \Delta S &\sim k_B \ln(g^l) \quad \Delta F = (2J - k_B T \ln g)l \end{aligned}$$

"Lower critical dimension"  $d_L$   $\xrightarrow{\text{Ising } d_L=1}$   
 $d \leq d_L$  no phase transition

2D Ising model (square lattice)

1944 Onsager

$$Z = \sum_{\{\sigma_i\}} e^{\beta J \sum_{\langle ij \rangle} \sigma_i \sigma_j} = \sum_{\{\sigma_i\}} e^{K \sum_{\langle ij \rangle} \sigma_i \sigma_j} \quad K \equiv \beta J$$

Compute  $Z$  in two limits

1. Low Temp limit

$$Z \simeq e^{-\beta E_0} + e^{-\beta E_1} g_1 + e^{-\beta E_2} g_2 + \dots$$

① GS

$$\begin{array}{ccc} + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{array} \quad \text{or} \quad \begin{array}{ccc} - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{array} \quad \begin{aligned} E_{GS} &= -2NJ \\ \text{deg: } &2 \end{aligned}$$

② 1<sup>st</sup> excited state

$$\begin{array}{ccc} + & + & + \\ + & - & + \\ + & + & + \end{array} \quad E_1 = E_{GS} + 2J \times 4$$

deg:  $N$

$$Z = 2e^{2NK} \left[ 1 + Ne^{-8K} + 2Ne^{-12K} + \frac{N^2 + 9N}{2} e^{-16K} + \dots \right]$$

③ 2<sup>nd</sup> excited state

$$\begin{array}{ccc} + & + & + \\ + & - & - \\ + & + & + \end{array} \quad E_2 = E_{GS} + 2J \times 6$$

$2N$

④ 3<sup>rd</sup> excited state

$$\begin{array}{ccc} + & + & + \\ + & - & + \\ + & + & + \end{array} \quad E_3 = E_{GS} + 2J \times 8 \quad \frac{N(N-5)}{2}$$

2. high temp limit

$$\begin{aligned} e^{K\sigma_i\sigma_j} &\xrightarrow{(\sigma_i\sigma_j)^2=1} \cosh K + \sigma_i\sigma_j \sinh K \\ &= \cosh K (1 + \sigma_i\sigma_j \tanh K) \end{aligned}$$

$$\begin{aligned} Z &= \sum_{\{\sigma_i\}} \prod_{(ij)} \cosh K (1 + \sigma_i\sigma_j \tanh K) \\ &= (\cosh K)^{2N} \sum_{\{\sigma_i\}} \prod_{(ij)} (1 + \sigma_i\sigma_j \tanh K) \end{aligned}$$

$$= 2^N (\cosh K)^{2N} \left[ 1 + N(\tanh K)^4 + 2N(\tanh K)^6 + \frac{1}{2}(N^2 + 9N)(\tanh K)^8 + \dots \right]$$

identify  $\tanh K$  with  $e^{-2\tilde{K}}$  (identical series at low  $T$  & high  $T$ !)

$$e^{-2\tilde{K}} = \tanh K \Leftrightarrow (\sinh 2K)(\sinh 2\tilde{K}) = 1$$

$\Rightarrow$  Kramers-Wannier Duality

1D Ising model

$$\begin{aligned} \text{PBC: } Z &= \sum_{\{\sigma_i\}} \prod_i e^{K\sigma_i\sigma_{i+1}} \\ &= (\cosh K)^N \sum_{\{\sigma_i\}} \prod_i (1 + \sigma_i\sigma_{i+1} \tanh K) \end{aligned}$$

$$= 2^N (\cosh k)^N [1 + (\tanh k)^N] \xrightarrow{N \rightarrow \infty} (2 \cosh k)^N$$

Correlation function

$$\begin{aligned} \langle \sigma_i \sigma_{i+n} \rangle &= \frac{\sum_{\{\sigma_i\}} \sigma_i \sigma_{i+n} e^{K \sum_i \sigma_i \sigma_{i+1}}}{Z} \\ &= \frac{(\tanh k)^n + (\tanh k)^{N-n}}{1 + (\tanh k)^N} \end{aligned}$$

regime  $1 \ll n \ll N$   $\langle \sigma_i \sigma_{i+n} \rangle = (\tanh k)^n = \exp(-\frac{n}{\xi})$

$$\xi = -\frac{1}{\ln(\tanh k)} \xrightarrow{K \rightarrow \infty} \xi \sim 2e^{\frac{2J}{k_B T}} \quad \text{correlation length grows as } T \rightarrow 0$$

$\Rightarrow$  no LRO in 1D  $\xi \approx L$  (finite)

correlation decay experimentally with separation

\* In the presence of a critical point:  $\xi \sim t^{-\nu}$ , correlation length diverges!

\* Exactly at  $T_c$  ( $t=0$ )

$$\begin{aligned} G(r) &\sim \frac{1}{r^{d-2+\eta}} \rightarrow \text{critical point} \\ \langle \sigma_i \sigma_{i+r} \rangle &\quad \uparrow \text{dim} \\ \text{Scale Invariance} \end{aligned}$$

$$G(\lambda r) = \lambda^* G(r)$$

Critical exponents

So far heat capacity  $C \sim |t|^{-\alpha}$

magnetization  $m \sim (-t)^\beta$

susceptibility  $\chi \sim |t|^{-\gamma}$



$$\text{at } t=0 \quad m \sim h^{1/8}$$

$$\text{correlation length } \xi \sim t^{-\nu}$$

$$\text{correlation function at } t=0: G(r) \sim \frac{1}{r^{d-2+\eta}}$$

	$\alpha$	$\beta$	$\gamma$	$\delta$	$\nu$	$\eta$
2D Ising	0	$\frac{1}{8}$	$\frac{7}{4}$	15	1	$\frac{1}{4}$
3D Ising	0.11	0.32	1.24	4.8	0.63	0.04
MFT	0	$\frac{1}{2}$	1	3	$\frac{1}{2}$	0

$$\alpha + 2\beta + \gamma = 2$$

$$\delta - 1 = \frac{\gamma}{\beta}$$

$$\gamma = \nu(2-\eta) \quad \alpha = 2 - d\nu$$

$$\text{Landau theory (MFT)} \quad f(m) = \frac{t}{2}m^2 + um^4 + vm^6 - h \cdot m$$

Ginzburg - Landau Theory: order parametre  $m(x)$

$$F[m(x)] = \int d^d x \left[ \frac{t}{2} m^2(x) + u m^4(x) + v m^6(x) + \frac{K}{2} (\nabla m)^2 + \frac{L}{2} (\nabla^2 m)^2 + \dots \right] m(x)$$

$$Z = \int Dm(x) \exp \left[ - \int d^d x \left[ \frac{t}{2} m^2 + u m^4 + v m^6 + \frac{K}{2} (\nabla m)^2 \right] \right]$$

Validity of MFT  $d \geq 4$  MFT OK ✓

$$\vec{m}(x) = (m_1, m_2, \dots, m_n) \quad n=3 \quad H = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$$

Continuous Symmetry  $d=2$  Mermin - Wagner