Differential Manifolds and Topology

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§1 Introduction

Let's look at a simple example. Let $f(x, y, z) = x\cos z + y \sin z$, and c a constant. Consider the surface $S_c = f^{-1}(c)$. Since $f'(x, y, z) = (\cos z, \sin z, -x\sin z + y\cos z) \neq 0$, S_c is indeed a manifold.

The main motivation of differential manifolds is that we want to apply calculus on manifolds like S_c, like we did in differential geometry for curves and surfaces.

Some more examples are $\mathbb{R}P^2, \mathbb{C}P^n$, we can't do calculus on them by taking coordinates on them, hence we developed a new theory called differential manifold.

Example 1.0.1
The complex projective space $\mathbb{C}P^n$ is defined as all the 1 dimensional complex linear subspace of $\mathbb{C}^{n+1}$, the elements can be written in homogenous coordinates $z := [z_0, z_1, \dots, z_n]$.
The atlas contains $n+1$ charts:

$$U_j := \{z \in \mathbb{C}P^n : z_j \neq 0\}, \quad \varphi_j : U_j \to \mathbb{C}^n.$$

Where

$$\varphi_j(z) = \left(\frac{z_0}{z_j}, \dots, \frac{\widehat{z_j}}{z_j}, \dots, \frac{z_n}{z_j}\right).$$

Here $\widehat{a}$ means a is omitted.

§2 Differential manifolds

§2.1 Basic definitions

Definition 2.1.1 (Differential manifolds)**.** Let M be a Hausdorff space, $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ be an **atlas**, satisfying:

- $\{U_\alpha\}_{\alpha\in A}$ is an open covering of M.
- $\phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n$ is a homomorphism, V_α is an open set;
- $\phi_{\alpha\beta} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ are C^r functions. They are called **transition functions**, this condition is called the compactibilty of charts.

We say M is a C^r **differential manifold** or just **differential manifold** if it's given a such atlas.

Definition 2.1.2 (Maps between manifolds)**.** Let M, N be C^r differential manifolds.
We say a map $f : M \to N$ is C^r if for all $p \in M$, suppose $p \in U_\alpha, f(p) \in W_\beta$, we have $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ is C^r at $\phi_\alpha(p)$.
Note that $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ is just a map in $\mathbb{R}^n$, so we can talk about differetiation of it.
We say M and N are C^r **homeomorphic** if there's a bijective map $f : M \to N$, and f, f^{-1} are both C^r maps.

Note that since $f(U_\alpha)$ may not be a chart in N, actually we need to take a neighborhood of p and discuss things in this neighborhood.
To avoid this annoying argument, we define a **maximal atlas** to be the atlas which contains all the compatible charts. Any atlas can be extended to a maximal atlas, and the maximal atlas is called a C^r structure on M.

Example 2.1.3 (On maximal atlas)
Let $X = \mathbb{R}$, $U_1 = \mathbb{R}, \phi_1 : \mathbb{R} \to \mathbb{R}$ by $\phi_1(x) = x$. Let $U_2 = \mathbb{R}, \phi_2 : \mathbb{R} \to \mathbb{R}$ by $\phi_2(x) = x^3$.
We can see that $\{(U_1, \phi_1)\}$ and $\{(U_2, \phi_2)\}$ are not compactible, so they are different atlas. Extending them to maximal atlas, we'll get two different smooth structures on X. But they are differential homeomorphic by $x \mapsto x^{\frac{1}{3}}$. ($\phi_2 \circ f \circ \phi_1^{-1}(x) = x$)

Remark 2.1.4 — You might hear somebody say that "there is only one smooth structure on $\mathbb{R}$", this usually means "up to a differential homeomorphism".

(By the way, Riemann first states the concept of manifolds)

Remark 2.1.5 — Sometimes we also require C_2 or quasi-compact on the base space. In different cases, the transition function might be required as piecewise linear / smooth etc.
In this course we assume the dimension of a manifold is finite and fixed.

§2.2 Tangent vectors and tangent spaces

In multi-dimensional calculus, we studied the integral and differentiations, also we studied the calculus of *vector fields*, including gradients, divergence and curl. We learned some relations like $\operatorname{curl}\operatorname{grad} f = \vec{0}$, $\operatorname{div}\operatorname{curl}\vec{F} = 0$. At last, we learned Stokes' formula, which we'll generalize it to manifolds in this course.

Definition 2.2.1 (Tangent vectors)**.** You can think of this definition intuitively as "The equivalence calss of tangency of parametrized curves".

Let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a parametrized curve. We say γ is **smooth** if for any chart (U, ϕ),

$$\phi \circ \gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n \in C^\infty.$$

Define the equivalence relation $\gamma_1 \sim \gamma_2$ on point p to be

$$\gamma_i(0) = p, \quad (\phi \circ \gamma_1)'(0) = (\phi \circ \gamma_2)'(0),$$

for all charts (U, ϕ) where $p \in U$.

We call the equivalence class of $\sim$ a **tangent vector** at p. Denote all the tangent vectors at p by T_pM.

Proposition 2.2.2

The tangent space T_pM is a vector space.

Proof. Let $[\gamma_1], [\gamma_2] \in T_pM$, for a chart (U, ϕ) where $p \in U$, WLOG $\phi(p) = 0 \in \mathbb{R}^n$.

Let
$$\widetilde{\gamma} := \phi^{-1}(\phi \circ \gamma_1 + \phi \circ \gamma_2) : (-\varepsilon, \varepsilon) \to U \subset M.$$

We can check that $[\gamma_1] + [\gamma_2] := [\widetilde{\gamma}]$ is a well-defined addtion, i.e. independent of charts and representatives.

Similarly we define $c[\gamma_1] := [\phi^{-1}(c\phi \circ \gamma_1)]$ as scalar multiplication. Indeed this two operations give the structure of vector spaces. □

You can also think of tangent vectors as directional derivatives: Let $f \in C^\infty(U)$, $v = [\gamma] \in T_pM$, $\gamma = \gamma(t)$. Define

$$v(f) := \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma)\Big|_{t=0}.$$

Since tangent vectors form a vector space, this "directional derivative" has many properties in common with those in $\mathbb{R}^n$, such as $(v_1 + v_2)(f) = v_1(f) + v_2(f)$.

Definition 2.2.3. Let $f : M \to N$ be a smooth map. There is an induced map

$$\begin{aligned} f_* : T_pM &\to T_{f(p)}N \\ [\gamma] &\mapsto [f \circ \gamma] \end{aligned}$$

called **tangent map**, it's a homomorphism of vector spaces. It's also denoted as $\mathrm{d}f$.

Remark 2.2.4 — For all charts $(U, (x_1, \cdots, x_n))$ at p, we can write a basis of T_pM, say

$$\left((\phi^{-1})_* \frac{\partial}{\partial x_1}, \ldots, (\phi^{-1})_* \frac{\partial}{\partial x_n} \right) \Big|_p$$

At p and $f(p)$, we can write f_* as a matrix with respect to the coordinate basis, this matrix is precisely the Jacobi matrix of f when f is a map between Euclid spaces.

Strictly, $\frac{\partial}{\partial x_i}$ is the tangent vector of $\gamma_i : t \mapsto (0, \ldots, t, \ldots, 0)$ at 0, where t is on the i-th entry. It's an analogy of partial derivatives in $\mathbb{R}^n$, so they only exist in a coordinate system.

§2.3 Vector fields

Definition 2.3.1 ((Tangent) Vector fields)**.** Let M be a manifold, we say $X : M \to TM$ is a **(tangent) vector field** if $X|_p \in T_pM$, and for any chart $(U, x_1, \cdots, x_n)$ we have

$$\varphi \circ X = v_1 \frac{\partial}{\partial x_1} + \cdots + v_n \frac{\partial}{\partial x_n}.$$

Where $v_1, \ldots, v_n \in C^\infty(U)$.

Denote all the vector fields on M by $\mathscr{X}(M)$.

We can see that $\mathscr{X}(M)$ is a vector space on $\mathbb{R}$ with infinite dimensions.

A vector field can be imagined as the "speed" of each point, if the points are moving according to this speed, we can get transformations on the manifold, with one parameter corresponding the "time". This operation is very similar to the integral when solving ODE.

To state these ideas mathematically,

Definition 2.3.2 (One-parameter transformation group)**.** An **one-parameter transformation group** on M is a collection of smooth homeomorphisms

$$\phi_t : M \to M, \quad t \in \mathbb{R}$$

where ϕ_t is smooth with respect to t. (View this in charts or as a smooth map $M \times \mathbb{R} \to M$)

The group operation is given by $\phi_s \cdot \phi_t = \phi_{s+t}$.

We can also think of ϕ as a group homomorphism $\mathbb{R} \to \mathit{Diffeo}(M)$.

Such groups will induce a vector field on M,

$$(X\big|_p)(f) := \frac{\mathrm{d}(f \circ \phi_t)}{\mathrm{d}t}\bigg|_{t=0}.$$

It can be thought of "infinitesimal one-parameter transformation".

Since the transformations might be wired when t gets large (like vortex in the sea), we define the **local one-parameter transformation group** to be

$$\phi : W \times (-\varepsilon, \varepsilon) \to U, \quad W \subset U \subset M$$

where W, U are open subsets, and only require $\phi_{s+t} = \phi_s \cdot \phi_t$ inside $(-\varepsilon, \varepsilon)$.

Similar to the local solution of ODE, we have:

Proposition 2.3.3

For each $X \in \mathscr{X}(M)$ and $p \in M$, there exists open neighborhoods $W \subset U$ and a local ϕ_t s.t. the vector field induced by ϕ_t on W is exactly X.

Proof. We skipped the proof since it's similar to ODE. □

We know that vector fields and one-parameter transformation group is almost the same thing, and they're connected by operations similar to taking derivatives or integrals. Thus sometimes we write $X = \dot{\phi}$ to show this relation.

In the meantime we know general transformations on M are not commutative, in algebra we introduced the commutator to study the un-commutativity, in differential manifolds, the analogy is Lie derivatives.

Definition 2.3.4 (Lie derivatives)**.** Let $X, Y \in \mathscr{X}(M)$, define the **Lie derivative**:

$$\mathcal{L}_X(Y)\Big|_p := \lim_{t\to 0} \frac{(\phi_{-t})_*(Y|_{\phi_t(p)}) - Y|_p}{t}.$$

Where ϕ is the local transformation group induced by X, i.e. $X = \dot{\phi}$ locally at p.

This operation can be viewed as $\mathcal{L}_X : \mathscr{X}(M) \to \mathscr{X}(M)$. It's called "derivative" since it has properties similar to derivatives:

- $\mathcal{L}_X(a_1Y_1 + a_2Y_2) = a_1\mathcal{L}_X(Y_1) + a_2\mathcal{L}_X(Y_2)$, $a_1, a_2 \in \mathbb{R}$.
- $\mathcal{L}_X(fY) = X(f)Y + f\mathcal{L}_X(Y)$, $f \in C^\infty(M)$. This is called Leibniz's law.
- $(\mathcal{L}_X(Y))(f) = X(Y(f)) - Y(X(f))$, $f \in C^\infty(M)$. $(*)$

The last equation reveals the similarlity between Lie derivatives and commutators (if we view vector fields as operators on smooth functions), so we'll often write $[X, Y] := \mathcal{L}_X(Y)$. It satisfies linearity, alternativity and *Jacobi equation*:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [Y, X]] = 0.$$

Therefore $[\cdot, \cdot] : \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M)$ is a Lie braket on $\mathscr{X}(M)$ that makes it an infinite dimensional **Lie algebra** on $\mathbb{R}$.

Here we'll give a proof of $(*)$.

Proof. Consider $F(t) = f(\phi_{-t}(p))$ for a fixed $p \in M$. By chain rule,

$$F(t) - F(0) = \int_0^1 \frac{\mathrm{d}F(st)}{\mathrm{d}s}\,\mathrm{d}s = t\int_0^1 F'(u)|_{u=st}\,\mathrm{d}s.$$

i.e.

$$f \circ \phi_{-t} = f + tg_t,$$

where

$$g_t(p) = \int_0^1 F'(u)|_{u=st}\,\mathrm{d}s = \int_0^1 \frac{\mathrm{d}(f(\phi_{-u(p)}))}{\mathrm{d}u}\Big|_{u=st}\,\mathrm{d}s.$$

We have

$$g_0(p) = \int_0^1 \frac{\mathrm{d}f(\phi_{-u}(p))}{\mathrm{d}u}\Big|_{u=0}\,\mathrm{d}s = -X(f)|_p.$$

Meanwhile

$$\begin{aligned}((\phi_{-t})_* Y)|_p(f) &= (Y|_{\phi_t(p)})(f \circ \phi_{-t}) \\ &= (Y|_{\phi_t(p)})(f + t g_t) \\ &= (Y(f) + tY(g_t))|_{\phi_t(p)}.\end{aligned}$$

Hence

$$\begin{aligned}(\mathcal{L}_X(Y))|_p f &= \lim_{t\to 0} \frac{((\phi_{-t})_* Y)|_p(f) - (Y|_p)(f)}{t} \\ &= \lim_{t\to 0} \frac{Y(f)|_{\phi_t(p)} - Y(f)|_p}{t} + \lim_{t\to 0} Y(g_t) \\ &= X(Y(f))|_p + Y(-X(f))|_p \\ &= (X(Y(f)) - Y(X(f)))\,|_p\end{aligned}$$

□

Remark 2.3.5 — This proof seems like coming out of nowhere, but the main idea is to expand $f \circ \phi_{-t} = f + t \cdot ? + o(t)$. This idea leads to constructing integrals in the first step, and finially computed the ? is just $-X(f)$.

Example 2.3.6 (Lie derivatives in local coordinates)

Let $(U, x_1, \ldots, x_n)$ be a chart on M. The partial derivatives $\frac{\partial}{\partial x_i}\Big|_U \in \mathscr{X}(U)$ are a basis of $\mathscr{X}(U)$. Note that

$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0,$$

hence we can compute any Lie brakets $[X, Y]$ using Leibniz's law.

Note that in general, n vector fields which are linear independent everywhere are *not* a coordinate basis on M, since usually the Lie brakets don't equal to 0.

This leads to a question: if we take m vector fields $X_1, \ldots, X_m$ which are linear independent everywhere, is there a local m dimensional submanifold at every point $p \in M$ tangent to these vector fields?

We've checked in the homework that to define $\mathcal{L}_X(Y)$ at p, we need to know the value of X and Y on a neiborhood of p instead of just at p. The reason behind it is that a single vector cannot generate a transformation group, or a "flow".

Back to our example, we say $(\frac{\partial}{\partial x_i})_{i=1}^n$ is a natural frame on U when given the coordinate $x_1, \ldots, x_n$ on U. When $m = 1$, this is just the existence of integral curves. (need some regularity) When $m > 1$, a necessary condition is that

$$[X_i, X_j] = \sum_k c_{ij}^k X_k$$

for some functions c_{ij}^k. This is intuitively saying that the integral curves of some X_i must either totally lies in the submanifold or disjoint with the submanifold.

In fact, this condition is also sufficient, this result is **Frobenius Theorem** in literature. This condition is like the "integrable condition" in PDE.

Let's look at the case $m = 2$ for an example, let $X, Y \in \mathscr{X}(U)$, assume that $[X, Y] = aX + bY$,

$$\mathcal{L}_X(uX + vY) = X(u)X + X(v)Y + v(aX + bY).$$

Our goal is to find a vector field Z s.t. $\mathcal{L}_X(Z) = 0$, i.e. X and Z are commutative. Thus by solving $\mathcal{L}_X(uX + vY) = 0$, we'll get the desired Z. Meanwhile this equation can be easily solved by solving $X(v) + bv = 0$ for v and $X(u) + av = 0$ for u.

Therefore the condition can be transformed as given $[X, Y] = 0$. Hence the induced transformations ϕ_s and ψ_t commute with each other, so (s, t) will give the parameter coordinates on the submanifold.

As for general m, we can proceed by induction to build local coordinate system by requiring $[X_i, X_j] = 0$ for each pair i, j. The equations can also be solved one by one, just like the case of $m = 2$.

The family of submanifolds given by Frobenius theorem gives a **foliation** on M. Note that submanifolds in a foliation are pairwise disjoint.

When the equations have no solution, we say $(X_1, \dots, X_m)$ is a **contact structure** on M. (i.e. a m-dimension subspace at each point) A classical example of a contact structure on $\mathbb{R}^3$ can be found on the internet (by searching "contact structure").

§2.4 Cotangent vectors and exterior forms

In calculus or physics, we often encouter symbols like $\mathrm{d}x$ which represents "infinitesimal increment" of x. Probably you were never been told the strict definition of these mysterious $\mathrm{d}x$.

In calculus we write $\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)$, where the $\mathrm{d}x$ seems to have nothing to do with this formula. In fact, it is only useful when we're changing the coordinates, like

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\,\mathrm{d}x_i = \sum_{i,j} \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial y_j}\,\mathrm{d}y_j.$$

Therefore we can write something like:

$$\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)\begin{pmatrix}\mathrm{d}x_1\\ \vdots \\ \mathrm{d}x_n\end{pmatrix} = \left(\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}\right)\begin{pmatrix}\mathrm{d}y_1\\ \vdots \\ \mathrm{d}y_n\end{pmatrix}.$$

We can see that $\mathrm{d}x_i$ are just to ensure this transform relation. Since we know $\frac{\partial}{\partial x_i}$ are a basis of vector field on manifold, so we can realize $\mathrm{d}x_i$ as the *dual space* of tangent space.

Definition 2.4.1 (Dual space)**.** Recall that in algebra, the **dual space** of a vector space V over $\mathbb{R}$ is defined as $V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$. If we take an element in V and V^*, we'll get a real number from them, this is how the word "dual" comes.

If there's a basis $e_1, e_2, \dots, e_n$ in V, there's a dual basis $\varepsilon_1, \dots, \varepsilon_n \in V^*$ such that $\varepsilon_i(e_j) = \delta_{ij}$, i.e. $\varepsilon_i(e_j) = 1$ if $i = j$, and 0 otherwise.

Definition 2.4.2 (Cotangent vectors)**.** Let $p \in M$, define

$$T_p^*M := \mathrm{Hom}_{\mathbb{R}}(T_pM, \mathbb{R})$$

to be the **cotangent space** at p. Its elements are called **cotangent vectors** at p.

Let $f : M \to N$ be a smooth map, we know there's a tangent map $f_* : T_pM \to T_{f(p)}N$. By algebra knowledge, there's a dual map of f_*, namely the **cotangent map** $f^* : T_{f(p)}^*N \to T_p^*M$.

In a chart $(U, x_1, \dots, x_n)$, let the dual basis of $(\frac{\partial}{\partial x_i})|_p$ be denoted as $(\mathrm{d}x_i)|_p$.

Recall that we can't define a vector field $\frac{\partial}{\partial f}$ for a single function f, in contrast, the notation $\mathrm{d}f$ does give a "cotangent vector field" on M. Since we can define $\mathrm{d}f = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\,\mathrm{d}x_i$ on each chart, and verify it's the same cotangent vector at a fixed point under different charts. This property is called *form invariance* in calculus. In fact we can write $\mathrm{d}f$ explicitly:

$$\mathrm{d}f(v) = \frac{\mathrm{d}(f \circ \gamma_t)}{\mathrm{d}t}\Big|_{t=0} = v(f)$$

for each tangent vector v.

On a chart $(U, x_1, x_2, \dots, x_n)$, we can write a cotangent field as $\omega = \omega_1\,\mathrm{d}x_1 + \cdots + \omega_n\,\mathrm{d}x_n$, where $\omega_i \in C^\infty(U)$. Strictly we should add a pullback, say $\omega = \varphi_U^*(\sum_i \omega_i\,\mathrm{d}x_i)$, but since this won't cause any ambiguity, we usually omit the φ_U^*.

Cotangent fields are also called **exterior differential forms of degree 1**, or just **1-forms**, and we denote the set of 1-forms on M by $A^1(M)$. You might already guess that there are general p-forms as well.

Indeed, in linear algebra, let V be a vector space on $\mathbb{R}$, the notation $\Lambda^p V^*$ denotes the set of alternating p-linear functions on V. Let $\{e_i\} \subset V$ be a basis, and $\{\varepsilon_i\} \subset V^*$ the dual basis. Then $\Lambda^p V^*$ has a basis consisting of $\varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_p} =: \varepsilon_I$, $I = \{i_1, \dots, i_p\} \subset \{1, 2, \dots, n\}$. Using this basis, for $\varphi \in \Lambda^p V^*$ we can write

$$\varphi = \sum_{|I|=p} \varphi_I \varepsilon_I, \quad \varphi_I \in \mathbb{R}.$$

Where ε_I is defined as

$$\varepsilon_I(e_{j_1}, \dots, e_{j_p}) = \begin{cases} \operatorname{sgn}\begin{pmatrix} i_1 & \cdots & i_p \\ j_1 & \cdots & j_p \end{pmatrix}, & I = J \\ 0, & I \neq J \end{cases}$$

If we do the same construction at every point of M, we get the space $\Lambda^p T^* M$. Then the **p-forms** are the elements of $A^p(M) := \Gamma(M, \Lambda^p T^* M)$. Note that 0-forms are just smooth functions, i.e. $C^\infty(M) = A^0(M)$.

For $\varphi \in A^p(M)$ we can write

$$\varphi = \sum_{|I|=p} \varphi_I\,\mathrm{d}x_I, \quad \varphi_I \in C^\infty(M).$$

§2.5 Wedge product and exterior differential operator

After constructing exterior forms, let's focus on the operations of exterior forms.

Like the wedge product in linear algebra, exterior forms can also perform wedge product:

Definition 2.5.1. Define the **wedge product** on exterior forms to be

$$\wedge : A^r(M) \times A^s(M) \to A^{r+s}(M),$$

For $\varphi \in A^r(M)$ and $\psi \in A^s(M)$,

$$(\varphi \wedge \psi)(X_1, \dots, X_{r+s}) = \frac{1}{r!s!} \sum_{\sigma \in S_{r+s}} \operatorname{sgn}(\sigma) \varphi(X_{\sigma(1)}, \dots, X_{\sigma(r)}) \psi(X_{\sigma(r+1)}, \dots, X_{\sigma(r+s)})$$

The coefficient $\frac{1}{r!s!}$ is used to eliminate repeating terms in the summation, or we can change S_{r+s} to $S_{r+s}/S_r \times S_s$. (Actually we can write φ and ψ using the basis and write $\varphi \wedge \psi$ explicitly)

There's another operator related to calculus, the **exterior differentiation** d.

Definition 2.5.2. The operator $\mathrm{d} : A^p(M) \to A^{p+1}(M)$ satisfies:

- $\mathbb{R}$-linearity: $\mathrm{d}(a\varphi + b\psi) = a\,\mathrm{d}\varphi + b\,\mathrm{d}\psi$, $a, b \in \mathbb{R}$, $\varphi, \psi \in A^r(M)$.
- Leibniz's law: $\mathrm{d}(\varphi \wedge \psi) = \mathrm{d}\varphi \wedge \psi + (-1)^r \varphi \wedge \mathrm{d}\psi$, $\varphi \in A^r(M)$, $\psi \in A^s(M)$.
- $\mathrm{d}(\mathrm{d}\varphi) = 0$, $\varphi \in A^r(M)$.
- For $f \in A^0(M)$, $\mathrm{d}f$ is the total differentiation of f.

We claim that these 4 properties uniquely determine the operator d.

Theorem 2.5.3

Such d exists and is unique.

Proof. Since we know $\mathrm{d}f = \sum_i \frac{\partial f}{\partial x_i}\,\mathrm{d}x_i$, it's natural to write (in a chart)

$$\mathrm{d}\varphi = \mathrm{d}\left(\sum_I \varphi_I\,\mathrm{d}x_I\right) = \sum_I \mathrm{d}(\varphi_I\,\mathrm{d}x_I) = \sum_I (\mathrm{d}\varphi_I \wedge \mathrm{d}x_I + \varphi_I \wedge \mathrm{d}(\mathrm{d}x_I)) = \sum_I \mathrm{d}\varphi_I \wedge \mathrm{d}x_I.$$

Here we've used all the properties required. $(\mathrm{d}(\mathrm{d}x_I) = \mathrm{d}(\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}) = 0)$

This formula on local coordinates means d must be unique.

As for existence, essentially we need to check the local expression above is compatible in different charts.

This approach is OK, but here we'll use an intrinsic construction that is independent of coordinates. For $\omega \in A^p(M)$,

$$\begin{aligned}\mathrm{d}\omega(X_1, \ldots, X_{p+1}) = &\sum_{j=1}^{p+1} (-1)^{j-1} X_j(\omega(X_1, \ldots, \widehat{X_j}, \ldots, X_{p+1})) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_n)\end{aligned}$$

To check this is right, the easiest way is to write it in coordinates, i.e. take $X_s = \frac{\partial}{\partial x_{i_s}}$. □

Remark 2.5.4 — Again, this construction seems jumps out of nowhere. If we look at 1-forms, $\omega \in A^1(M)$,

$$(\mathrm{d}\omega)(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$$

Now let $\omega = g\,\mathrm{d}f$, $\mathrm{d}\omega = \mathrm{d}g \wedge \mathrm{d}f$, and note that

$$X(g\,\mathrm{d}f(Y)) = X(gY(f)) = X(g)Y(f) + g \cdot XY(f)$$

$$Y(g\,\mathrm{d}f(X)) = Y(gX(f)) = Y(g)X(f) + g \cdot YX(f)$$

We see that $(\mathrm{d}g \wedge \mathrm{d}f)(X, Y) = X(g)Y(f) - Y(g)X(f)$ and the remaining term is exactly the Lie bracket. From this we can guess the general expression of $\mathrm{d}\omega$.

We can generalize the tangent and cotangent fields to tensors: $\mathscr{T}^{r,s}(M) = \Gamma(M, T^{r,s}(M))$ the (r,s)-tensor. Here $T^{r,s}M := \underbrace{TM \otimes \cdots \otimes TM}_{r} \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_{s}$.

Let V be a finite dimensional vector space on $\mathbb{R}$, the elements in $\underbrace{V \otimes \cdots \otimes V}_{r} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{s}$ are multi-linear maps

$$\underbrace{V^* \times \cdots \times V^*}_{r} \times \underbrace{V \times \cdots \times V}_{s} \to \mathbb{R}$$

Thus in a chart $(U, x_1, \dots, x_n)$, let $\varphi \in \Gamma(U, T^{r,s}M)$, we can write φ as

$$\varphi = \sum \varphi_I \frac{\partial}{\partial x_{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_{i_r}} \otimes \mathrm{d}x_{j_1} \otimes \cdots \otimes \mathrm{d}x_{j_s},$$

where the summation is taken over $i_1, \dots, i_r, j_1, \dots, j_s \in \{1, \dots, n\}$. Note that $a \otimes b$ and $b \otimes a$ are independent elements.

Remark 2.5.5 — For general definition of tensor product, let R be a commutative ring, 1 the identity of R. Let M, N be R-modules, we can define the tensor product $M \otimes_R N$. The elements are of the form $\sum m_i \otimes n_i$, where we require the linearity of each components, and $\lambda m \otimes n = (\lambda m) \otimes n = m \otimes (\lambda n)$. The vector spaces are just modules over a field.

Remark 2.5.6 (On general vector bundles) **—** A "bundle" means that we attach a fibre (usually a vector space) on each point, and the fibres varies in a way that we think as "continuously".

An $\mathbb{R}$-vector bundle E of rank n is defined as $(p : E \to X, \{(U_\alpha, \Phi_\alpha)\})$, such that $U_\alpha \subset X$ is open, with $\Phi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ satisfying

- $\Phi_\alpha(q) = (p(q), \varphi_\alpha(q))$,
- On $U_\alpha \cap U_\beta$, $\Phi_\beta \circ \Phi_\alpha = \mathrm{id}_{U_\alpha \cap U_\beta} \times \varphi_{\alpha\beta}$, where $\varphi_{\alpha\beta} \in \mathrm{GL}(n, \mathbb{R})$.

Under this definition, $\varphi_{\alpha\beta}\varphi_{\beta\gamma}\varphi_{\gamma\alpha} = \mathrm{id}$. This is called the cocycle condition.

Like we do with groups, vector bundles also have homomorphisms, subbundles and quotient bundles.

Example 2.5.7

A Riemann metric on M is a $(0,2)$ symmetric tensor, and positive definite everywhere, i.e. can be locally written as

$$g = \sum_{i,j} g_{ij}\, \mathrm{d}x_i \otimes \mathrm{d}y_j$$

with $g_{ij} = g_{ji}$ and $(g_{ij})_{n \times n}$ are positive everywhere. When $n = 2$ we sometimes write $g = E\,\mathrm{d}x^2 + F\,\mathrm{d}x\,\mathrm{d}y + G\,\mathrm{d}y^2$, where $\mathrm{d}x_i\,\mathrm{d}x_j := \frac{1}{2}(\mathrm{d}x_i \otimes \mathrm{d}x_j + \mathrm{d}x_j \otimes \mathrm{d}x_i)$.

To construct a Riemann metric, we need the unit decomposition theorem.

Theorem 2.5.8 (Unit decomposition)

Let M be a smooth quasicompact manifold, then for all open covering $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ there exists a locally finite open refinement with a unit decomposition.

Proof. For a compact set K and an open set $U \supseteq K$, we can construct a function $f \in C^\infty(M)$, s.t. $\operatorname{supp} f \subset U$, and $f|_K \equiv 1, 0 \le f \le 1$.

By quasicompactness, take a locally finite precompact open covering of M which refines the one we already have (precompact means the closure is compact), and still denote it as $\{(U_\alpha, \varphi_\alpha)\}$, then we can take $W_\alpha \subset U_\alpha$ s.t. W_α precompact, and construct f_α w.r.t. W_α as stated. Hence $\lambda_\alpha = \frac{f_\alpha}{\sum_\beta f_\beta}$ satisfies the condition. □

§3 Integrals on manifolds

In this section we generalize integrals and Stokes formula to the smooth manifolds.

§3.1 Integrals of exterior forms

Let M be an oriented smooth manifold with dimension n, and $\omega \in A^n(M)$. We want to define $\int_M \omega$.

First we take a unit decomposition λ_α on the atlas $\{(U_\alpha, \varphi_\alpha)\}$. Here we require the atlas compatible with the orientation of M. Define

$$\int_M \omega := \sum_\alpha \int_{\mathbb{R}^n} (\varphi_\alpha^{-1})^*(\lambda_\alpha \omega)$$

The integral of n-forms on $\mathbb{R}^n$ is defined as usual way.

Since the value stays invariant under the refinement of unit decomposition, for two different decomposition, we can consider their common refinement to show they give the same value. Thus the integral is well-defined.

For manifolds with boundary, we use the “outer normal vector fisrt”(ONF) rule to determine the orientation on the boundary.

Specifically, if $(e_1, e_2, \ldots, e_{n-1})$ is a frame of ∂M, then $(\nu, e_1, \ldots, e_{n-1})$ is a frame compatible with the orientation of M.

Theorem 3.1.1 (Stokes formula)
Let M be an n-dimensional orientable smooth manifold with boundary, and $\omega \in A_{cpt}^{n-1}(M)$. Determine the orientation on ∂M by ONF, then

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega.$$

Proof. The idea is using unit decomposition and hence assume $M \subset \mathbb{R}^n$. We only need to check it on local charts. □

When the manifold is not orientable, we have other forms of integrals, such as integrals with respect to a measure. However, unlike the exterior forms, there aren't close connections between measures of different dimensions, we can't generalize Stokes' formula in this way.

Back to the Stokes formula, there's one thing we need to know about: when the boundary ∂M is piecewise smooth, does this formula still holds?

Intuitively this should be right, since we can use a series of smooth boundaries to approximate the singular point, and this difference is small under integration. This technique will be formalized in differential topology part, so here we just take it as granted.

§3.2 Closed and exact forms

After we have Stokes formula, we would like to consider some special forms.

Definition 3.2.1. Let ω be a differential form, if $\mathrm{d}\omega = 0$, we say it's a **closed form**, if $\omega = \mathrm{d}\varphi$ for some differential form φ, we say ω is an **exact form**.

Clearly exact forms are always closed, while closed forms may not be exact.

Example 3.2.2

Let $M = \mathbb{R}^2 \setminus \{O\}$, and $\omega = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2+y^2}$, then ω is closed but not exact. (Homework problem)

Let's dig deeper into the connections between closed and exact forms. Naturally we'll ask: when does a closed 1-form on M is exact?

Now if we want to construct φ s.t. $\mathrm{d}\varphi = \omega$, we would naturally want to do this by integration on paths (since ω is 1-form).

Therefore we fix a basepoint $x_0 \in M$, and for each $x \in M$, we choose a path $\alpha : [0,1] \to M$ connecting x_0 and x, and define $\varphi(x) = \int_\alpha \omega = \int_0^1 \alpha^*\omega$. Obviously there are many different paths, but if two paths forms a boundary of some regions in M, by Stokes formula on this region we get the integral is invariant of paths. (i.e. $0 = \int_D \mathrm{d}\omega = \int_\alpha \omega - \int_{\alpha'} \omega$)

Proposition 3.2.3

If M is simply connected, then any closed 1-forms on M are exact.

However, if two paths are not homotopic (i.e. there's a "hole" between the paths), things may go wrong, the integrals may change, like the above example.

When $M = \mathbb{R}^2 \setminus \{P_1, \dots, P_m\}$, indeed there exists $\eta_i \in A^1(M)$ s.t. $\mathrm{d}\eta_i = 0$ and

$$\int_{\partial B(\epsilon, P_j)} \eta_i = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

The construction is just applying a translation and scaling of the above example. Therefore we can decompose any form $\omega \in A^1(M)$ as

$$\omega = \mathrm{d}\varphi + a_1\eta_1 + \cdots + a_m\eta_m.$$

This is actually a simple example of de Rham cohomology.

§3.3 de Rham cohomology

Let M be a smooth manifold,

$$Z^p(M) := \{\omega \in A^p(M) : \mathrm{d}\omega = 0\}, \quad B^p(M) := \{\omega \in A^p(M) : \exists \varphi, \omega = \mathrm{d}\varphi\}.$$

We can define the de Rham cohomology group

$$H^p_{\mathrm{dR}}(M) := \frac{Z^p(M)}{B^p(M)} = \frac{\ker(d : A^p \to A^{p+1})}{\mathrm{im}(d : A^{p-1} \to A^p)}.$$

It's also a vector space over $\mathbb{R}$. In general it might be of infinte dimension. But in the future we can prove when M is compact, $\dim H^p_{\mathrm{dR}}(M) < \infty$; When M is quasi-compact, the dimension is countable.

Example 3.3.1
When $p \geq n+1$, $H^p_{\mathrm{dR}}(M) = 0$.

Another example is

$$H^p_{\mathrm{dR}}(\mathbb{R}^2 \setminus \{P_1, \dots, P_m\}) \cong \begin{cases} \mathbb{R}, & p = 0 \\ \mathbb{R}^m, & p = 1 \\ 0, & p \geq 2. \end{cases}.$$

We can think of de Rham cohomology class as the "topological obstacles" that keep closed forms from being exact forms, while homology class counts the holes in the space linearly.

Remark 3.3.2 — Simplicial homopogy groups H_n: description and facts. See textbooks for details.

Let X be a topology space fused together by finitely many *simplicial complexes* (generalization of triangles in higher dims). We can define the boundary operator ∂ on each complex, also we define the *group of p-chains* $C_p(X)$ to be the abelian group freely generated by p-dim complexes.

Hence $\partial : C_p(X) \to C_{p-1}(X)$ is a homomorphism and $\partial^2 = 0$. Therefore

$$H_p(X) := \frac{Z_p(X)}{B_p(X)} = \frac{\ker \partial_p}{\operatorname{im} \partial_{p+1}}$$

is called the **simplicial homology group** of X.

We have the following fact:

Theorem 3.3.3 (de Rham isomorphism)
Let M be a smooth manifold, we have the canonical isomorphism

$$H^p_{\mathrm{dR}}(M) \cong \mathrm{Hom}(H_p(M), \mathbb{R}).$$

Some related facts:

- The homology group $H_p(X)$ is a homotopic invariance.

  It can be dirived from the following:

  - $H_p(X)$ is invariant under subdivision of simplicial complexes.
  - For different partitions, we can take their common subdivision to induce the isomorphism. This is enough to deduce homeomorphic invariance.
  - For general homotopy, we can use "combinatorical" maps to approxiamte them.

- For smooth manifold M, we can always find a smooth triangularization.

  E.g. when M is compact, let g be a Riemann metric, we can cover M using finitely many small convex balls, and construct the triangularization by the intersections of balls, i.e. centers as 0-complex, intersection of n balls corresponds to a $n-1$-complex formed by these n vertices.

Now we can find $A^p(M) \to \mathrm{Hom}(C_p(M), \mathbb{R})$ as

$$\omega \mapsto \left(c = \sum c_i \sigma_i \mapsto \int_c \omega = \sum c_i \int_{\sigma_i} \omega \right),$$

where $C_p(M)$ is taken from the smooth triangularization of M. Therefore the integral gives a homomorphism

$$\int : H^p_{\mathrm{dR}} \to \mathrm{Hom}(H_p(M), \mathbb{R}),$$

and the theorem states that this is in fact an isomorphism.

Remark 3.3.4 — The homology theory nowadays often talks about "singular homology", which use a simplified language (hence more abstract) that can be easily connected with cohomology and algebra.

The homology group also has something to do with Euler characteristic:

$$\chi(M) = \sum_p (-1)^p \operatorname{rank}(H_p(M))$$

(recall that $\operatorname{rank} G = r$ means $G = \mathbb{Z}^r \times T$ for an abelian group)

This is an example of transfering numerical invarance to algebraic invariance, which is called "categorification" in differential topology.

§4 Differential topology

§4.1 Whitney theorems

The basic question for this section: given a smooth manifold M, can we place M into $\mathbb{R}^n$?

Example 4.1.1

If M is compact, we can take unit decomposition $(U_\alpha, \varphi_\alpha)$, $\alpha = 1, \dots, k$ with $\sum_\alpha \lambda_\alpha = 1$. Consider a map

$$f : M \to \underbrace{\mathbb{R}^{n+1} \times \cdots \times \mathbb{R}^{n+1}}_{k}$$

$$p \mapsto ((\lambda_i(p)\varphi_i(p), \lambda_i(p)))_{i=1}^k$$

We can check that $f(x) = f(y) \implies \varphi_\beta(x) = \varphi_\beta(y)$ if $x \in U_\beta$, thus $x = y$. In fact f is also an **immersion**, i.e. $T_pM \xrightarrow{f_*} T_{f(p)}\mathbb{R}^N$ is an injective homomorphism.

By the fact that an injective immersion of a compact space is an embedding, we know f is an embedding from M into $\mathbb{R}^N$.

This means we always can embed a compact manifold into an Euclid space with sufficiently high dimensions. In fact, we can do it with less dimensions.

Definition 4.1.2. Let M be a smooth manifold, we say $f : M \to N$ is an **immersion** if $f_* : T_pM \to T_{f(p)}N$ is injective for all $p \in M$. Sometimes we write $f : M \looparrowright N$ for immersions.

We say f is an **embedding** if it's also a homeomorphism between M and $f(M)$.

Example 4.1.3

Let $f : \mathbb{R} \to \mathbb{R}^2$, then the image of f is a smooth curve. If the curve has self-intersection, then it's not an embedding.

The case for immersions are more interesting. The map $f(x) = e^{ix}$ and the *topologist's sine curve* are both immersions but not embeddings. (Since the topologist's sine curve is not locally path connected, it's not homeomorphic to $\mathbb{R}$.)

If the image of f has a singular point (i.e. the tangent line doesn't exist), then f is not an immersion.

We say a map f is **proper** if the pre-image of compact sets are also compact. Note that if f maps $\mathbb{R}$ to an open segment in the above example, then it's not proper. So we can think of a proper map maps ends to ends.

Proposition 4.1.4

An injective proper map into a locally compact Hausdorff space is a homeomorphism to its image.

Proof. Left as exercise. This proposition is a key property of proper maps. □

Proposition 4.1.5

Let M be a C_2 smooth manifold (second countable), there exists a proper smooth map $f : M \to \mathbb{R}^n$, $\forall n \in \mathbb{N}^*$.

Proof. It's sufficient to construct $f : M \to \mathbb{R}$ proper and smooth, then compose it with linear maps $\mathbb{R} \to \mathbb{R}^n$.

Take a unit decomposition $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$, with $\sum \lambda_\alpha = 1$. Recall that we require the covering is locally finite, and $\operatorname{supp}(\lambda_\alpha)$ is a compact set in U_α. Also $W_\alpha = \lambda_\alpha^{-1}(1)$ forms a covering.

Moreover, we can take a *countable* unit decomposition, let $f(p) := \sum_j j\lambda_j(p)$ for $j \in \mathbb{N}$. We can check this map is indeed proper. □

Remark 4.1.6 — The key trick is to take a countable unit decompostion, and this requires the C_2 condition on M, which is stronger than the quasi-compact condition of unit decomposition. This difference comes from the fact that we want to do something to the charts one by one, in this case it's taking the sum $\sum_j \lambda_j$.

Theorem 4.1.7 (Whitney immersion)

Let M be a C_2 smooth manifold with $\dim M = m$. Let $f : M \to \mathbb{R}^n$ be a smooth map with $n \geq 2m$. Then for all $\varepsilon > 0$, there exists an immersion $g : M \to \mathbb{R}^n$ satisfies:

$$\|g(p) - f(p)\| < \varepsilon, \quad \forall p \in M.$$

Theorem 4.1.8 (Whitney embedding)

In the above theorem, if $n \geq 2m+1$, we can also require g is injective. In particular, if f is proper, we can make g proper, hence an embedding.

Note that C_2 is a necessary condition of embedding into $\mathbb{R}^n$.

Proof outline of Whitney immersion. For the sake of convenience, we assume M is compact here. (For the general case, the proof is the same with a minor adjust of induction)

We take the unit decomposition $\{W_i\} \prec \{U_i\}$ as usual.

Observe that if the condition is satisfied on some open set, we can modify f outside a neighborhood of a compact set in it, and still keep the conditions satisfied in this compact set.

Therefore our plan is to do this on each U_i, s.t. the difference we made is no more than $\frac{\varepsilon}{2^i}$, and do not affect the immersion property already attained in $W_1 \cup \cdots \cup W_{i-1}$.

On each U_i, we can think of $U = B(0,2)$, $W = B(0,1) \subset \mathbb{R}^m$. The function $\lambda : \mathbb{R}^m \to \mathbb{R}$ satisfies $\operatorname{supp}(\lambda) \subset U$, $\lambda|_W = 1$. For $f : U \to \mathbb{R}^n$, we want to modify f inside a range of ε s.t. the modification supports on a compact set in U.

To do so, consider $g(x) = f(x) + \lambda(x)Ax$, where $A \in Mat_{n\times m}(\mathbb{R})$ has sufficiently small coefficients, say $\sum |a_{ij}|^2 < \frac{\varepsilon}{100}$. We hope $Dg = Df + A$ has rank m everywhere in W (hence it's an immersion). (Remember that λ is 1 on W)

Consider

$$\begin{aligned} U \times Mat_{n\times m}(\mathbb{R}) &\to Mat_{n\times m}(\mathbb{R}) \\ (x, A) &\mapsto -Df|_x + A \end{aligned}$$

We hope that for some A, $Dg : W \to Mat_{n\times m}$ avoids all the matrices with rank less than m.

Lemma 4.1.9

For $k \leq m$, the matrices with rank lower than m forms a submanifold M_k with dimension $nk + (m-k)k$.

Proof. WLOG the columns $v_1, \dots, v_k$ are linearly independent. If $B + \delta X$ also has rank k, then $v_1, \dots, v_k$ can be arbitarily changed, thus has n degree of freedom; while the rest must lie in the space spanned by $v_1, \dots, v_k$, hence has degree of freedom k. □

Thus $\dim U + \dim M_k = m + nk + mk - k^2 < nm$ for $k \leq m-1$ when $n \geq 2m$.

So all matrices A s.t. $Dg(W)$ intersects with M_k forms a null set (the dimension is less than nm, and it's a smooth manifold), hence we can always take A s.t. Dg has rank m everywhere. □

Remark 4.1.10 — In fact we only need f to be C^2 to apply this proof.

The statement "$f : M \to N$ is C^1 and $\dim M < \dim N$, then the image is a null set in N" cannot be reduced to C^0, Peano curve is a counterexample for this.

Next we'll prove the Whitney theorem for injective immersion.

Proof of Theorem 4.1.8. Suppose $n \geq 2m+1$.

Similarly, if $f : M \to \mathbb{R}^n$ is already an embedding on U, then we'll show the small modification will not affect the compact set $W \subset U$.

Note that by Whitney immersion theorem, we may assume f is an immersion, and immersions are local embeddings, so we can always start from a small open set U.

Therefore we need to show that for disjoint open sets $U, V \subset \mathbb{R}^m$, if we have

$$f : U \cup V \to \mathbb{R}^n$$

s.t. $f|_U$ is an embedding, we want to modify it to g s.t. $g|_U = f|_U$, and $g(V) \cap f(U) = \emptyset$.

Now since we have an extra dimension, it's natural to translate the image $f(V)$ by a fixed length in that dimension.

Let $g(x) = f(x) + \lambda(x) \cdot \vec{\beta}$, where λ is the unit decomposition on V. Consider

$$U \times V \to \mathbb{R}^n, \quad (x, y) \mapsto -f(x) + g(y).$$

By dimensional reasons, the image has measure zero in $\mathbb{R}^n$. Thus we can take $\vec{\beta}$ s.t. it's not in the image. (Moreover we can take $\|\vec{\beta}\| < \varepsilon$.)

The rest can be implemented by induction, see for details on the textbook. □

Remark 4.1.11 (Locally canonical representation of immersions) — Let $f : M \to N$ be an immersion. Then for any $p \in M$, there exists a local coordinate $U = U(p) \subset M$ and $W = W(f(p)) \subset N$, s.t. f is of the form

$$f(x_1, \ldots, x_m) = (y_1, \ldots, y_n)$$

with $y_i = x_i$ when $i \leq m$ and 0 otherwise.

This is essentially the implicit function theorem.

In history, what we stated above is called the "easy" Whitney immersion / embedding. The "hard" version can reduce the dimension to $2m - 1$ and $2m$, respectively. However in this case we cannot ensure the resulting map is close enough to original map. Therefore the range of easy version are also called "*stable range*".

For some specific manifold M, it might can be immersed / embedded into lower dimensional space than the given bound. But the bound in hard Whitney theorems is indeed the best bound. (e.g. $\mathbb{R}P^m$ when $m = 2^k$ cannot be smoothly immersed into $\mathbb{R}^n$ for $n = 2m - 2$. This need the knowledge of characteristic classes)

§4.2 Diffeotopy

Intuitively, there are many different ways to embed a manifold to a given space, so we need a way to say which ones are the same.

Definition 4.2.1 (Diffeotopy). Let $f_0, f_1 : M \to N$ be two smooth embeddings. We say they are **diffeotopic** if there exists a smooth map

$$F : M \times [0, 1] \to N$$

such that $f_t(x) := F(x, t)$ are smooth embeddings for all t, and $f_0(x) = F(x, 0)$, $f_1(x) = F(x, 1)$.

The embeddings which are not diffeotopic represents the different *knotting* ways of M in N. Naturally we would ask: when are the embeddings plain, i.e. there is only one diffeotopic class?

If we apply Whitney embeddings to $M \times [0, 1]$, we know that $f_0, f_1 : M \hookrightarrow \mathbb{R}^{2m+3}$ are always diffeotopic, meaning that M will not "knot" in Eucild space of $(2m + 3)$ or higher space.

Remark 4.2.2 — Warning: $M \times [0,1]$ is in fact a manifold with boundary, and we are requiring the boundary maps to f_0 and f_1. We will discuss this issue later in this course.

§4.3 Tubular neighborhood thoerem

Now we have Whitney embeddings, we can think every manifold as a submanifold in $\mathbb{R}^n$, which greatly simplifies our proofs and discussions. (As $\mathbb{R}^n$ has many canonical structures like orthogonality and coordinates)

The idea of tubular neighborhood can be expressed by an simple example. Think of an embedding $S^1 \hookrightarrow \mathbb{R}^3$ whose image is a circle or knot, then we can find a neighborhood of S^1 which looks like a "tube".

This "tube" can be constructed as a union of a portion of normal plane at every point of S^1, so the tube is a "product" of S^1 and a disk. In the sense of "products", the canonical representation of immersions can be regarded as locally tubular neighborhood, since it's a product of M and the latter $n-m$ coordinates.

Another example is $S^1 \hookrightarrow M$, where M is a Mobius band, note that the neighborhood is different from the case when it's embedded into an Eucild plane. So our description of tubular neighborhood must take this difference into consideration, thus we will use *normal bundle* instead of product spaces.

The smooth property of the manifold is to prevent something called *wild knots*, that is, a series of knots converging to a point and come back to the starting point.

Definition 4.3.1 (Normal bundles)**.** Let TM be the tangent bundle of M, if we embedded M into $\mathbb{R}^n$,

$$(TM)^\perp := \{(x,v) \in M \times \mathbb{R}^n : v \in (T_xM)^\perp\}$$

is called the **concrete normal bundle**. The projection map $\pi : (TM)^\perp \to M$ is just $(x,v) \mapsto x$.

Note that each fibre is a vector space, and it can be locally trivialized (locally written as a product space, equivalent to giving an atlas). So it's indeed a vector bundle.

Theorem 4.3.2 (Tubular neighborhood theorem, alpha version)
Let M be a smooth manifold embedded into $\mathbb{R}^n$, (denote the embedding by ι) there exists a smooth function $\varepsilon : M \to (0, +\infty)$ such that the map $\psi : (TM)^\perp \to \mathbb{R}^n$ by $(x,v) \mapsto x+v$ is a homoemorphism onto its image, restricted on the neighborhood of zero section Z:

$$\Delta_\varepsilon := \{(x,v) \in (TM)^\perp : \|v\| < \varepsilon(x)\}.$$

Proof. Consider the tangent map $\psi_* : T_{(x,v)}(TM)^\perp \to T_{x+v}\mathbb{R}^n$, note that $T_{(x,v)}(TM)^\perp \cong T_xM \oplus (T_xM)^\perp \cong \mathbb{R}^n$. By definition ψ_* is the identity map on zero section Z.

This induces that ψ is an immersion in a neighborhood of Z. We only need to show it's also an injection in some neighborhood.

For any $p \in M$, we can take compact sets $p \in W \subset W'$ and open set $W' \subset U$ s.t. ι is an embedding on W', and $\iota(M \setminus \mathring{W}')$ is disjoint with $\iota(W)$. (Hence the distance has a lower bound δ)

Again we first consider the case when M is compact, we can take the finite covering $W_\alpha \subset W'_\alpha \subset U_\alpha$ as stated, we can take $\delta := \min_\alpha \delta_\alpha$, so $\varepsilon = \delta$ the constant map satisfies the condition.

For the case when M is C_2, we proceed by induction. Suppose we already constructed ϵ_n on $U_1 \cup \cdots \cup U_n$, such that ψ is injective on $\Delta_{\epsilon_n}(W_1 \cup \cdots \cup W_n)$.

In the $(n+1)$-th step, take $\delta_{n+1} < \min\{\frac{\epsilon_n}{3}(x)\}$, and

$$\epsilon_{n+1}(x) = \epsilon_n(x) + \lambda_{n+1}\delta_{n+1}.$$

Therefore taking n to infinity we'll get $\varepsilon : M \to [0, +\infty)$ which satisfies the condition. □

We have other "polished" versions of tubular neighborhood theorem which comes in handy.

Theorem 4.3.3

Let M be a C_2 smooth manifold, $\iota : M \hookrightarrow \mathbb{R}^n$ is a smooth embedding. There exists a smooth retraction of neighborhood $\Omega \to \iota(M)$ and a smooth homeomorphism $(TM)^\perp \to \Omega$ s.t. the following diagram commutes.

$$\begin{tikzcd} (TM)^\perp \arrow[r, "\text{diffeo.}"] \arrow[d, "\pi"] & \Omega \arrow[d, "\text{retraction}"] \\ M \arrow[r, "\iota"] & \iota(M) \end{tikzcd} \tag{1}$$

Here π is the bundle projection.

Proof. Let $\eta : \Delta_\epsilon(M) \to (TM)^\perp$ by

$$(x, v) \mapsto \left(x, \frac{v}{\sqrt{\epsilon(x)^2 - \|v\|^2}} \right)$$

Then $\psi \circ \eta^{-1}$ gives the map $(TM)^\perp \to \Omega$. □

Theorem 4.3.4

Let M be a C_2 smooth manifold with embedding $\iota : M \hookrightarrow N$. Here we require N to be a Riemanian manifold. Then the statement of above theorem also holds.

Sketch of proof. Embed N into $\mathbb{R}^p$ smoothly, we need to show a "double layer" tubular neighborhood theorem, with respect to

$$TM \oplus (TM)^\perp_{TN} \oplus (TN)^\perp \cong T\mathbb{R}^p.$$

Using the same technique we can prove this result. □

An application of this tubular neighborhood theorem is the smoothing of maps and homotopies.

Theorem 4.3.5

If $f : M \to N$ is a continuous map, then there exists a smooth map $\widetilde{f} : M \to N$ s.t.

$$\widetilde{f}(M) \subset U, \quad \forall U \supseteq f(M).$$

Proof. Let $N \hookrightarrow \mathbb{R}^p$, we can find a smooth map $g : M \to \mathbb{R}^p$ s.t. $g(M)$ lies in a tubular neighorhood Ω of N. Let $\widetilde{f} : M \to N$ be the composition of g and the retraction $\Omega \to N$.

The second condition automatically satisfies when we replace N with U. □

Definition 4.3.6 (Smooth homotopy)**.** Let $H : M \times [0,1] \to N$ be a C^∞ map and a homotopy. If there exists $\varepsilon > 0$, such that $H_t = H_0$ and $H_{1-t} = H_1$ for $t \in (0, \varepsilon)$, we say H is a **smooth homotopy**.

The reason we're requiring more is that we hope the concatenation of smooth homotopies is still smooth.

Theorem 4.3.7

Let $f : M \times [0,1] \to N$ be a continuous homotopy. If f_0, f_1 are smooth, Then there exists a smooth homotopy $\widetilde{f} : M \times [0,1] \to N$ sufficiently close to f.

Remark 4.3.8 — Smooth homotopies do not require all the f_t's are immersions, e.g. $S^1 \to \mathbb{R}^2$ by common way and a twisted way. We use the term "**regular homotopy**" for homotopies that are immersions everywhere.

§4.4 Transversality

Well, transversality mainly talks about how two manifolds are intersected. Taking 1-dim manifold (i.e. curves) for an example, if the intersection point has "multiplicity" greater than 1, then the curves are not transversal.

Another example of non-transversal is the saddle surface intersecting a plane at the saddle point, where the intersection are two lines.

Definition 4.4.1. Let $f : M \to N$ be a C^∞ map between manifolds. Let $A \subset M$, $S \subset N$ be regular C^∞ submanifolds. We say f and S **transverse** on A if for all $p \in A \cap f^{-1}(S)$,

$$f_* T_p M + T_{f(p)} S = T_{f(p)} N.$$

We write this as $f \pitchfork_A S$.

Example 4.4.2

When $A \cap f^{-1}(S) = \emptyset$, automatically $f \pitchfork_A S$. When $S = N$, it's trivial that $f \pitchfork_A S$. When $A = M$, we denote the transversality as $f \pitchfork S$.

A real basic example is when $S = \{q\}$, then $f \pitchfork q$ means "f submerges q ", or q is a *regular value* of f.

Theorem 4.4.3 (Regular value theorem)

Let $f : M \to N$ be a smooth map, $q \in N$. If $f \pitchfork q$, we have $f^{-1}(q)$ is a regular smooth submanifold of M.

Moreover $\dim f^{-1}(q) = \dim M - \dim N$ when $f^{-1}(q) \neq \emptyset$.

The proof is essentially the implicit function theorem, and it's much like the proof when $M = \mathbb{R}^n$, $N = \mathbb{R}$.

Definition 4.4.4. We say $f : M \to N$ is a **submersion** at $p \in M$ if the tangent map $f_* : T_p M \to T_{f(p)} N$ is surjective.

We also call p a **regular point** of f. If $q \in N$ satisfies the points in $f^{-1}(q)$ are all regular, we say q is a **regular value**.

The opposite of regular points/values are named as **critical points/values**. Denote all the critical points of f by $Crit(f)$.

Theorem 4.4.5 (Disk lemma)

Let M be a compact manifold, $\dim M = \dim N$, $f : M \to N$, and q is a regular value of f. There exists an open neighborhood $V = V(q) \subset N$ and an open sets $f^{-1}(V) = U_1 \cup \cdots \cup U_k$, where U_i's are pairwise disjoint, such that

$$f|_{U_i} : U_i \xrightarrow{\sim} V$$

are smooth homeomorphisms.

Proof. Since $f^{-1}(q)$ is a 0-dim regular submanifold, it is an isolated set in M. By the compactness of M, it must be finite, say $f^{-1}(q) = \{p_1, \dots, p_k\}$.

By the canonical form of submersions, we can take $W_i = W_i(p_i)$ s.t. $f|_{W_i}$ is a smooth homeomorphism onto $f(W_i) =: V_i$.

Let

$$V = \left(\bigcap_{i=1}^{k} V_i\right) \setminus f\left(M \setminus \bigcup_{i=1}^{k} W_i\right).$$

We can directly check that it satisfies the conditions. □

You might notice that the statement is very similar to the covering space. However f need not be a covering map since there's an issue of connected components.

Let's look at some examples which provide more insight of critical points.

Example 4.4.6

Consider a map $f : 2T^2 \to T^2$ by "pinching" a part of the surface which contains a hole to a single point, then $Crit(f)$ is not contractible, hence we'll lose some information when we look at the image of f.

Because of this kind of wiredness, we can't determine the preimage of critical values like we do in regular value theorem.

However, there are still something we can do with these points.

Definition 4.4.7 (Non-degenerate isolated critical points)**.** Let $f : M \to \mathbb{R}$ be a smooth map, $\dim M = m$. We say $p \in Crit(f)$ is **non-degenerate**, if there's a local chart $U \ni p$ s.t. f is of the form

$$-(x_1^2 + \cdots + x_i^2) + (x_{i+1}^2 + \cdots + x_m^2)$$

where the coordinate of p is $(0, 0, \dots, 0)$.

Here the number i is independent of the choice of charts, called the **index** of p. E.g. $m = 2$, the parabolic surface opens upward has a non-degenerate critical point of index 0, the saddle surface has a point of index 1.

Obviously p is isolated critical point, as there's no other critical points in U.

If all the critical points of f are non-degenerate, we call f a **Morse function**. For example, the projection map from T^2 to the x-axis.

In fact this is connected to topological properties of compact manifolds. If M is compact, then

$$\chi(M) = \sum_{i=1}^{m} (-1)^i \# Crit_i(f) \implies \chi(M) \leq \# Crit(f).$$

Moreover Morse function can give a cell decomposition of M, hence will induce the so-called Morse homology.

Remark 4.4.8 — Morse functions and non-degenerate critical points are not required in this course.

Example 4.4.9

Consider $f : \mathbb{C}^2 \to \mathbb{C}$ by $(z, w) \mapsto zw$. The regular values of f are $c \in \mathbb{C}, c \neq 0$.

The crtical value is $c = 0$,

$$f^{-1}(0) = (\mathbb{C} \times \{0\}) \cup (\{0\} \times \mathbb{C})$$

Because of the existence of the critical point, there are nontrivial automorphisms of $f^{-1}(c)$ by rotate c around the origin several loops.

This is known as Lefschetz fibration in literature.

Theorem 4.4.10 (Sard)

Let $f : M \to N$ be a C^∞ map, then the critical values of f form a null set in N, hence the regular values are dense in N. (Manifolds are C_2.)

We will skip the proof now for some reasons. The proof will essentially use the condition C^∞ instead of C^r.

There is a more general theorem:

Theorem 4.4.11 (Transversal preimage theorem)

Let $f : M \to N$ be a C^∞ map, $S \subset N$ is a regular closed submanifold. If $f \pitchfork S$, then $f^{-1}(S) \subset M$ is a regular C^∞ manifold, and $\operatorname{codim} f^{-1}(S) = \operatorname{codim} S$.

The key step of the proof is the "local canonical representation" of transverse, i.e. there exists $U = U(p)$ and $V \supseteq f(U)$ s.t. f is of the form $(u, v) \mapsto (\eta(u, v), v) \in \mathbb{R}^s \times \mathbb{R}^{n-s}$, which is an application of implicit function theorem.

Theorem 4.4.12 (Transverse approximation theorem)

Let $f : M \to N$ be a C^∞ map, $S \subset N$ is a closed regular smooth manifold. Then there exists a smooth map $g : M \to N$ approaches f within arbitarily small difference such that $g \pitchfork S$ and $f \simeq g$. (Homotopy equivalent)

Moreover if $f \pitchfork_K S$, where $K \subset M$ compact, we have $f|_K \equiv g|_K$.

The key point is that when $K \subset M$ is compact and $f \pitchfork_K S$, we have $f \pitchfork_U S$ for some $U \supset K$ open.

Comparing to Sard's theorem, we can make small modifications to make f into g in this theorem.

Next we'll go through the "boundary" versions of above thoerems.

Theorem 4.4.13 (Collar neighborhood theorem)

Let M be a smooth compact manifold with boundary, then ∂M has an open neighborhood homeomorphic to $\partial M \times [0, 1)$.

Theorem 4.4.14

Let $f : M \to N$, C^∞. If M is a manifold with boundary, N without boundary, $S \subset N$ regular. If $f \pitchfork S$, and $\partial f := f|_{\partial M} \pitchfork S$, then $f^{-1}(S) \subset M$ regular and $\partial f^{-1}(S) = f^{-1}(S) \cap \partial M$. When it is nonempty, $\operatorname{codim}_M f^{-1}(S) = \operatorname{codim} S$.

Theorem 4.4.15

Let $f : M \to N$, C^∞. If M is a manifold with boundary, N without boundary, $S \subset N$ regular and closed. Then there exists g arbitarily close to f, such that $g \pitchfork S$ and $\partial g \pitchfork S$.

Moreover if $f \pitchfork_K S$ holds for a compact set K, we can require $g|_K = f|_K$.

These theorems have other versions in fibre bundle sections and other cases, but they can't be implied by each other, so we have to repeat this annoying proof again. That's why the books about differential topology are always longer.

§4.5 Connected sums

Recall that last semester we talked about connected sums in classifications of closed surfaces. In differential topology, we need to clarify the differential structures, so the question arise: under what conditions are the connected sum $M \# N$ well-defined?

First let's look at some examples.

Example 4.5.1

If M has two connected components, then $M \# N$ is not well-defined, since N can connected to either component of M.

If $M, N \cong \mathbb{C}P^2$, then there's an orientation issue of $M \# N$, it can be $\mathbb{C}P^2 \# \mathbb{C}P^2$ or $\mathbb{C}P^2 \# (-\mathbb{C}P^2)$, which aren't smoothly homeomorphic. This can be proved by the "intersection form".

From these examples we see that the connectedness and orientations matters the connected sum. Another issue comes from the differential structure at the place where the tube connects the original manifolds. Also we need to check whether the connected sums are homeomorphic when we choose different disks to cut out.

When M, N are connected and oriented, by collar neighborhood theorem, we can cut out a smooth small disk and connects the collar neighborhood together to get the charts around the cut.

To be exact, let $B \subset M, B' \subset N$ be n-dimensional balls, and the boundary of $M \setminus B$, $N \setminus B'$ is homeomorphic to S^{n-1}. Take the collar neighborhood of them and fuse them together (compatible with orientations) to form the tubular neighborhood of S^{n-1}, then $(M \setminus B) \cup_{S^{n-1}} (N \setminus B')$ is a smooth manifold.

To prove the manifold does not depend on the choice of balls,

Theorem 4.5.2 (Homogeneity)

Let M be a smooth manifold, $U \subset M$ is a connected open set. For any $p, q \in U$, there exists smooth homeomorphisms $h_t : M \to M$, $t \in [0,1]$, such that $h_0 = \mathrm{id}_M$, $h_1(p) = q$, and h_t is identity map outside some compact subset of U.

Proof. When U is an open ball in a chart, we can explicitly construct the h_t's.(Omitted here)

The conclusion in the theorem forms an equivalence relation on U, and we can prove the equivalence class is both open and closed, thus by connectedness we're done. □

Remark 4.5.3 — In fact the topological connected sum are harder to define.

The set of smooth manifolds up to smooth homeomorphisms is a half group under the connected sum. It becomes a group after quotient out the *cobordism* relation.

§4.6 Intersection numbers

Some motivations:

- Whether $f_0, f_1 : M \to N$ is homotopic?
- Is $f : M \to N$ contractible?
- Does $f : M \to M$ has a fixed point?
- How many fixed points are there?
- Is there a singular point in a smooth vector field?
- Does a smooth function has critical points?
- Whether M and N are smoothly homeomorphic?

To answer these questions, people want to find some invariants to give a sufficient condition of these statements. Typical ones are Euler characteristcs, fundamental groups and homology groups. Here we introduce another one: intersection numbers.

Example 4.6.1 (Fundamental theorem of algebra)

Let $f \in \mathbb{C}[x]$, then f has a zero in $\mathbb{C}$.

We can smoothly extend f to $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, then $\deg f$ is precisely the degree of f as a smooth map. Later we'll see that $\deg f \neq 0 \implies f$is surjective.

Example 4.6.2 (Lefschetz number)

Let $f : M \to M$, M is a compact smooth manifold. Let $\Delta_M = \{(x, x) \in M \times M\}$, then the fixed points are just $\Gamma(f) \cap \Delta_M$.

The Lefschetz number $L(f)$ is defined as the "signed sum" of transverse intersection points of $\Gamma(f)$ and Δ_M. (e.g. when $M = \mathbb{R}$, the points $f(x) = x$ with $f'(x) > 1$ or $f'(x) < 1$ are treated as negative or positive)

In fact $L(\mathrm{id}_M) = \chi(M)$, and $L(f) \neq 0 \implies f$ has fixed points.

There are two types of intersection numbers, we'll discuss them one by one.

§4.6.1 Modulo 2 intersection number

Our goal is: for smooth manifold M, N, M compact, $S \subset N$ a regular closed submanifold, and a continuous map $f : M \to N$, if $\dim S + \dim M = \dim N$, we will find a number

$$I_2(f, S) \in \mathbb{Z}/2\mathbb{Z}$$

which only depends on M, N, S and the homotopy class of f. Here we do not require the manifolds to be orientable.

The way to construct it is as follows:

- First we do this for $f \in C^\infty$ and $f \pitchfork S$.
- Use the transversal approximation to generalize it to C^0.

Intuitively, when $f \in C^\infty$ and $f \pitchfork S$, if $f^{-1}(S)$ is a finite set, we simply let

$$I_2(f, S) := \# f^{-1}(S) \bmod 2.$$

The reason we're taking modulo 2 is to ensure invariance under homotopic equivalences of f.

Proposition 4.6.3

If $f_0, f_1 \in C^\infty$, $f_0, f_1 \pitchfork S$, $f_0 \simeq f_1$. then $I_2(f_0, S) = I_2(f_1, S)$.

Proof. Let $F : M \times [0, 1] \to N$ be a homotopy from f_0 to f_1.

By transversal approxiamtion, we can assume $F \pitchfork S$.

Now by transversality, $F^{-1}(S)$ is a union of finitely many loops and segments (compact 1d regular submanifold), thus $\partial F^{-1}(S)$ consists of finite points, and $\# \partial F^{-1}(S)$ is even. Therefore $I_2(f_0, S) = I_2(f_1, S)$. □

Note that here we used the classification of compact 1-dimensional smooth manifolds.

In fact, take the universal convering space $\widetilde{X}/X$, we can prove that $\widetilde{X} \approx \mathbb{R}$. Then $\mathrm{Gal}(\widetilde{X}/X)$ is an infinite cyclic group (or trivial group), taking a smooth Riemann metric on $\widetilde{X}$, the rest is the same with topological case (i.e. C^0 case).

Next for $f : M \to N \in C^0$, take a transversal approxiamtion $g \simeq f$, g is C^∞ and $g \pitchfork S$. Define $I_2(f, S) := I_2(g, S)$. This is well-defined by above proposition.

§4.6.2 Oriented intersection number

Sometimes we will find that, if we can assign a "direction" for curves, we can say the intersection is "positive" or "negative", which provides a more efficient way to count the number of intersections.

This can be generalized to any orientable manifolds. Let M, N be smooth oriented manifolds, M is compact. Let $f : M \to N$ be a continuous map, $S \subset N$ is regular closed oriented submanifold with $\dim M + \dim S = \dim N$, then we want to define $I(f, S) \in \mathbb{Z}$.

Still we'll do this in two steps. When $f \in C^\infty$ and $f \pitchfork S$, for any $p \in f^{-1}(S)$, by transversality

$$f_* T_p M \oplus T_{f(p)} S = T_{f(p)} N.$$

Using the orientation of M, S, N, the orientation on two sides may be the same or the opposite, which corresponds to $I(f, S; p) = \pm 1$. Hence

$$I(f, S) = \sum_{p \in f^{-1}(S)} I(f, S; p)$$

Proposition 4.6.4

If $f_0, f_1 \in C^\infty$, $f_0, f_1 \pitchfork S$, $f_0 \simeq f_1$. then $I(f_0, S) = I(f_1, S)$.

Proof. Using the notation of the previous proof, $F^{-1}(S)$ is an oriented submanifold now.

For each segment or loop in $F^{-1}(S)$, consider their boundary (which lies in $\partial F^{-1}(S) \subset M \times \{0, 1\}$).

Observe that the orientation of $\partial(M \times [0, 1])$ derived from F and the orientation given by $M = M \times \{0\} = M \times \{1\}$ is the same on $M \times \{1\}$ and the opposite on $M \times \{0\}$.

We need to case-study all the possible segments in $F^{-1}(S)$, here we only consider one as an example. If the segment has two endpoints in $M \times \{1\}$, then the outer normal vector on two endpoints is the same on "ending point", opposite on "starting point" of the segment.

By definition we can check the contribution to $I(f_1, S)$ is $+1$ for same orientation and -1 for opposite orientation.

Summing up all these segments we'll get $I(f_0, S) = I(f_1, S)$. $\square$

Next we give some examples of intersection numbers.

Example 4.6.5 (Self-intersection number)

Let $S \subset N$ be a regular closed submanifold.

$$I(S, S) := I(i_S, S) \in \mathbb{Z}$$

is defined to be the **self-intersection number** of S.

For example, let $S \simeq S^1$ be a submanifold of $2T^2$ (which has trivial normal bundle). Clearly i_S has a homotopy equivalence that has no intersection with S, so $I(S, S) = 0$.

In contrast, let S be a loop on a Mobius band, i_S has a homotopy equivalence that intersects with S at only one point transversally, thus $I_2(S, S) = 1$. Note that Mobius band is not orientable, so we can't talk about $I(S, S)$.

Example 4.6.6 (Mapping degree)

Let M, N be compact smooth manifolds with the same dimensions. Let $f : M \to N$ be a smooth map (or continuous map), we can define

$$\deg_2(f) := I_2(f, q) \in \mathbb{Z}_2, \quad q \in N.$$

Moreover if M, N are oriented,

$$\deg(f) := I(f, q) \in \mathbb{Z}$$

is defined as the degree of f.

This is independent of the choice of q, so in fact $\deg(f) = \# f^{-1}(q)$ for any regular value $q \in N$, where the counting is modulo 2 or signed.

Some examples of mapping degrees:

Example 4.6.7

Let $p \in \mathbb{C}[z]$ as a map $\widehat{p} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. The degree of the map $\deg \widehat{p} = \deg(p)$ as a polynomial.

The conjugation map $z \mapsto \overline{z}$ has degree -1.

Example 4.6.8 (Pinch)

Let M be a surface, and $A \subset M$ is has a boundary homeomorphic to S^1. Then the map $f : M \to N$ with $f(A) = \{pt\}$, and $f|_{M \setminus A}$ embedding is called a "pinch". Clearly it has degree 1.

Example 4.6.9

We can construct a map $f_m : S^n \to S^n$ s.t. $\deg f_m = m$, for all $m \in \mathbb{Z}$.

When $n = 1$, it's just $e^{i\theta} \mapsto e^{im\theta}$, and to lift it in higher dimensions, we can think of S^n as two cones fused together with common base S^{n-1}.

§4.7 Poicare-Hopf Theorem

Let M be a n-dimensional smooth manifold, and v is a continuous vector field on M.

Definition 4.7.1 (Isolated singular points)**.** We say $p \in M$ is an isolated singular point, if there exists $u = u(p) \subset M$ s.t.

$$v(p) = \vec{0}, \quad v(q) \neq \vec{0}, \forall q \in u \setminus \{p\}.$$

There are a lot of situations which can happen around an isolated singular points, I'm not able to draw them as pictures. But in general it's pretty much like magnitic fields in physics, this might help you imagine some of them.

If we take a small ball $B_\varepsilon(p)$, the vectors at $\partial B(p)$ gives a map $\partial B(p) \to S^{n-1}$ by their directions, and since $\partial B(p)$ can be identified as S^{n-1}, there's an induced map $S^{n-1} \to S^{n-1}$, and its degree is roughly the index of p.

Definition 4.7.2. Let p be an isolated singular point, take a chart containing p s.t. p is the origin. Let $D_\varepsilon(0)$ be a sufficiently small open ball in $\mathbb{R}^n$, Then we have the map

$$\partial D_\varepsilon(0) \to S^{n-1}, \quad x \mapsto \frac{v(x)}{\|v(x)\|}.$$

Define the **index** of p to be

$$ind_p(v) := \deg(\partial D_\varepsilon(0) \to S^{n-1}).$$

Since we use the oriented mapping degree, we require the manifold M is oriented.

This definition involves the choices of ε and the chart.

Theorem 4.7.3 (Poincare-Hopf)

Let M be a smooth oriented compact manifold, let v be a continuous vector field whose singular points are all isolated.

$$\sum_{p:v(p)=0} ind_p(v) = \chi(M),$$

where $\chi(M)$ is the Euler characteristic.

Remark 4.7.4 — Since we haven't define $\chi(M)$ in the courses, we'll regard $\chi(M)$ as $I(E_0, E_0)$ where E_0 is the image of zero section in TM, and the intersection number is taken in TM.

Later we can check it's identical with $\chi(M)$ defined in other ways, such as simplicial subdivision.

Example 4.7.5

Let $\phi_t : M \to M$ be a one-parameter transformation group, and $v = \frac{\partial \phi}{\partial t}|_{t=0}$. If $v(p) = 0$ and $d\phi|_p : T_pM \to T_pM$ doesn't have eigenvalue 1, (i.e. it's nondegenerate) then

$$ind_p(v) = \operatorname{sgn}(\det(I - (d\phi)_p)) = \pm 1.$$

For a vector field v, denote $\widehat{v}(x) := (x, v(x)) \in TM$. Intuitively, since every section is homotopic to zero section,

$$I(E_0, E_0) = I(\widehat{v}, E_0) \text{ “=” } \sum_p ind_p(v)$$

Definition 4.7.6 (Local intersection number)**.** Let M, N be oriented smooth manifold, $S \subset N$ oriented, regular, smooth submanifold, and $\dim M + \dim S = \dim N$.

Let $f : M \to N$ be a continuous map, $\Omega \subset M$ is open, $\overline{\Omega}$ is compact in M.

Let $\widetilde{f} \pitchfork_\Omega S$, $\widetilde{f} \simeq f$ and the homotopy H satisfies

$$H\big|_{[0,1]\times\partial\Omega} \cap S = \emptyset.$$

(This is to say the “endpoints” of f can't cross S during the homotopy, so that the intersection number stays the same.) The **local intersection number** is defined as

$$I(f, S, \Omega) := \sum_{p \in \widetilde{f}^{-1}(S)\cap\Omega} \operatorname{sgn}(f, S)_p$$

Since $\widetilde{f}^{-1}(S) \cap \Omega$ is finite by the compactness of $\overline{\Omega}$, this intersection number is well-defined.

Remark 4.7.7 — We can define $I(f, S, \Omega)$ for $f : \overline{\Omega} \to N$, with the requirements that $f \in C^0(\overline{\Omega})$, and $\widetilde{f} \pitchfork_\Omega S$ and $\widetilde{f}(\partial\Omega) \cap S = \emptyset$.

- Homotopic invariance:

  Clearly if H gives a homotopy $f_0 \simeq f_1$, if $H^{-1}(S) \cap ([0,1] \times \partial\Omega) = \emptyset$, we have $I(f_0, S, \Omega) = I(f_1, S, \Omega)$.

- Region additivity:

  Let $\Omega_1, \dots, \Omega_k \subset M$ pairwise disjoint. Then

  $$I\left(f, S, \bigcup_{i=1}^{k} \Omega_i\right) = \sum_{i=1}^{k} I(f, S, \Omega_i).$$

- "Cut out": Let $K \subset \Omega$ be a compact subset, $f(K) \cap S = \emptyset$, then

  $$I(f, S, \Omega) = I(f, S, \Omega \setminus K).$$

Proof of Theorem 4.7.3. Like we pointed out, if $p_1, \dots, p_s$ are the singular points of v, take pre-compact open neighborhoods $\Omega_i = \Omega_i(p_i)$, such that they are pairwise disjoint.

$$\chi(M) = I(E_0, E_0) = I(\widehat{v}, E_0) = \sum_{i=1}^{s} I(\widehat{v}, E_0, \Omega_i)$$

All we need to show is $I(\widehat{v}, E_0, \Omega_i) = ind_{p_i}(v)$.

We may assume that each Ω_i is an open ball $\mathring{D}_\varepsilon(0) \subset \mathbb{R}^n$ in some chart.

First, WLOG that v is smooth in a neighborhood of ∂D_ϵ. Choose a vector $\xi \in v(\partial D_\epsilon)$, the index is the intersection number $I(v, \xi)$. Hence we can define a vector field $w(x) = \xi$ on $\mathbb{R}^n$. Intuitively $I(v, \xi) = I(v, w, D_{(0.9\epsilon, 1.1\epsilon)})$.

Let $v^*(x) = \frac{\|x\|}{\epsilon} v(x)$. Then $ind_p(v) = I(v^*, w, D_{(0.9\epsilon, 1.1\epsilon)})$. (Need to check v^* and w transverse)

We have

$$ind_p(v) = I(v, w, D_{(0.9\epsilon, 1.1\epsilon)}) = I(v, E_0, D_{0.9\epsilon, 1.1\epsilon}) = I(v, E_0, D_{1.1\epsilon}) = I(v, E_0, D_\epsilon).$$

Therefore we're done. ∎

Remark 4.7.8 — The index is in fact a special case of **winding number**.

Let $f : M \to \mathbb{R}^n$ continuous, M is a $n-1$ dimensional compact oriented smooth manifold. For $q \notin f(M)$,

$$W(f, q) := \deg\left(M \to S^{n-1}, \quad p \mapsto \frac{f(p) - q}{\|f(p) - q\|}.\right)$$

This is a homotopic invariance of f.

Poincare-Hopf theorem means that different definitions of Euler characteristic are the same, at least for orientable compact manifolds.

For a finite simplicial complex X, the Euler characteristic is defined as

$$\chi(X) := \sum_{i=0}^{\infty} (-1)^i \#\{i \text{ dim simplicial}\}.$$

Theorem 4.7.9

Let M be a smooth compact manifold.

- M always have a smooth simplicial partition.
- $I(E_0, E_0) = \chi(M)$.

Here the smooth simplicial partition means a smooth homeomorphism $g : X \to M$ s.t. in each m-dim simplex, the map $\Delta^m \to X \to M$ can be extended to a C^∞ map on a neighborhood of Δ^m.

Proof outline. (1) can be proved by taking a Riemann metric and a cover consisting of convex balls.

Next, for any simplicial partition of M, we can construct a vector field such that $\sum_p ind_p(v) = \chi(M)$. This can be done by setting the centroid of each simplex as a singular point, and vectors always pointing from high-dim centroid to low-dim centroids. This construction asserts that each i-dim centroid has index $(-1)^i$, as desired. □

There are other equivalent definitions of Euler characteristics, such as

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{F}} H_i(X, \mathbb{F})$$

where $\dim_{\mathbb{F}} H_i(X, \mathbb{F})$ is the i-th Betti number over the field $\mathbb{F}$.

Since the homology group $H_i(X, \mathbb{F})$ is homotopically invariant, the Euler characteristic is also a homotopic invariance.

Recall that we learned de Rham isomorphism, thus when M is a smooth compact manifold,

$$\chi(M) = \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{R}} H^i_{\mathrm{dR}}(M)$$

§4.8 Gauss-Bonnet Theorem on hypersurfaces in $\mathbb{R}^{n+1}$

Let n be an even integer, $M \subset \mathbb{R}^{n+1}$ is a regular compact connected submanifold.

In this case, M is orientable, which determines a bounded reigion in $\mathbb{R}^{n+1}$. Let the outward normal vector determine the orientation of M, and $K(x)$ is the Gauss curvature at the point x.

Theorem 4.8.1 (Gauss-Bonnet)

Using the above notations, we have

$$\int_M K(x) \, \mathrm{d}Area(x) = \frac{Area(\mathbb{S}^n)}{2} \chi(M).$$

Here $\mathrm{d}Area(x)$ is the volume form on M, $Area(\mathbb{S}^n)$ is the surface area of n-dim sphere.

Specifically, let $g = \sum_{i,j} g_{ij} \, \mathrm{d}x_i \otimes \mathrm{d}x_j$ be the Riemann metric on M induced by Euclidean metric,

$$\mathrm{d}Area(x) := \sqrt{\det(g_{ij})} \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n.$$

> **Remark 4.8.2 —** Recall that Gauss curvature is the Jacobian determinant of Gauss-Winegarten map, i.e. $K(x) = J_G(x)$, where
> $$G : M \to \mathbb{S}^n, \quad x \mapsto \vec{n}_x.$$
> Where $\vec{n}_x$ is the outward normal vector at x.

Proof. There's a fact about mapping degree: Let $f : M \to N$, $\omega \in A^n(N)$,

$$\int_M f^*\omega = \deg(f) \int_N \omega.$$

This result matches our intuition. To prove it you just need to use disk lemma and definitions of integrals.

This tells us that

$$\int_M K \, \mathrm{d}Area = \deg(G) \int_{\mathbb{S}^n} \mathrm{d}Area_{\mathbb{S}^n}.$$

Since "obviously" $G^* \, \mathrm{d}Area_{\mathbb{S}^n} = K \, \mathrm{d}Area$.

> **Remark 4.8.3 —** This needs to be checked carefully for signs, which will use the connectivity of M to reduce the checking to only one point.

Thus we only need to show that $\deg(G) = \frac{\chi(M)}{2}$. This will be done by constructing a vector field with the sum of indexes of singular points equal to $2\deg(G)$.

The construction is as follows: Fix a vector $\vec{a} \in \mathbb{S}^n$, let $v(x)$ be the orthogonal projection of $\vec{a}$ onto T_xM.

$$v(x) = 0 \iff \vec{n}_x = \pm\vec{a}.$$

Therefore if we take $\vec{a}$ such that $\pm\vec{a}$ are both regular values of G, the vector field v only contains isolated singular points. Now by Sard's theorem, the critical values form a null set, clearly such $\vec{a}$ exists.

Next we take a look at the index of these singular points. Auctually, they are all *non-degenerate* singular points. (Otherwise we'll get G_* is degenerate on the principal normal section, contradicts with the regular value condition)

Hence we have

$$ind_x(v) = \mathrm{sgn}_x(\pm G), \quad G(x) = \pm\vec{a}.$$

Since n is even, $\mathrm{sgn}_x(G) = \mathrm{sgn}_x(-G)$, summing it up we'll get

$$\chi(M) = \sum_x ind_x(v) = 2\sum_x \mathrm{sgn}_x(G) = 2\deg(G).$$

□

At last we look at the statement that "$\mathbb{R}^{n+1} \setminus M$ consist of two connected component, one is bounded and the other is unbounded". (Will use intersection number and tubular neighborhood)