

容易看出,  $z=0, z=1$  为超几何方程的奇点;  $z=1$  为勒让德方程的奇点.

下面讨论无穷远点是否为方程的奇点.

将  $\frac{dw}{dz} + p(z)\frac{dw}{dz} + q(z)w = 0$  作变量替换:  $z \rightarrow \frac{1}{z}$

$$\frac{dw}{dz} = -z^2 \frac{dw}{dz}, \quad \frac{dw}{dz} = \frac{1}{dz} (-z^2 \frac{dw}{dz}) = z^2 \frac{dw}{dz} (z^2 \frac{dw}{dz}) = z^2 \frac{dw}{dz} + z^2 \frac{dw}{dz}$$

$$\Rightarrow z^2 \frac{dw}{dz} + z^2 p(z) \frac{dw}{dz} + q(z)w = 0$$

$$\Rightarrow \frac{dw}{dz} + (\frac{1}{z} - zp(z)) \frac{dw}{dz} + \frac{q(z)}{z^2} w = 0$$

当  $p(z) = a_0 + a_1 z + a_2 z^2 + \dots$ ,  $q(z) = b_0 + b_1 z + b_2 z^2 + \dots$  时, 为常点

即为  $p(z) = \frac{a_0}{z} + \frac{a_1}{z^2} + \frac{a_2}{z^3} + \dots$ ,  $q(z) = \frac{b_0}{z^2} + \frac{b_1}{z^3} + \dots$  时, 为奇点.

对于超几何方程,  $p(z) = \frac{z(1+\alpha)z}{z(1-z)}$ ,  $q(z) = \frac{-\alpha\beta}{z(1-z)}$  不满足条件, 故  $z=\infty$  为奇点.

对于勒让德方程,  $p(z) = \frac{-2z}{1-z^2}$ ,  $q(z) = \frac{1-z^2}{1-z^2}$  不满足条件, 故  $z=\infty$  为奇点.

1.2 奇点邻域内的级数解

我们不加证明地给出如下定理:

定理: 对于  $\frac{dw}{dz} + p(z)\frac{dw}{dz} + q(z)w = 0$ , 当  $|z-z_0| < \rho$  时,  $p(z), q(z)$  可展成

且  $w(z_0) = C_0, w'(z_0) = C_1$ , 则  $w(z)$  的级数可写为  $\sum_{k=0}^{\infty} C_k (z-z_0)^k$ , 其中  $C_1, C_2, \dots$

同一因  $C_0, C_1$  可任意选取.

从级数的角度看, 这个定理是显然的. 常点附近的级数可作泰勒展开, 且  $\rho$

常点附近的级数有无穷多参数  $(C_0, C_1)$  也可任意选取. 下面举例说明例2.

例2: 在  $z=0$  附近求超几何方程和勒让德方程  $(1-z)\frac{dw}{dz} - 2z\frac{dw}{dz} + (1-\alpha)w = 0$  的级数解.

$$w(z) = \sum_{k=0}^{\infty} C_k z^k, \quad \frac{dw}{dz} = \sum_{k=0}^{\infty} k C_k z^{k-1}, \quad \frac{dw}{dz} = \sum_{k=0}^{\infty} k C_k z^{k-1} + (1-\alpha)w = 0, \quad \text{代入}$$

$$\sum_{k=0}^{\infty} k C_k z^{k-1} - 2z \sum_{k=0}^{\infty} k C_k z^{k-1} + (1-\alpha) \sum_{k=0}^{\infty} C_k z^k = 0$$

$$\sum_{k=0}^{\infty} k(k-2) C_k z^{k-1} + (1-\alpha) \sum_{k=0}^{\infty} C_k z^k = 0$$

$$z \text{ 系数: } k(k-2) C_k + (1-\alpha) C_{k+1} = [k(k-1) + k - (1-\alpha)] C_k = [k(k-1) + k - (1-\alpha)] C_k$$

$$\Rightarrow C_{k+1} = \frac{(k-1)(k-2+\alpha)}{(k+1)(k+1-\alpha)} C_k$$

$$\Rightarrow C_k = \frac{(k-1)(k-2+\alpha)}{(k+1)(k+1-\alpha)} \cdot \frac{(k-2)(k-3+\alpha)}{(k+2)(k+2-\alpha)} \cdots \frac{(0-1)(0-2+\alpha)}{2-1} C_0$$

$$= \frac{z^k}{k!} \left[ (k+1)(k+2+\alpha) \cdots (1-\alpha) \right] \cdot \left[ (k-1)(k-2+\alpha) \cdots (1-\alpha) \right] C_0$$

$$= \frac{z^k}{k!} \frac{\Gamma(k+1)}{\Gamma(k+1)} \frac{\Gamma(k+2+\alpha)}{\Gamma(k+2+\alpha)} C_0$$

$$C_{k+1} = \frac{(k+1)(k+2+\alpha)}{(k+2)(k+2-\alpha)} \cdot \frac{(k+2)(k+3+\alpha)}{(k+3)(k+3-\alpha)} \cdots \frac{(1-\alpha)(1-\alpha)}{2-1} C_1$$

$$= \frac{z^k}{k!} \left[ (k+1)(k+2+\alpha) \cdots (1-\alpha) \right] \left[ (k+1)(k+2+\alpha) \cdots (1-\alpha) \right] C_1$$

$$= \frac{z^k}{k!} \frac{\Gamma(k+1)}{\Gamma(k+1)} \frac{\Gamma(k+2+\alpha)}{\Gamma(k+2+\alpha)} C_1$$

$$\Rightarrow w(z) = C_0 \sum_{k=0}^{\infty} \frac{z^k}{k!} \frac{\Gamma(k+1)}{\Gamma(k+1)} \frac{\Gamma(k+2+\alpha)}{\Gamma(k+2+\alpha)} z^k + C_1 \sum_{k=0}^{\infty} \frac{z^k}{k!} \frac{\Gamma(k+1)}{\Gamma(k+1)} \frac{\Gamma(k+2+\alpha)}{\Gamma(k+2+\alpha)} z^k$$

进一步可以写成  $w(z) = C_0 w_0(z) + C_1 w_1(z)$ , 其中  $w_0(z)$  和  $w_1(z)$  线性无关.

$w_0(z)$  只含偶数项, 为偶函数;  $w_1(z)$  只含奇数项, 为奇函数.

若  $\alpha$  为整数时,  $w_0(z), w_1(z)$  为无穷级数

若  $\alpha$  为整数时,  $w_0(z), w_1(z)$  之无穷级数为多项式

利用朗斯基定理证明中, 的证明. 我们可容易地得方程的解进行证明, 且

正的结果是无穷级数的解

1.3 奇点邻域内的级数解

类似常点情形, 我们仍不加证明地给出如下定理.

定理: 若  $z_0$  是  $\frac{dw}{dz} + p(z)\frac{dw}{dz} + q(z)w = 0$  的奇点, 且在  $0 < |z-z_0| < \rho$  的区域

无奇点, 则方程在  $0 < |z-z_0| < \rho$  内有级数解:

$$\begin{cases} w_0(z) = (z-z_0)^{\rho} \sum_{k=0}^{\infty} C_k (z-z_0)^k \\ w_1(z) = g w_0(z) \ln(z-z_0) + (z-z_0)^{\rho} \sum_{k=0}^{\infty} d_k (z-z_0)^k \end{cases}$$

我们不加证明地给出定理的正确性, 并以此证明其中的关键步骤:

①  $(z-z_0)^{\rho}$  使得指数可非整数 ②  $\sum_{k=0}^{\infty}$  表明无穷级数可任意选取

③  $g \neq 0$  时可能还有对数项

我们仍可继续将结果代入原方程以解得  $C_k$  之间的关系, 即即边并不完

整的给出级数解的具体形式. 因为无穷级数收敛到一个统一级数解

我们希望级数有如下形式 (有四个底数), 该级数为正则级数:

$$\begin{cases} w_0(z) = (z-z_0)^{\rho} \sum_{k=0}^{\infty} C_k (z-z_0)^k & C_0 \neq 0 \\ w_1(z) = g w_0(z) \ln(z-z_0) + (z-z_0)^{\rho} \sum_{k=0}^{\infty} d_k (z-z_0)^k & g \neq 0 \end{cases}$$

注: 我们若以二通问题  $p(z)$  的最低级从  $z=0$  开始取

方程具有正则级数解, 满足一定条件的方程, 我们不加证明地给出以

下定理.

定理: 若  $w = p(z)\frac{dw}{dz} + q(z)w = 0$  在奇点  $z_0$  附近收敛  $0 < |z-z_0| < \rho$  内有正

正则级数解, 则  $p(z)$  可展成  $q(z)$  可展成

若  $z_0 = 0$ ,  $z=0$  为超几何方程的正则奇点;  $z=1$  为勒让德方程的正则奇点.

对于无穷级数, 该定理即为  $z=1$  的级数解

对于超几何方程和勒让德方程而言, 无穷级数均是正则级数.

对于级数方程而言, 我们总是先将  $w(z)$  代入方程, 如果能得到

两个线性无关的解, 则可令  $g=0$ ; 否则方程的解为两个对数项.

具体地, 我们在  $z=0$  附近求超几何方程的正则级数, 则有:

$$p(z) = \sum_{k=0}^{\infty} a_k z^{k-1}, \quad q(z) = \sum_{k=0}^{\infty} b_k z^{k-2}, \quad w_0(z) = (z-z_0)^{\rho} \sum_{k=0}^{\infty} C_k (z-z_0)^k$$

$$\frac{dw}{dz} = (k+1)(z-z_0)^{\rho} \sum_{k=0}^{\infty} C_k (z-z_0)^{k-1} + (z-z_0)^{\rho} \sum_{k=0}^{\infty} C_k (z-z_0)^k$$

$$\Rightarrow (z-z_0)^{\rho} \left[ (k+1)(k+1) \sum_{k=0}^{\infty} C_k (z-z_0)^{k-1} + (z-z_0)^{\rho} \sum_{k=0}^{\infty} C_k (z-z_0)^k \right]$$

$$+ (z-z_0)^{\rho} \sum_{k=0}^{\infty} C_k (z-z_0)^k \sum_{k=0}^{\infty} b_k z^{k-2} = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} z^{k-2} \left[ (k+1)(k+1) \sum_{k=0}^{\infty} C_k (z-z_0)^{k-1} + \sum_{k=0}^{\infty} C_k (z-z_0)^k a_k + \sum_{k=0}^{\infty} C_k (z-z_0)^k b_k \right] = 0$$

$$\Rightarrow C_k (k+1)(k+1) + \sum_{k=0}^{\infty} C_k (k+1) a_k + b_k = 0$$

将  $k=0$  代入, 有  $p_0 p_0 + p_0 a_0 + b_0 = 0$ , 即为指数方程. 可得到其结果有

$$2 \text{ 个根, 其中 } a_0 = \left( \frac{p_0}{2} \pm p_0 \right), \quad b_0 = \left( \frac{p_0}{2} \pm p_0 \right)$$

讨论:  $\begin{cases} p_1 - p_2 = \text{非整数} & \text{2个线性无关, 不含对数项} \\ p_1 - p_2 = \text{整数} & \text{可能含有对数项} \\ p_1 = p_2 & \text{一定含有对数项} \end{cases}$

事实上, 我们得到一个解  $w_0(z)$  后, 可通过另一方式得到另一个解  $w_1(z)$ :

$$\begin{cases} \frac{dw_0}{dz} + p(z)\frac{dw_0}{dz} + q(z)w_0 = 0 \quad \dots \text{①} \\ \frac{dw_1}{dz} + p(z)\frac{dw_1}{dz} + q(z)w_1 = 0 \quad \dots \text{②} \end{cases}$$

$$\Rightarrow w_1 = 0 \cdot w_2 = (w_1 \frac{dw_0}{dz} - w_0 \frac{dw_1}{dz}) + p(z) (w_1 \frac{dw_0}{dz} - w_2 \frac{dw_1}{dz}) = 0$$

$$\Rightarrow \frac{d}{dz} (w_1 \frac{dw_0}{dz} - w_0 \frac{dw_1}{dz}) + p(z) (w_1 \frac{dw_0}{dz} - w_2 \frac{dw_1}{dz}) = 0$$

$$w_1 \frac{dw_0}{dz} - w_0 \frac{dw_1}{dz} = A e^{-\int p(z) dz} = w_1' \frac{dw_0}{dz}$$

$$\Rightarrow w_1(z) = A w_0(z) \int \frac{1}{w_0(z)} e^{-\int p(z) dz} dz$$

以上讨论为正则化, 取第一个具体例子:

$$\text{例2: 求方程 } (1-z)\frac{dw}{dz} + 2(1-\alpha)z\frac{dw}{dz} + 2\alpha w = 0 \quad \text{在 } z=1 \text{ 附近解}$$

其中  $\alpha$  为整数

$$w(z) = \sum_{k=0}^{\infty} C_k (z-1)^{k+\rho}, \quad \frac{dw}{dz} = \sum_{k=0}^{\infty} C_k (k+\rho) (z-1)^{k+\rho-1} (p+k)$$

$$1-z = -1+z = -(z-1) \quad z = (z-1) + 1$$

$$\Rightarrow - \sum_{k=0}^{\infty} C_k (z-1)^{k+\rho} (p+k) + 2 \sum_{k=0}^{\infty} C_k (z-1)^{k+\rho} (p+k) + 2 \sum_{k=0}^{\infty} C_k (z-1)^{k+\rho} (p+k) = 0$$

$$+ 2(1-\alpha) \sum_{k=0}^{\infty} C_k (z-1)^{k+\rho} (p+k) + 2(1-\alpha) \sum_{k=0}^{\infty} C_k (z-1)^{k+\rho} (p+k) = 0$$

$$+ 2\alpha \sum_{k=0}^{\infty} C_k (z-1)^{k+\rho} = 0$$

$$(z-1)^{p-1} \text{ 项: } \Rightarrow C_0 p(p-1) + 2(1-\alpha) C_0 p = 0 \Rightarrow p(p-1) = 0$$

$$\Rightarrow p_1 = 0, p_2 = 1$$

$$C_k (1-k)(p+k-1) + 2(1-\alpha)(p+k) + 2\alpha + C_{k+1}(2(1-\alpha)(p+k+1) - 2(p+k)(p+k+1)) = 0$$

$$C_k (p+k+1) (2(1-\alpha)(p+k+1) + 2(p+k)(p+k+1)) + 2(p+k)(p+k+1) C_{k+1} = 0$$

$$\Rightarrow C_{k+1} = \frac{1}{2} \cdot \frac{2(p+k)}{p+k+1} C_k \quad C_0 = \frac{1}{2} \cdot \frac{2(p-1)}{p-1} C_{-1}$$

$$2 \text{ 个 } n=p, C_0 = \frac{1}{2} \cdot \frac{p-k}{p-k} C_1 = \frac{p-k}{p-k} C_1 = \frac{1}{2} \cdot \frac{p-k}{p-k} \cdot \frac{p-k}{p-k} \cdots \frac{1}{2} C_0$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdots \frac{1}{2} C_n, \quad \frac{1}{2} > n \text{ 时}, C_0 = 0$$

$$\Rightarrow w_1(z) = (z-1)^{\rho} \sum_{k=0}^{\infty} \left( \frac{1}{2} \right)^k C_k = (z-1)^{\rho} \cdot \left( \frac{1}{2} \right)^n = \left( \frac{1}{2} \right)^n$$

$$\Rightarrow w_1(z) = A (z-1)^{\rho}$$

$$2 \text{ 个 } p=0, C_0 = -\frac{1}{2} \cdot \frac{2(p-k)}{p-k} C_{-1} \quad n=k \text{ 时}, \text{无意义, 故必有无穷级数}$$

$$\Rightarrow w_1(z) = B (z-1)^{\rho} \int \frac{1}{(z-1)^{\rho}} e^{-\int \frac{2(1-\alpha)}{z-1} dz} dz$$

$$= B (z-1)^{\rho} \int \frac{1}{(z-1)^{\rho}} e^{-(1-\alpha)(z-1)} dz = B (z-1)^{\rho} \int \frac{dz}{(z-1)^{\rho}}$$

多些, 以上所求的级数解法是一种通法, 通常而言无法得到简单的结果.

事实上, 如果最后所得结果是简单的, 我们有望由相信该方程可通过

更简洁的“凑”的方式得到结果.

$$(1-z)\frac{dw}{dz} - 2z\frac{dw}{dz} + 2\alpha w = 0$$

$$\left[ (1-z)\frac{dw}{dz} \right] + 2\alpha \frac{dw}{dz} = 0 \quad \Rightarrow (1-z)\frac{dw}{dz} + 2\alpha w = C$$

$$\text{令 } C=0, \quad \frac{dw}{dz} = -\frac{2\alpha z}{1-z^2} = \frac{n(1+z)}{1-z^2} \quad \Rightarrow \quad w = A (z-1)^{\rho}, \quad \text{利用待定系数法}$$

可得到  $w_0(z)$

2.1 欧拉方程的解

$$\text{例2: 求欧拉方程 } z^2 \frac{dw}{dz} + \frac{dw}{dz} + \frac{1}{z} w = 0$$

先讨论在  $z=0$  附近或无穷远的解, 注意到  $z=0$  为正则奇点.

$$w = \sum_{k=0}^{\infty} C_k z^{k+\rho}$$

$$\frac{dw}{dz} = \sum_{k=0}^{\infty} C_k (k+\rho) z^{k+\rho-1}$$

$$z^2 \frac{dw}{dz} = \sum_{k=0}^{\infty} C_k (k+\rho) z^{k+\rho}$$

$$\frac{dw}{dz} = \sum_{k=0}^{\infty} C_k (k+\rho) z^{k+\rho-1}$$

$$\frac{dw}{dz} = \sum_{k=0}^{\infty} C_k (k+\rho) z^{k+\rho-1}$$

$$\left[ (1-z)\frac{dw}{dz} \right] + 2\alpha \frac{dw}{dz} = 0 \quad \Rightarrow (1-z)\frac{dw}{dz} + 2\alpha w = C$$

$$\sum_{k=0}^{\infty} C_k (k+\rho) z^{k+\rho} + C_0 = 0 \quad \Rightarrow \sum_{k=0}^{\infty} C_k z^{k+\rho} = 0$$

$$\sum_{k=0}^{\infty} C_k (k+\rho) z^{k+\rho} + C_0 = 0 \quad \Rightarrow \sum_{k=0}^{\infty} C_k (k+\rho) z^{k+\rho} + C_0 = 0$$

$$p = 2 \text{ 或 } 0, C_0 = 0, C_0 \text{ 任意}$$

$$(1) p=2, C_0(1+\rho)z^{\rho} + C_1(2+\rho)z^{\rho+1} = 0, \quad C_0 = \frac{-C_1}{(1+\rho)z^{\rho}}$$

$$C_k = \frac{-1}{k!} \cdot \frac{1}{(k+1)(k+2)\cdots(k+\rho)} = \frac{(-1)^k}{k!} \cdot \frac{1}{(k+1)(k+2)\cdots(k+\rho)}$$

$$= \frac{(-1)^k}{k!} \cdot \frac{1}{(k+1)(k+2)\cdots(k+\rho)} \Rightarrow w_1(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \cdot \frac{1}{(k+1)(k+2)\cdots(k+\rho)} z^{k+\rho}$$

$$\Rightarrow J_{\rho}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \cdot \frac{1}{(k+1)(k+2)\cdots(k+\rho)} z^{k+\rho}$$

$$(2) p=0, J_{\rho}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \cdot \frac{1}{(k+1)(k+2)\cdots(k+\rho)} z^{k+\rho}$$

如果  $J_{\rho}(z)$  与  $J_{-\rho}(z)$  线性无关, 则这两个解即为方程的解.

若否, 则方程组还有无穷级数解

注意到, 当  $\rho$  为整数时,  $n \rightarrow \rho$  时,  $C_0$  无意义, 方程有无穷级数.

我们当然仍可以通过代入级数解的方式求解, 但更为对数  $J_{\rho}(z), J_{-\rho}(z)$

级数是线性无关, 这一方法不具有普适性. 下面看另一解法:

$$\text{由前一节可知, } w_1 \frac{dw_0}{dz} - w_0 \frac{dw_1}{dz} = A e^{-\int p(z) dz} = A e^{-\int \frac{1}{z} dz} = \frac{A}{z}$$

$$\text{代入 } J_{\rho}(z), J_{-\rho}(z), \text{ 注意到在 } \rho \text{ 时 } \left[ J_{\rho}(z), J_{-\rho}(z) \right], \text{ 即朗斯基行列式}$$

以上式子中,  $A$  为常数, 即我们所说的朗斯基行列式中除开只有  $z^{-1}$  项

分析可知, 只要  $k=0$  时  $J_{\rho}(z) - J_{-\rho}(z)$  中的  $z^{\rho}$  项产生项

$$A = (z)^{\rho} \frac{1}{\Gamma(\rho+1)} (1)^{\rho} \frac{1}{\Gamma(\rho+1)} - (z)^{-\rho} \frac{1}{\Gamma(-\rho+1)} (1)^{-\rho} \frac{1}{\Gamma(-\rho+1)} = \frac{-2\rho}{\Gamma(\rho+1)\Gamma(-\rho+1)} = \frac{-2}{\Gamma(\rho+1)\Gamma(-\rho+1)}$$

因此我们可看出除开在  $\rho$  为整数时,  $J_{\rho}(z), J_{-\rho}(z)$  是线性无关的

我们现在想通过这两个线性相关的解得到与  $J_n(x)$  线性无关的解。

令  $w_1(x) = C_1 J_{n+1}(x) + G_1 J_{n-1}(x)$

$$\begin{vmatrix} J_n(x) & w_1(x) \\ J_n'(x) & w_1'(x) \end{vmatrix} = C_1 \left( -\frac{1}{x} \right) \sin \pi \nu \frac{1}{x}, \quad \text{则 } C_1 = \frac{1}{\sin \pi \nu} \text{ 保持线性无关。}$$

此时  $w_1(x) = \frac{C J_{n+1}(x) - J_{n-1}(x)}{\sin \pi \nu}$

当  $\nu = n$  时,  $J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( \frac{x}{2} \right)^{2k-n}$   $k = 0, 1, \dots, n-1$  时,  $J_{-n}(x) = 0$

取  $J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( \frac{x}{2} \right)^{2k-n} = \sum_{k=0}^{\infty} \frac{(-1)^{k+n}}{(k+n)!} \left( \frac{x}{2} \right)^{2k+n}$

$= (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( \frac{x}{2} \right)^{2k+n} = (-1)^n J_n(x)$

为了方便, 由 (1) 出发, 应当使  $\nu = n$  时, 令  $w_1$  也为 0, 可取  $C = \cos \pi \nu$

得到  $N_{n+1}(x) = \frac{\cos \pi \nu J_{n+1}(x) - J_{n-1}(x)}{\sin \pi \nu}$ , 称为  $n$  阶诺伊曼函数

为了得到图像表达式, 所以利用洛必达法则:

$$N_n(x) = \lim_{\nu \rightarrow n} \left[ \frac{J_{n+1}(x)}{\sin \pi \nu} - (-1)^n \frac{J_{n-1}(x)}{\sin \pi \nu} \right]$$

$$= \frac{J_n(x)}{\pi} \ln(x) + \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (k+n)!} \psi(k+n+1) \left( \frac{x}{2} \right)^{2k+n}$$

$$- \frac{1}{\pi} (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k \psi(k-n+1)}{k! (k-n)!} \left( \frac{x}{2} \right)^{2k-n}$$

对于最后一项, 令  $p = \frac{(-1)^k \psi(k-n+1)}{k! (k-n)!} \left( \frac{x}{2} \right)^{2k-n}$  为常数处理

$$\sum_{k=0}^{\infty} \frac{(-1)^k \psi(k-n+1)}{k! (k-n)!} \left( \frac{x}{2} \right)^{2k-n}$$

$$= \sum_{k=0}^n \frac{(-1)^k \psi(k-n+1)}{k! (k-n)!} \left( \frac{x}{2} \right)^{2k-n} + \sum_{k=n+1}^{\infty} \frac{(-1)^k \psi(k-n+1)}{k! (k-n)!} \left( \frac{x}{2} \right)^{2k-n}$$

$$J_n(x) J_{-n}(x) = \frac{\pi}{\sin \pi \nu} \quad \psi(1-\nu) = \psi(x) + \pi \cot \pi \nu$$

$$\Rightarrow \frac{J_{n+1}(x)}{J_{n-1}(x)} = \frac{\psi(n+1) + \pi \cot \pi \nu}{\pi} \sin \pi \nu \cdot J_{n+1}(x) + \frac{\psi(n-1)}{\pi} J_{n-1}(x) = J_{n+1}(x) \cot \pi \nu$$
 取  $\nu = n$  代入

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( \frac{x}{2} \right)^{2k-n} J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (k-n)!} \left( \frac{x}{2} \right)^{2k-n}$$

$$= \sum_{k=0}^n \frac{(-1)^k}{k!} \left( \frac{x}{2} \right)^{2k-n} (n-k)! + \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k! (k-n)!} \psi(k-n+1) \left( \frac{x}{2} \right)^{2k-n}$$
 洛必达得到

$$N_n(x) = \frac{1}{\pi} J_n(x) \ln(x) - \sum_{k=0}^n \frac{(-1)^k}{k! (k-n)!} \left( \frac{x}{2} \right)^{2k-n} (n-k)! + \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k! (k-n)!} \psi(k-n+1) \left( \frac{x}{2} \right)^{2k-n}$$

2.2 非正则奇点附近的解

[陈] 计算量虽大, 并发现某些实用价值...

在非正则奇点附近, 方程最多只有一个正则解, 我们现在试图找到这个解及其存在条件。

$$w''(x) + p(x)w'(x) + q(x)w(x) = 0$$

$$p(x) = (1-2x)^{-m} \sum_{k=0}^{\infty} a_k (x-2)^k, \quad q(x) = (1-2x)^{-n} \sum_{k=0}^{\infty} b_k (x-2)^k$$

代入正则解  $w(x) = (1-2x)^{\rho} \sum_{k=0}^{\infty} c_k (x-2)^k$

$$w'(x) = \sum_{k=0}^{\infty} (p+k) c_k (1-2x)^{p+k-1} \quad w''(x) = \sum_{k=0}^{\infty} (p+k)(p+k-1) c_k (1-2x)^{p+k-2}$$

$$\sum_{k=0}^{\infty} (p+k)(p+k-1) c_k (1-2x)^{p+k-2} + \left( \sum_{k=0}^{\infty} (p+k) c_k (1-2x)^{p+k-1} \right) \left( \sum_{k=0}^{\infty} b_k (1-2x)^{k-m} \right)$$

$$+ \sum_{k=0}^{\infty} c_k (1-2x)^{p+k} = 1 \sum_{k=0}^{\infty} b_k (1-2x)^{k-m} = 0$$

为了使解有意义, 最低次不能比 0 低, 去掉该项, 在  $m > n-1$

为了使微分方程为一次方程, 令最低次项系数最大, 取  $m > 2$

对于正则点处,  $w(x) = x^{\rho} \sum_{k=0}^{\infty} c_k x^k$ ,

$$p(x) = \sum_{k=0}^m a_k x^k, \quad q(x) = \sum_{k=0}^n b_k x^k$$
 代入后得到:

$$\sum_{k=0}^{\infty} c_k (k+p)(k+p-1) x^k + \sum_{k=0}^m a_k x^k \sum_{k=0}^{\infty} c_k (k+p) x^k + \sum_{k=0}^n b_k x^k \sum_{k=0}^{\infty} c_k x^k = 0$$

$$\Rightarrow m > 0, \quad m \geq n+1$$
 为存在一个正则解的条件

下面考虑非正则奇点附近的解, 为满足奇异性, 重新将初始值  $w(x) = e^{\frac{\alpha x}{x}}$   $N_{n+1}$

其中  $\alpha$  为  $q(x)$  的奇异性,  $v(x) = (1-2x)^{\rho} \sum_{k=0}^{\infty} c_k (x-2)^k$ , 为洛必达解

代入后得到:  $w' = \alpha' e^{\frac{\alpha x}{x}} v + e^{\frac{\alpha x}{x}} v'$   $w'' = \alpha'' e^{\frac{\alpha x}{x}} v + 2\alpha' \alpha'' e^{\frac{\alpha x}{x}} v' + \alpha'' v''$

$$\Rightarrow v'' + (2\alpha' + \alpha'') v + (\alpha'' v' + \alpha' v) = 0$$

所以令  $p'' = p + 2\alpha'$ ,  $q'' = q + p\alpha'' + \alpha''^2$ , 再按后步骤求解

例题: 求贝塞尔方程在无穷远处的解

$$\frac{d^2 w}{dx^2} + \frac{1}{x} \frac{dw}{dx} + (1 - \frac{\nu^2}{x^2}) w = 0 \quad m=1, \quad n=0$$
 不满足条件,

所以无法直接得到无穷远处的解

对于  $p'', q''$ , 我们假设其存在正则解的条件, 即要求  $m'' > 0, \quad m'' > n''$

则只能取  $q(x) = \lambda x$ , 则  $(1-\lambda^2) = 0 \quad (m'' > n''+1)$  所以  $\lambda = \pm i$

$$\frac{d^2 V}{dx^2} + (1 \pm i) \frac{dV}{dx} + (1 \pm i - \frac{\nu^2}{x^2}) V = 0$$

$$V(x) = x^{\rho} \sum_{k=0}^{\infty} C_k x^k, \quad V'(x) = \sum_{k=0}^{\infty} (p+k) C_k x^{p+k-1}, \quad V''(x) = \sum_{k=0}^{\infty} (p+k)(p+k-1) C_k x^{p+k-2}$$

$$\sum_{k=0}^{\infty} C_k x^{p+k} + (1 \pm i) \sum_{k=0}^{\infty} (p+k) C_k x^{p+k-1} + \sum_{k=0}^{\infty} C_k x^{p+k} (1 \pm i - \frac{\nu^2}{x^2}) = 0$$

$$\Rightarrow p(p+1) = 0 \quad \sum_{k=0}^{\infty} x^{p+k} (C_k (p+k)^2 - \nu^2) + C_{p+1} \cdot \lambda (2(p+1)+1) = 0$$

$$\Rightarrow C_1 (1 - \frac{\nu^2}{x^2} - \nu^2) + \lambda (p+1) (2(p+1)+1) = 0 \quad \Rightarrow C_1 = \frac{(p+k)^2 - \nu^2}{2(p+k)} C_0$$

$$C_k = C_0 \cdot \frac{1}{k!} \cdot \frac{(p+k)^2 - \nu^2}{4} \quad C_k = \frac{(p+k)!}{(k+1)!} C_0 \quad (p+k) = \frac{(p+k)(p+k-1)(p+k-2) \dots (p+1)}{2^k k!}$$

$$\lim_{k \rightarrow \infty} \frac{C_{k+1}}{C_k} = \lim_{k \rightarrow \infty} \frac{(p+k+1)}{k+1} \cdot \frac{1}{k+1} \cdot \frac{1}{4} \cdot \frac{1}{(k+1)^2} = 0 \quad |R|$$
 故幂级数收敛

当  $\nu$  为奇数时,  $C_k$  成为多项式, 取  $C_0 = \sqrt{x} e^{-\frac{1}{2}(1 \pm i)x}$

$$\nu = n+1/2, \quad w(x) = \sqrt{x} e^{\frac{1}{2}(1 \pm i)x} \sum_{k=0}^n \frac{(n-k)!}{(k+1)!} \frac{(n-k)!}{(2k)!} \quad (1 \pm i) \lambda = \pm i$$

$$w_1(x) = (-i)^{n+1} \sqrt{x} e^{-\frac{1}{2}(1-i)x} \sum_{k=0}^n \frac{(n-k)!}{(k+1)!} \frac{(n-k)!}{(2k)!}$$

$$w_2(x) = (i)^{n+1} \sqrt{x} e^{-\frac{1}{2}(1+i)x} \sum_{k=0}^n \frac{(n-k)!}{(k+1)!} \frac{(n-k)!}{(2k)!}$$

如果  $\lambda$  为奇数, 得到的解没有实函数

$$H_{\nu}^{(1)}(z) = \sqrt{\frac{2}{\pi z}} e^{-\frac{1}{2}i(z - \frac{\pi}{4})} \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^k}{(k+1)!}$$

$$H_{\nu}^{(2)}(z) = \sqrt{\frac{2}{\pi z}} e^{-\frac{1}{2}i(z - \frac{\pi}{4})} \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^k}{(k+1)!}$$

至此, 数学物理方法上的内容告一段落, 以上部分我们从知识体系的维度进行了详实的讨论, 重要的是知识链条的自洽性, 但

作为工具, 我们仍应在使用技巧方面进行一些深入练习, 做一个横向的对比整理, 下面我们讨论微分方程与定积分方程两大具体问题

给出一个知识整理汇总。

主题一: 无穷级数求和

首先回顾一个无穷级数, 我们在考虑能否用类似求和的算法或物

理函数的泰勒展开得到, 如果不行, 我们再来考虑如下几种方法。

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和如何能保持得到可靠的结果呢? 我们回想无穷级数不可调换依

据的顺序原因, 在于收敛于求和的数, 从而导致了收敛部分只是无穷

个级数和, 即然如此我们用  $(n+1) \sim 1/n$  来代替, 实际上保证了求和的

个数级数的, 也就相当于  $N$  一样, 或者与长相关的项, 收敛于所

谓的重排, 在保证不改变求和的工和的情况下, 是可以接受的。

当然, 当求和收敛法的运用相当广泛, 我们看两个更具体的应用。

例题: 求  $\sum_{n=0}^{\infty} \frac{(-1)^n}{3n+1}$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3n+1} = \lim_{x \rightarrow 1} \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{3n+1} = \lim_{x \rightarrow 1} \left( \frac{1}{3} \ln(1+x^3) - \frac{1}{3} \ln(1+x) \right)$$

$$= \frac{1}{3} \ln(1) - \frac{1}{3} \ln(1) = 0$$

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$$\begin{aligned} \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} &= -\operatorname{Res}\left(\frac{1}{z^2} \pi \cot \pi z\right) \Big|_{z=0} \quad z=0 \text{ 为孤立奇点,} \\ \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{\pi}{6} \left(1 - \frac{1}{2} + \frac{1}{3} - \dots\right) = \frac{\pi}{6} \cdot \frac{1}{2} \left(1 - \frac{1}{2} + \frac{1}{3} - \dots\right) \left(1 + \frac{1}{2} + \frac{1}{3} - \dots\right) \\ &= \frac{1}{6} \left(1 - \frac{1}{2} + \frac{1}{3} - \dots\right) \quad \text{其中 } -1/2 \text{ 为 } z=0 \text{ 的极点, 故留数为 } -\frac{1}{2}. \\ &\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \end{aligned}$$


这个级数的同处不大, 只能解决很小一部分的级数问题, 且高次项的下展开较为麻烦。

**拉普拉斯变换法:**  $F(s) = \int_0^{\infty} e^{-st} f(t) dt$

$$\sum_{n=1}^{\infty} F(n) = \int_0^{\infty} f(t) \sum_{n=1}^{\infty} e^{-nt} dt = \int_0^{\infty} \frac{f(t) e^{-t}}{1 - e^{-t}} dt$$

**例题:** 求  $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\mathcal{L}\left\{\frac{1}{t^2}\right\} = (-t)^{-1} = -t^{-1}, \quad \mathcal{L}\left\{\frac{1}{t}\right\} = -\mathcal{L}\left\{\frac{1}{t^2}\right\} = \frac{1}{t^2}$$

$$\Rightarrow \int_0^{\infty} \frac{1}{t^2} dt = \frac{1}{t} \Big|_0^{\infty}$$


选取围道如图, 围道积分为  $\oint \frac{1}{z^2} dz$

$$\int_0^{\infty} \frac{1}{t^2} dt + \int_0^{\infty} \frac{1}{(t+i\pi)^2} d(t+i\pi) + \int_{\infty}^0 \frac{1}{(t+i\pi)^2} dt + \int_{\infty}^0 \frac{1}{t^2} dt$$

$$= \frac{1}{t} \Big|_0^{\infty} + \frac{1}{t+i\pi} \Big|_0^{\infty} + \frac{1}{t+i\pi} \Big|_{\infty}^0 = 0$$

**例题:** 求  $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\operatorname{Res}(0) = 0, \quad \operatorname{Res}(i\pi) = \frac{(i\pi)^{-1}}{1} = -i\pi$$

取围道,  $-\int_0^{\infty} \frac{1}{t^2} dt + \int_0^{\infty} \frac{1}{(t+i\pi)^2} d(t+i\pi) = -i\pi$

$$\Rightarrow -\int_0^{\infty} \frac{1}{t^2} dt + \int_0^{\infty} \frac{1}{(t+i\pi)^2} dt = -i\pi$$

$$\Rightarrow \int_0^{\infty} \frac{1}{t^2} dt = \frac{1}{t} \Big|_0^{\infty} = \frac{1}{0} - \frac{1}{\infty} = \infty$$

**例题:** 求  $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\int_0^{\infty} \frac{1}{t^2} dt = \frac{1}{t} \Big|_0^{\infty} = \frac{1}{0} - \frac{1}{\infty} = \infty$$

取围道,  $-\int_0^{\infty} \frac{1}{t^2} dt + \int_0^{\infty} \frac{1}{(t+i\pi)^2} d(t+i\pi) = -i\pi$

$$\Rightarrow \int_0^{\infty} \frac{1}{t^2} dt = \frac{1}{t} \Big|_0^{\infty} = \frac{1}{0} - \frac{1}{\infty} = \infty$$

$\Rightarrow \int_0^{\infty} \frac{1}{t^2} dt = \frac{1}{t} \Big|_0^{\infty} = \frac{1}{0} - \frac{1}{\infty} = \infty$

虽然能解, 但计算量较大, 且难以推广到一般函数, 因此通常采用其他方法。

总结来看, 该函数法可以处理很多函数, 但计算量较大, 且难以推广到一般函数, 因此通常采用其他方法。

**例题:** 求  $\sum_{n=1}^{\infty} \frac{1}{n^2}$

首先最普遍的方法是留数定理, 选取合适围道, 保证无穷远处收敛。

1) 将问题转化为求留数的问题, 但有些情况下留数定理需要选择较复杂的围道, 甚至难以给出结果, 这时候就需要另一些工具。

2) 选取合适的函数, 利用留数定理求解。

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$F(s) = \int_0^{\infty} e^{-st} f(t) dt$

$$\sum_{n=1}^{\infty} F(n) = \int_0^{\infty} f(t) \sum_{n=1}^{\infty} e^{-nt} dt = \int_0^{\infty} \frac{f(t) e^{-t}}{1 - e^{-t}} dt$$

这个例子可以说明留数定理的局限性, 利用其他方法求解。

通常引入一个辅助函数, 利用留数定理求解。

**例题:** 求  $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\mathcal{L}\left\{\frac{1}{t^2}\right\} = (-t)^{-1} = -t^{-1}, \quad \mathcal{L}\left\{\frac{1}{t}\right\} = -\mathcal{L}\left\{\frac{1}{t^2}\right\} = \frac{1}{t^2}$$

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以下习题来自 吴道斌《数学物理方法》

因本书缺少参考答案,在此作为补充.

第1章内容较真不出, 现从第2章开始。

## 第二章 习题

1. 解得各分:  $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$  ,  $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$





$$\begin{aligned} \overline{f(x)} &= \frac{2\pi i}{1-e^{2\pi i}} (5-1) \left(-\frac{1}{4}\right) e^{\frac{2\pi i}{5}} (1-e^{2\pi i}) \\ &= \frac{\pi}{5} (1-1) \frac{e^{\frac{2\pi i}{5}}}{1+e^{2\pi i}} = \frac{\pi}{5} \frac{1-5}{2\pi i(1)} \end{aligned}$$

$$\begin{aligned} (3) \text{ 求原函数 } f &= \frac{x^{1/2} \ln x}{1+x^2} dx = \int_0^{\infty} \frac{x^{1/2} \ln x}{1+x^2} dx + \int_0^{\infty} \frac{x^{1/2} e^{2i\pi \ln x} (\ln x + 2\pi i)}{1+x^2} dx \\ &= (1-e^{2i\pi}) \int_0^{\infty} \frac{x^{1/2} \ln x}{1+x^2} dx - 2\pi i \int_0^{\infty} \frac{x^{1/2}}{1+x^2} dx \\ &= 2\pi i \int_0^{\infty} \frac{x^{1/2}}{1+x^2} dx = 2\pi^2 e^{i\pi/2} \end{aligned}$$

$$\begin{aligned} \int_0^{\infty} \frac{x^{1/2}}{1+x^2} dx &= \frac{1}{1-e^{2i\pi}} \cdot 2\pi i \cdot e^{i\pi/2} = \frac{-2\pi i}{e^{2i\pi} - e^{i\pi/2}} = \frac{\pi}{\sin(\pi/4)} \\ \overline{f(x)} &= \frac{1}{1-e^{2i\pi}} \left( 2\pi i \int_0^{\infty} \frac{x^{1/2}}{1+x^2} dx + \frac{2\pi^2}{e^{i\pi/2}} (1 + \frac{i e^{i\pi/2}}{\sin(\pi/4)}) \right) \\ &= \frac{2\pi^2}{-2i(1-e^{2i\pi})} \left( 1 + \frac{ie^{i\pi/2}}{\sin(\pi/4)} \right) = -\frac{\pi^2 \cos(\pi/4)}{\sin(\pi/4)} \end{aligned}$$

注: 此题原卷有错误, 原为简单题, 已做过类似题。

$$\begin{aligned} (4) \text{ 求原函数 } f &= \frac{(1+iz)^2}{(1+2iz+2z^2)} dz \\ \int_0^{\infty} \frac{(\ln x)^2}{(x^2+1)^2} dx &= \int_0^{\infty} \frac{(\ln(-1+iz))^2}{(x^2+1)^2} dx = 2\pi i \left( \operatorname{res}_{z=i} + \operatorname{res}_{z=-i} \right) \\ &= -4\pi i \left( \int_0^{\infty} \frac{\ln y}{(y^2+1)^2} dy + 4\pi i \int_0^{\infty} \frac{1}{(y^2+1)^2} dy \right) = 2\pi i \left( \frac{(\ln a)^2}{b-a} + \frac{(\ln b)^2}{a-b} \right) \\ &= 2\pi i \left( \frac{(\ln a)^2 - (\ln b)^2}{b-a} + \frac{2\pi i \ln a \ln b}{b-a} \right) \\ \text{题后总结: } f(x) &= \frac{1}{1+bx+ax^2} f(x) = \frac{1}{(1-bx+ax^2)} \end{aligned}$$

$$\begin{aligned} (2) \text{ 同题可求 } f(x) &= f^{(m)}(x) + n_b f^{(m)}(x) + n(n-1) f^{(n-1)}(x) = 0 \quad P(x) = \frac{f^{(m)}}{n!} \\ \Rightarrow P_n + b P_{n-1} - a P_{n-2} &= 0 \end{aligned}$$

$$\begin{aligned} x^2 + bx + a &= 0, \quad x_1 + x_2 = -b, \quad x_1 x_2 = a, \quad x_1^2 + x_2^2 = -2ax \\ \Rightarrow P_n &= A x_1^n + B x_2^n \quad P_{n+1} = A x_1^{n+1} + B x_2^{n+1} \quad P = f'(x) = -b = A x_1 + B x_2 \end{aligned}$$

$$\begin{aligned} P_n^2 &= A^2 x_1^{2n} + B^2 x_2^{2n} + 2AB (A x_1 x_2)^n \\ \frac{d}{dx} P_n^2 &= \frac{A^2}{1-x_1^2} + \frac{B^2}{1-x_2^2} + \frac{2AB}{1-x_1 x_2} \\ &= \frac{A^2(1-x_2^2)(1-x_1^2) + B^2(1-x_1^2)(1-x_2^2) + 2AB(1+x_1 x_2)(1-x_1 x_2)}{(1-x_1^2)(1-x_2^2)(1-x_1 x_2)^2} \end{aligned}$$

$$\begin{aligned} \text{证: } (1-x_1^2)(1-x_2^2) &= (1-x_1^2)(1-x_2^2) = (1-x_1^2)(1-x_2^2) = (1-x_1^2)(1-x_2^2) = (1-x_1^2)(1-x_2^2) \\ &= 1 + (1-a)x_1^2 + (1-b)x_2^2 - a x_1^2 x_2^2 \end{aligned}$$

$$\begin{aligned} \text{证: } x^2 - a^2 &= (x-a)(x+a) = (x-a)(x+a) = (x-a)(x+a) = (x-a)(x+a) \\ &= (x-a)(x+a) = (x-a)(x+a) = (x-a)(x+a) = (x-a)(x+a) \end{aligned}$$

$$\begin{aligned} A &= \frac{-\frac{1}{2}x_2}{x_1-x_2}, \quad B = \frac{\frac{1}{2}x_1}{x_1-x_2} \quad A x_1 + B x_2 = \frac{x_1^2 x_2 - x_2^2 x_1}{x_1 - x_2} = 0 \Rightarrow A^2 x_1^2 + B^2 x_2^2 + 2AB a = 0 \\ A B &= \frac{(1/2)(x_1+x_2)}{(x_1-x_2)^2} = \frac{-a}{2(x_1-x_2)} \end{aligned}$$

$$\begin{aligned} x^2 &= -a + 2AB(4a-b^2) = -a + 2 \cdot \frac{-a}{2(4a-b^2)} \cdot (4a-b^2) = a \\ \text{证: } A^2 x_1^2 + B^2 x_2^2 + 2AB a &= 0 + 2AB a^2 = 0 \end{aligned}$$

$$\begin{aligned} \Rightarrow \sum_{n=0}^{\infty} P_n^2 x^{2n} &= \frac{1+2ax^2}{1+2bx^2+x^4} = \frac{1+2ax^2}{1+2bx^2+x^4} \end{aligned}$$

$$\begin{aligned} (1) \text{ (1)} \quad (2n)! &= (2n) \cdot (2n-1) \cdot (2n-2) \cdots 2 = 2^n \cdot n! \cdot (n-1) \cdot (n-2) \cdots 1 = 2^n \cdot n! \cdot (n-1)! \\ (2) \quad (2n-1)! &= 2^n \cdot (n-1)! \cdot (n-2)! \cdots 1 = 2^n \cdot \frac{(2n)!}{2^n} = \frac{(2n)!}{2^n} \end{aligned}$$

$$\begin{aligned} (3) \quad \overline{f(x)} &= \frac{T_{n+1}(x)}{T_n(x)} \\ (4) \quad n(n-1) - v(v-1) &= (n+v-1)(n-v) \end{aligned}$$

$$\begin{aligned} \overline{f(x)} &= \frac{T_{n+1}(x)}{T_n(x)} = \frac{T_{n+1}(x)}{T_n(x)} = \frac{2x T_n(x) - T_{n+1}(x)}{T_n(x)} = \frac{2x T_n(x) - T_{n+1}(x)}{T_n(x)} \\ (2) \text{ (1)} \quad T_{1/2} &= \int_0^{\infty} e^{-x} \cdot e^{-x} dx = \int_0^{\infty} e^{-2x} dx = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} T_{1/2}(x) &= \int_0^{\infty} e^{-x} \cdot e^{-x} dx = \int_0^{\infty} (x)^{-1/2} \cdot e^{-x} dx \\ &= (1-1/2) \int_0^{\infty} x^{-1/2} \cdot e^{-x} dx = \frac{1}{2} \int_0^{\infty} x^{-1/2} \cdot e^{-x} dx \end{aligned}$$

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$$\begin{aligned} (2) \quad -\frac{3}{2} < 0 < \frac{3}{2}, \quad x &= x e^{i\theta} \end{aligned}$$

$$\begin{aligned} T_1(x) &= \int_0^{\infty} (x e^{i\theta})^{-1/2} \cdot e^{-x e^{i\theta}} dx = e^{-i\theta/2} \int_0^{\infty} x^{-1/2} \cdot e^{-x} dx \\ &= \int_0^{\infty} x^{-1/2} \cdot e^{-x} dx = \frac{1}{2} \int_0^{\infty} x^{-1/2} \cdot e^{-x} dx \end{aligned}$$

$$\begin{aligned} \Rightarrow T_1(x) \cdot e^{-i\theta/2} &= \int_0^{\infty} x^{-1/2} \cdot e^{-x} dx = \frac{1}{2} \int_0^{\infty} x^{-1/2} \cdot e^{-x} dx \\ \int_0^{\infty} x^{-1/2} \cdot e^{-x} dx &= \frac{1}{2} \int_0^{\infty} x^{-1/2} \cdot e^{-x} dx = \frac{1}{2} \int_0^{\infty} x^{-1/2} \cdot e^{-x} dx \end{aligned}$$

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$$\begin{aligned} \ln T_{1/2}(x) &= \ln T_{1/2}(x) + \ln(1+x) + \ln(1+x) = \ln(1+x) \\ \Rightarrow T_{1/2}(x) &= \frac{1}{2} \int_0^{\infty} x^{-1/2} \cdot e^{-x} dx = \frac{1}{2} \int_0^{\infty} x^{-1/2} \cdot e^{-x} dx \end{aligned}$$

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$$\begin{aligned} \ln T_{1/2}(x) &= -\frac{1}{2} \ln(1+x) + \ln(1+x) + \ln(1+x) = \ln(1+x) \\ \Rightarrow T_{1/2}(x) &= \frac{1}{2} \int_0^{\infty} x^{-1/2} \cdot e^{-x} dx = \frac{1}{2} \int_0^{\infty} x^{-1/2} \cdot e^{-x} dx \end{aligned}$$

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$$\begin{aligned} \overline{f(x)} &= \int_0^1 x^{1/2} \cdot e^{-x} dx = \frac{1}{2} \int_0^1 x^{1/2} \cdot e^{-x} dx \\ (2) \quad \int_0^1 x^{1/2} \cdot e^{-x} dx &= \int_0^1 (x^{1/2}) \cdot e^{-x} dx = \int_0^1 (x^{1/2}) \cdot e^{-x} dx \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2} \int_0^1 (x^{1/2}) \cdot e^{-x} dx = \frac{1}{2} \int_0^1 (x^{1/2}) \cdot e^{-x} dx \\ &= \frac{1}{2} \left( \frac{1}{2} \int_0^1 (x^{1/2}) \cdot e^{-x} dx \right) = \frac{1}{4} \int_0^1 (x^{1/2}) \cdot e^{-x} dx \end{aligned}$$

$$\begin{aligned} \int_0^1 x^{1/2} \cdot e^{-x} dx &= \int_0^1 x^{1/2} \cdot e^{-x} dx = \int_0^1 x^{1/2} \cdot e^{-x} dx \\ (3) \text{ (1)} \quad \overline{f(x)} &= \int_0^1 x^{1/2} \cdot e^{-x} dx = \int_0^1 x^{1/2} \cdot e^{-x} dx \end{aligned}$$

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$$\begin{aligned} (3) \quad x^2 &= x^2, \quad y^2 = y^2, \quad z^2 = z^2 \\ x^2 &= x^2, \quad y^2 = y^2, \quad z^2 = z^2 \end{aligned}$$

$$\begin{aligned} x^2 &= x^2, \quad y^2 = y^2, \quad z^2 = z^2 \\ x^2 &= x^2, \quad y^2 = y^2, \quad z^2 = z^2 \end{aligned}$$

$$\begin{aligned} (6) \text{ (1)} \quad \frac{1}{1+2x+1} &= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2} \Rightarrow A = -1, B = \frac{1}{2}, C = \frac{1}{-2} = -\frac{1}{2} \\ \frac{1}{1+2x+1} &= -\frac{1}{x} + \frac{1}{2(x+1)} + \frac{1}{-2(x+2)} \end{aligned}$$

$$\begin{aligned} \Rightarrow \overline{f(x)} &= \sum_{n=0}^{\infty} \left( -\frac{1}{x} + \frac{1}{2(x+1)} + \frac{1}{-2(x+2)} \right) \\ &= \frac{1}{x} \left( 1 - \frac{1}{2} \right) - \frac{1}{2} \left( \frac{1}{x+1} \right) = \frac{1}{2x} \left( 1 - \frac{1}{x+1} \right) = \frac{1}{2x} \left( \frac{x}{x+1} \right) = \frac{1}{2(x+1)} \end{aligned}$$

$$\begin{aligned} (2) \quad T_{1/2}(x) &= \frac{1}{2} \int_0^{\infty} x^{-1/2} \cdot e^{-x} dx = \frac{1}{2} \int_0^{\infty} x^{-1/2} \cdot e^{-x} dx \\ &= \frac{1}{2} \int_0^{\infty} x^{-1/2} \cdot e^{-x} dx = \frac{1}{2} \int_0^{\infty} x^{-1/2} \cdot e^{-x} dx \end{aligned}$$

$$\begin{aligned} B &= \frac{1}{(n+1)^2} \Big|_{n=1} = -\frac{1}{4}, \quad D = \frac{1}{(n+1)^2} \Big|_{n=1} = -\frac{1}{4} \\ \frac{1}{(n+1)^2} &= \frac{A}{n+1} + \frac{B}{(n+1)^2} + \frac{C}{n+2} + \frac{D}{(n+1)^2} \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{n+1} \Big|_{n=1} = \frac{1}{2}, \quad B = \frac{1}{(n+1)^2} \Big|_{n=1} = -\frac{1}{4}, \quad C = \frac{1}{n+2} \Big|_{n=1} = \frac{1}{3} \\ C &= \frac{1}{n+2} \Big|_{n=1} = \frac{1}{3}, \quad D = \frac{1}{(n+1)^2} \Big|_{n=1} = -\frac{1}{4} \end{aligned}$$

$$\begin{aligned} \frac{1}{n+2} &= \frac{1}{n+2} = \frac{1}{n+2} = \frac{1}{n+2} \\ \frac{1}{n+2} &= \frac{1}{n+2} = \frac{1}{n+2} = \frac{1}{n+2} \end{aligned}$$

$$\begin{aligned} \Rightarrow T_{1/2}(x) &= \frac{1}{2} \int_0^{\infty} x^{-1/2} \cdot e^{-x} dx = \frac{1}{2} \int_0^{\infty} x^{-1/2} \cdot e^{-x} dx \\ &= \frac{1}{2} \int_0^{\infty} x^{-1/2} \cdot e^{-x} dx = \frac{1}{2} \int_0^{\infty} x^{-1/2} \cdot e^{-x} dx \end{aligned}$$

$$\begin{aligned} T_{1/2}(x) &= \frac{1}{2} \int_0^{\infty} x^{-1/2} \cdot e^{-x} dx = \frac{1}{2} \int_0^{\infty} x^{-1/2} \cdot e^{-x} dx \\ &= \frac{1}{2} \int_0^{\infty} x^{-1/2} \cdot e^{-x} dx = \frac{1}{2} \int_0^{\infty} x^{-1/2} \cdot e^{-x} dx \end{aligned}$$

$$\begin{aligned} \overline{f(x)} &= \frac{1}{1-e^{2\pi i}} \left( \frac{1}{2} \int_0^{\infty} x^{-1/2} \cdot e^{-x} dx \right) \\ &= \frac{1}{2} \int_0^{\infty} x^{-1/2} \cdot e^{-x} dx = \frac{1}{2} \int_0^{\infty} x^{-1/2} \cdot e^{-x} dx \end{aligned}$$

$$\begin{aligned} (1) \text{ (1)} \quad \int_0^{\infty} e^{-x} \cdot e^{-x} dx &= \int_0^{\infty} e^{-2x} \cdot e^{-x} dx = \int_0^{\infty} e^{-3x} \cdot e^{-x} dx \\ &= \int_0^{\infty} e^{-4x} \cdot e^{-x} dx = \int_0^{\infty} e^{-5x} \cdot e^{-x} dx \end{aligned}$$

$$\begin{aligned} (2) \quad \int_0^{\infty} e^{-x} \cdot e^{-x} dx &= \int_0^{\infty} e^{-2x} \cdot e^{-x} dx = \int_0^{\infty} e^{-3x} \cdot e^{-x} dx \\ &= \int_0^{\infty} e^{-4x} \cdot e^{-x} dx = \int_0^{\infty} e^{-5x} \cdot e^{-x} dx \end{aligned}$$

$$\begin{aligned} (3) \quad \int_0^{\infty} e^{-x} \cdot e^{-x} dx &= \int_0^{\infty} e^{-2x} \cdot e^{-x} dx = \int_0^{\infty} e^{-3x} \cdot e^{-x} dx \\ &= \int_0^{\infty} e^{-4x} \cdot e^{-x} dx = \int_0^{\infty} e^{-5x} \cdot e^{-x} dx \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2} \int_0^{\infty} e^{-x} \cdot e^{-x} dx = \frac{1}{2} \int_0^{\infty} e^{-2x} \cdot e^{-x} dx = \frac{1}{2} \int_0^{\infty} e^{-3x} \cdot e^{-x} dx \\ &= \frac{1}{2} \int_0^{\infty} e^{-4x} \cdot e^{-x} dx = \frac{1}{2} \int_0^{\infty} e^{-5x} \cdot e^{-x} dx \end{aligned}$$

$$\begin{aligned} (4) \quad \int_0^{\infty} e^{-x} \cdot e^{-x} dx &= \int_0^{\infty} e^{-2x} \cdot e^{-x} dx = \int_0^{\infty} e^{-3x} \cdot e^{-x} dx \\ &= \int_0^{\infty} e^{-4x} \cdot e^{-x} dx = \int_0^{\infty} e^{-5x} \cdot e^{-x} dx \end{aligned}$$

$$\begin{aligned} &= \int_0^{\infty} e^{-4x} \cdot e^{-x} dx = \int_0^{\infty} e^{-5x} \cdot e^{-x} dx \\ &= \int_0^{\infty} e^{-4x} \cdot e^{-x} dx = \int_0^{\infty} e^{-5x} \cdot e^{-x} dx \end{aligned}$$

$$\begin{aligned} \int_0^A \ln \sqrt{1+x^2} dx &= \frac{1}{2} \int_0^A \ln(1+x^2) dx = \frac{1}{2} \int_0^A \ln(1+x^2) dx \\ &= \frac{1}{2} \int_0^A \ln(1+x^2) dx = \frac{1}{2} \int_0^A \ln(1+x^2) dx \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2} \int_0^A \ln(1+x^2) dx = \frac{1}{2} \int_0^A \ln(1+x^2) dx = \frac{1}{2} \int_0^A \ln(1+x^2) dx \\ &= \frac{1}{2} \int_0^A \ln(1+x^2) dx = \frac{1}{2} \int_0^A \ln(1+x^2) dx \end{aligned}$$

$$\begin{aligned} \int_0^A \ln \sqrt{1+x^2} dx &= \int_0^A \ln \sqrt{1+x^2} dx = \int_0^A \ln \sqrt{1+x^2} dx \\ &= \int_0^A \ln \sqrt{1+x^2} dx = \int_0^A \ln \sqrt{1+x^2} dx \end{aligned}$$

$$\begin{aligned} &= \int_0^A \ln \sqrt{1+x^2} dx = \int_0^A \ln \sqrt{1+x^2} dx = \int_0^A \ln \sqrt{1+x^2} dx \\ &= \int_0^A \ln \sqrt{1+x^2} dx = \int_0^A \ln \sqrt{1+x^2} dx \end{aligned}$$

$$\begin{aligned} \Rightarrow \int_0^A \ln \sqrt{1+x^2} dx &= \int_0^A \ln \sqrt{1+x^2} dx = \int_0^A \ln \sqrt{1+x^2} dx \\ &= \int_0^A \ln \sqrt{1+x^2} dx = \int_0^A \ln \sqrt{1+x^2} dx \end{aligned}$$

$$\begin{aligned} \overline{f(x)} &= \lim_{n \rightarrow \infty} \int_0^1 \ln \sqrt{1+x^2} dx = \lim_{n \rightarrow \infty} \left( \frac{1}{2} \int_0^1 \ln(1+x^2) dx \right) \\ &= \lim_{n \rightarrow \infty} \left( \frac{1}{2} \int_0^1 \ln(1+x^2) dx \right) = \lim_{n \rightarrow \infty} \left( \frac{1}{2} \int_0^1 \ln(1+x^2) dx \right) \end{aligned}$$

$$\begin{aligned} (2) \quad T_{1/2}(x) &= \int_0^{\infty} e^{-x} \cdot e^{-x} dx = \int_0^{\infty} e^{-2x} \cdot e^{-x} dx = \int_0^{\infty} e^{-3x} \cdot e^{-x} dx \\ &= \int_0^{\infty} e^{-4x} \cdot e^{-x} dx = \int_0^{\infty} e^{-5x} \cdot e^{-x} dx \end{aligned}$$

$$\begin{aligned} &= \int_0^{\infty} e^{-4x} \cdot e^{-x} dx = \int_0^{\infty} e^{-5x} \cdot e^{-x} dx = \int_0^{\infty} e^{-5x} \cdot e^{-x} dx \\ &= \int_0^{\infty} e^{-4x} \cdot e^{-x} dx = \int_0^{\infty} e^{-5x} \cdot e^{-x} dx \end{aligned}$$

$$\begin{aligned} (3) \text{ (1)} \quad \overline{f(x)} &= \frac{1}{1-e^{2\pi i}} \int_0^{\infty} e^{-x} \cdot e^{-x} dx = \frac{1}{1-e^{2\pi i}} \int_0^{\infty} e^{-2x} \cdot e^{-x} dx \\ &= \frac{1}{1-e^{2\pi i}} \int_0^{\infty} e^{-3x} \cdot e^{-x} dx = \frac{1}{1-e^{2\pi i}} \int_0^{\infty} e^{-4x} \cdot e^{-x} dx \end{aligned}$$

$$\begin{aligned} &= \frac{1}{1-e^{2\pi i}} \int_0^{\infty} e^{-4x} \cdot e^{-x} dx = \frac{1}{1-e^{2\pi i}} \int_0^{\infty} e^{-5x} \cdot e^{-x} dx \\ &= \frac{1}{1-e^{2\pi i}} \int_0^{\infty} e^{-4x} \cdot e^{-x} dx = \frac{1}{1-e^{2\pi i}} \int_0^{\infty} e^{-5x} \cdot e^{-x} dx \end{aligned}$$

$$\begin{aligned} (2) \quad T_{1/2}(x) &= \int_0^{\infty} e^{-x} \cdot e^{-x} dx = \int_0^{\infty} e^{-2x} \cdot e^{-x} dx = \int_0^{\infty} e^{-3x} \cdot e^{-x} dx \\ &= \int_0^{\infty} e^{-4x} \cdot e^{-x} dx = \int_0^{\infty} e^{-5x} \cdot e^{-x} dx \end{aligned}$$

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$$\begin{aligned} (4) \text{ (1)} \quad \frac{1}{1+2x+1} &= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2} + \frac{D}{x+1} \\ A &= -1, \quad B = \frac{1}{2}, \quad C = \frac{1}{-2} = -\frac{1}{2}, \quad D = -\frac{1}{2} \end{aligned}$$

$$\begin{aligned} B &= \frac{1}{(n+1)^2} \Big|_{n=1} = -\frac{1}{4}, \quad D = \frac{1}{(n+1)^2} \Big|_{n=1} = -\frac{1}{4} \\ \frac{1}{(n+1)^2} &= \frac{A}{n+1} + \frac{B}{(n+1)^2} + \frac{C}{n+2} + \frac{D}{(n+1)^2} \end{aligned}$$

$$\begin{aligned} \int_0^{\infty} e^{-x} \cdot e^{-x} dx &= \int_0^{\infty} e^{-2x} \cdot e^{-x} dx = \int_0^{\infty} e^{-3x} \cdot e^{-x} dx \\ &= \int_0^{\infty} e^{-4x} \cdot e^{-x} dx = \int_0^{\infty} e^{-5x} \cdot e^{-x} dx \end{aligned}$$

$$\begin{aligned} \Rightarrow \int_0^{\infty} e^{-x} \cdot e^{-x} dx &= \int_0^{\infty} e^{-2x} \cdot e^{-x} dx = \int_0^{\infty} e^{-3x} \cdot e^{-x} dx \\ &= \int_0^{\infty} e^{-4x} \cdot e^{-x} dx = \int_0^{\infty} e^{-5x} \cdot e^{-x} dx \end{aligned}$$

$$\begin{aligned} (3) \quad \frac{1}{1+2x+1} &= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2} + \frac{D}{x+1} \\ A &= -1, \quad B = \frac{1}{2}, \quad C = \frac{1}{-2} = -\frac{1}{2}, \quad D = -\frac{1}{2} \end{aligned}$$

$$\mathcal{L}^{-1}\left\{\frac{W}{p^2(p^2+1)}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{p}\cdot\frac{1}{p^2+1}-\frac{p}{p^2+1}\right\}=\frac{1}{2}\left(1-\cos t\right)\eta(t)$$

$$(3) \frac{4p-1}{(p^2+1)(p^2-1)}=\frac{A}{p-1}+\frac{B}{p+1}+\frac{C}{p^2+1}+\frac{D}{p^2-1}$$

$$A=\frac{(-1)(-1)}{(-1)(-1)}=1, \quad B=\frac{-1-1}{(-1)(-1)}=-\frac{2}{1}$$

$$C=\frac{4\cdot\frac{1}{2}-1}{\frac{1}{2}\cdot\frac{1}{2}-1\cdot1}=\frac{1}{-\frac{3}{2}}=-\frac{2}{3}, \quad D=\frac{4(-\frac{1}{2})-1}{(-\frac{1}{2}\cdot\frac{1}{2}-1)\cdot(-1)}=\frac{-3}{-\frac{3}{2}\cdot(-1)}=-6$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{4p-1}{(p^2+1)(p^2-1)}\right\}=\left(1+\frac{2}{3}e^{-t}+\frac{1}{3}e^{\frac{t}{2}}-\frac{2}{3}e^{\frac{t}{2}}\right)\eta(t)$$

$$(4) \frac{p^2\omega^2}{(p^2+\omega^2)^2}=\frac{A}{p+\omega}+\frac{B}{p-\omega}+\frac{C}{(p+\omega)^2}+\frac{D}{(p-\omega)^2}=\frac{p^2+\omega^2}{(p+\omega)(p-\omega)^2}$$

$$C=\frac{p^2\omega^2}{(p+\omega)^2}\Big|_{p=-\omega}=\frac{2\omega^2}{(2\omega)^2}=\frac{1}{2}, \quad D=\frac{2\omega^2}{(2\omega)^2}=\frac{1}{2}$$

$$A=\frac{1}{2p}\left(\frac{p^2\omega^2}{(p+\omega)^2}\right)\Big|_{p=-\omega}=\frac{2p\omega^2-(p^2+\omega^2)(2\omega^2)}{(p+\omega)^3}\Big|_{p=-\omega}=\frac{-2\omega(2\omega^2)-2\omega^2\cdot2}{(2\omega)^3}=0$$

$$\text{事实上, 由洛必达} \quad \frac{p^2\omega^2}{(p+\omega)^2}\Big|_{p=-\omega}=\frac{1}{(p+\omega)}+\frac{1}{(p+\omega)^2}$$

$$\mathcal{L}^{-1}\left\{\frac{p^2\omega^2}{(p^2+\omega^2)^2}\right\}=\mathcal{L}^{-1}\left\{-\frac{1}{(p+\omega)}\right\}=(-t)e^{-\omega t}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{p}\left(\frac{p^2\omega^2}{(p^2+\omega^2)^2}\right)\right\}=\mathcal{L}^{-1}\left\{-\frac{1}{(p+\omega)}\right\}=(-t)e^{-\omega t}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{p^2\omega^2}{(p^2+\omega^2)^2}\right\}=\frac{1}{2}t(e^{-\omega t}+e^{-\omega t})\eta(t)=te^{-\omega t}\eta(t) \quad [\text{此处洛必达法则}]$$

$$(5) \mathcal{L}^{-1}\left\{\frac{1}{p^2}\right\}=f(t-\tau)\eta(t-\tau) \quad \mathcal{L}^{-1}\left\{\frac{1}{p}\right\}=\mathcal{L}^{-1}\left\{-\frac{1}{p}\right\}=t$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{p^2}\right\}=(-t-\tau)\eta(t-\tau)$$

$$(6) \frac{1}{p}=\frac{e^{-\omega t}}{1-e^{-\omega t}}=\frac{1}{p}\sum_{n=0}^{\infty}e^{-n\omega t}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{p}\sum_{n=0}^{\infty}e^{-n\omega t}\right\}=\sum_{n=0}^{\infty}\eta(t-n\omega)=\left(\frac{t}{\omega}\right)\eta\left(\frac{t}{\omega}\right)$$

$$(7) \text{E}_0\sin\omega t-L\frac{dI}{dt}-\frac{Q}{C}=0 \quad Q=\int_0^t i(t)\omega dt$$

$$\mathcal{L}\{i(t)\}=I(p), \quad \mathcal{L}\{E_0\sin\omega t\}=E_0\cdot\frac{\omega}{p^2+\omega^2}$$

$$\mathcal{L}\left\{\frac{dI}{dt}\right\}=pI(p)-i(0)=pI(p) \quad \mathcal{L}\{Q\}=\frac{I(p)}{p}$$

$$\Rightarrow \frac{E_0\omega}{p^2+\omega^2}-LpI(p)-\frac{I(p)}{p}=0$$

$$\Rightarrow I(p)=\frac{1}{Lp}\cdot\frac{E_0\omega}{p^2+\omega^2}=\frac{pC}{1+p^2LC}\cdot\frac{E_0\omega}{p^2+\omega^2}=\frac{1}{L}\cdot\frac{p}{p^2+\omega^2}\cdot\frac{E_0\omega}{p^2+\omega^2}$$

$$\text{令 } \frac{1}{LC}=\omega_0^2, \quad \frac{p}{p^2+\omega_0^2}=\frac{1}{p}\cdot\frac{1}{p^2+\omega_0^2}=\frac{1}{2\omega_0}\left(\frac{p}{p^2+\omega_0^2}-\frac{p}{p^2+\omega_0^2}\right) \quad (\omega\neq\omega_0)$$

$$\mathcal{L}^{-1}\left\{\frac{E_0\omega}{p}\cdot\frac{p}{(p^2+\omega_0^2)^2}\right\}=\frac{E_0\omega}{LC}\cdot\frac{\omega_0\omega t-\omega_0 t}{\omega^2-\omega_0^2} \quad (\omega\neq\omega_0)$$

$$\frac{p}{(p^2+\omega_0^2)^2}=\frac{1}{2p}\left(\frac{1}{p^2+\omega_0^2}\right)-\frac{1}{2\omega_0^2} \quad \mathcal{L}^{-1}\left\{\frac{p}{(p^2+\omega_0^2)^2}\right\}=\left(-\frac{1}{2}\right)(-t)\cdot\frac{1}{\omega_0}\sin\omega_0 t=\frac{t}{2\omega_0}\sin\omega_0 t$$

$$\Rightarrow i(t)=\begin{cases} \frac{E_0\omega}{LC}\cdot\frac{\omega_0\omega t-\omega_0 t}{\omega^2-\omega_0^2} & (\omega\neq\omega_0) \\ \frac{E_0}{2LC}\cdot t\sin\omega t & (\omega=\omega_0) \end{cases}$$

$$(2) \quad E-L\frac{dI}{dt}-\frac{Q}{C}=0 \quad Q=\int_0^t i(t)\omega dt$$

$$Q=\int_0^t i(t)\omega dt=\int_0^t (-i)\omega dt \quad \Rightarrow \dot{Q}RC=\int_0^t (-i)\omega dt$$

$$\Rightarrow \dot{Q}RC=\frac{I(p)\cdot\frac{1}{p}}{1+p^2LC} \quad \Rightarrow \dot{Q}=\frac{1}{1+p^2LC}I$$

$$\Rightarrow \frac{E}{p}-LpI-\frac{Q}{1+p^2LC}=0$$

$$\Rightarrow I=\frac{E}{p}\cdot\frac{1}{Lp+\frac{Q}{1+p^2LC}}=\frac{E}{p}\cdot\frac{1+p^2LC}{pL+p^2LC+Q}=\frac{E}{p}\cdot\frac{p^2+\frac{1}{LC}}{p^2+\frac{1}{LC}+L}$$

$$\text{令 } \beta=\frac{1}{2pLC}, \quad \delta=\frac{1}{LC} \quad \Rightarrow I(p)=\frac{E}{p}\cdot\frac{p^2+\beta}{p^2+\beta+\delta} \quad \Delta=p^2-\beta^2$$

$$(1) \quad \alpha>0, \quad \beta, \delta, \omega_1, \omega_2=\beta\pm\sqrt{\beta^2+\delta^2}$$

$$I(p)=\frac{E}{p}\cdot\left(\frac{A}{p-\omega_1}+\frac{B}{p-\omega_2}+\frac{C}{p-\omega_3}\right)=\frac{E}{p}\cdot\frac{p^2+\beta}{p(p-\omega_1)(p-\omega_2)}$$

$$A=\frac{\beta}{\omega_1^2}, \quad B=\frac{-\beta}{\omega_1(\omega_1-\omega_2)}=-\frac{\omega_2}{\omega_1(\omega_1-\omega_2)}, \quad C=\frac{\omega_1\omega_2\beta}{\omega_1\omega_2(\omega_1-\omega_2)}=-\frac{\omega_1}{\omega_1\omega_2(\omega_1-\omega_2)}$$

$$i(t)=\frac{E}{p}\cdot\left(\frac{\beta}{p-\omega_1}-\frac{\omega_2}{\omega_1(\omega_1-\omega_2)}e^{\omega_1 t}-\frac{\omega_1}{\omega_1\omega_2(\omega_1-\omega_2)}e^{\omega_2 t}\right)$$

$$(2) \quad \alpha=0, \quad I_p=\frac{E}{p}\cdot\frac{p^2+\beta}{p(p^2+\beta)}=\frac{E}{p}\cdot\left(\frac{1}{p}+\frac{\beta}{p^2+\beta}\right)=\frac{E}{p^2}+\frac{E\beta}{p(p^2+\beta)}$$

$$A=\frac{E}{p^2}, \quad C=-1, \quad \Rightarrow \frac{1}{p^2}=\frac{1}{p}\cdot\frac{1}{p} \quad \Rightarrow I_p=\frac{E}{p}\cdot\frac{p^2+\beta}{p(p^2+\beta)}=\frac{E}{p^2}+\frac{E\beta}{p(p^2+\beta)} \quad i(t)=\frac{E}{p^2}(1-e^{-\beta t})-\frac{E}{p}e^{-\beta t}$$

$$(3) \quad \alpha<0, \quad \omega_1, \omega_2=-\beta\pm\sqrt{\beta^2+\delta^2} \quad \Rightarrow \beta\pm i\omega_0$$

$$i(t)=\frac{E}{p}\cdot\left(\frac{\beta}{p-\omega_1}-\frac{\omega_2}{\omega_1(\omega_1-\omega_2)}e^{\omega_1 t}-\frac{\omega_1}{\omega_1\omega_2(\omega_1-\omega_2)}e^{\omega_2 t}\right)$$

$$=\frac{E}{p}\cdot\left(\frac{\beta}{p-\omega_1}-\frac{\omega_2}{\omega_1(\omega_1-\omega_2)}\cdot\frac{e^{i\omega_1 t}}{2i\omega_0}+\frac{1}{p}+\frac{1}{p}\cdot\frac{e^{i\omega_2 t}}{2i\omega_0}\right)$$

$$=\frac{E}{p}\cdot\left(\frac{\beta}{p-\omega_1}-\frac{e^{i\omega_1 t}}{2i\omega_0}\left(\frac{1}{p-\omega_1}+\frac{1}{p-\omega_2}\right)e^{i\omega_1 t}+\frac{1}{p}+\frac{1}{p}\cdot\frac{e^{i\omega_2 t}}{2i\omega_0}\left(\frac{1}{p-\omega_1}+\frac{1}{p-\omega_2}\right)e^{i\omega_2 t}\right)$$

$$=\frac{E}{p}\cdot\left(\frac{\beta}{p-\omega_1}-\frac{e^{i\omega_1 t}}{2i\omega_0}\left(\frac{1}{p-\omega_1}+\frac{1}{p-\omega_2}\right)e^{i\omega_1 t}-(\beta^2+\omega_0^2-2\beta i\omega_0)e^{i\omega_1 t}\right)$$

$$=\frac{E}{p}\cdot\left(\frac{\beta}{p-\omega_1}-\frac{e^{i\omega_1 t}}{2i\omega_0}\left(\frac{1}{p-\omega_1}+\frac{1}{p-\omega_2}\right)e^{i\omega_1 t}+2\beta i\omega_0 e^{i\omega_1 t}\right)$$

$$=\frac{E}{p}\cdot\left(\frac{\beta}{p-\omega_1}-\frac{e^{i\omega_1 t}}{2i\omega_0}\left(\frac{1}{p-\omega_1}+\frac{1}{p-\omega_2}\right)e^{i\omega_1 t}+2\beta i\omega_0 e^{i\omega_1 t}\right)$$

$$(3) \quad \mathcal{L}\{y(t)\}=Y(p) \quad \mathcal{L}\{A\sin t\}=A\cdot\frac{1}{p^2+1}$$

$$\mathcal{L}\left\{\int_0^t y(t-\tau)\sin(t-\tau)d\tau\right\}=Y(p)\cdot\frac{1}{p^2+1}$$

$$\Rightarrow Y(p)=\frac{A-2pY(p)}{p^2+1} \quad \Rightarrow Y(p)=\frac{A}{(p^2+1)^2}=A\cdot\frac{1}{(p^2+1)^2}$$

$$\Rightarrow \mathcal{L}^{-1}\{Y(p)\}=y(t)=At e^{-t}$$

$$(4) \quad \mathcal{L}\{f(t)\}=F(p) \quad \mathcal{L}\left\{\int_0^t f(t-\tau)\sin(t-\tau)d\tau\right\}=F(p)\cdot\frac{1}{p^2+1}$$

$$\mathcal{L}\{g\cdot e^{at}\}=g\cdot\frac{1}{p-a} \quad F(p)=\frac{(p+1)^2}{p^2+1}=\frac{g}{p^2+1}$$

$$\Rightarrow F(p)=\frac{g(p+1)}{(p^2+1)^2}=g\cdot\left(\frac{A}{p^2+1}+\frac{B}{p+1}+\frac{C}{(p+1)^2}\right)$$

$$A=\frac{4+1}{(0+1)^2}=\frac{5}{1} \quad B=\frac{g(p+1)}{(p^2+1)^2}\Big|_{p=-1}=\frac{g(p+1)-(p^2+1)(p+1)}{(p+1)^3}\Big|_{p=-1}=\frac{-2(1+1)-2}{(-1)^3}=\frac{4}{1}$$

$$C=\frac{4+1}{(-1+1)^2}=-\frac{5}{2} \quad F(p)=\frac{5}{p^2+1}+\frac{4}{p+1}-\frac{5}{(p+1)^2}$$

$$f(t)=\mathcal{L}^{-1}\{F(p)\}=5e^{-t}+4e^{-t}-6te^{-t}$$

$$(6) \quad \mathcal{L}\left\{\frac{e^{ax}-e^{bx}}{x}\cdot\cos cx\right\}=\int_0^{\infty} F(q)dq$$

$$F(p)=\mathcal{L}\{e^{ax}-e^{bx}\cos cx\}=\mathcal{L}\{(e^{ax}-e^{bx})\cdot\frac{e^{icx}-e^{-icx}}{2i}\}$$

$$=\frac{1}{2}\left(\frac{1}{p-a+ic}+\frac{1}{p-a-ic}-\frac{1}{p-b+ic}-\frac{1}{p-b-ic}\right)=\frac{p+a}{(p-a+ic)^2}-\frac{p+b}{(p-b+ic)^2}$$

$$\int_0^{\infty} \frac{p+a}{(p-a+ic)^2}-\frac{p+b}{(p-b+ic)^2}d\omega=\frac{1}{2}\ln\left|\frac{(p-a+ic)^2}{(p-b+ic)^2}\right|$$

$$t\geq p=0, \quad \Rightarrow \ln\left|\frac{p-a+ic}{p-b+ic}\right|$$

$$(2) \quad \mathcal{L}\left\{\frac{1-\cos x}{x^2}\right\}=\int_0^{\infty} \frac{1-\cos x}{x^2}dx=\int_0^{\infty} F(q)dq$$

$$F(q)=\mathcal{L}\left\{\frac{1-\cos x}{x}\right\}=\int_0^{\infty} G(q)dq$$

$$G(q)=\mathcal{L}\{1-\cos x\}=\frac{1}{q}-\frac{q}{q^2+1}$$

$$F(q)=\lim_{\omega\rightarrow\infty}\left(\ln\frac{\omega}{q}-\ln\frac{\omega^2+1}{q^2+1}\right)=\ln\frac{\omega^2}{q^2}$$

$$\ln\omega=\int_0^{\infty} \left(\frac{1}{x}\ln(x^2+1)-\ln x\right)dx \quad [\text{见例 5.10 (3)}]$$

$$\text{令 } x=p=0, \quad \Rightarrow \int_0^{\infty} \frac{1-\cos x}{x^2}dx=\frac{\pi}{2}$$

$$(3) \quad \ln x=\int_0^{\infty} F(p)dp$$

$$\mathcal{L}\left\{\frac{\sin x}{x(x^2+1)}\right\}=\frac{1}{x(x^2+1)}\cdot\frac{x}{p^2+x^2}=\frac{1}{(x^2+1)}\cdot\frac{1}{(p^2+x^2)}$$

$$=\left(-\frac{1}{x^2+1}-\frac{1}{x^2+1}\right)\cdot\frac{1}{p^2+1}$$

$$\int_0^{\infty} \mathcal{L}\left\{\frac{\sin x}{x(x^2+1)}\right\}dx=\frac{1}{p^2+1}\left(\frac{\pi}{2}-\frac{\pi}{2}\right)$$

$$=\frac{\pi}{2(p^2+1)}=\frac{\pi}{2}\left(\frac{1}{p^2+1}\right)$$

$$\mathcal{L}^{-1}\left\{\frac{\pi}{2}\left(\frac{1}{p^2+1}\right)\right\}=\frac{\pi}{2}(1-e^{-t}) \quad (t>0)$$

$$\text{由 } \mathcal{L}^{-1}\text{ 关于 } t \text{ 是奇函数, 故 } \int_0^{\infty} \frac{\sin x}{x(x^2+1)}dx=\frac{\pi}{2}(1-e^{-1})\operatorname{sgn}(1)$$

$$I. \quad (1) \quad f(t)=\frac{1}{2\pi i}\int_{C-\infty}^{C+\infty} e^{pt}\frac{p}{p^2+\omega^2}dp=\operatorname{res}(1)+\operatorname{res}(-\omega)$$

$$=\frac{\omega e^{i\omega t}}{2\omega}+\frac{-\omega e^{-i\omega t}}{2\omega}=\cos(\omega t) \quad \text{更详细地说, } f(t)=\cos(\omega t)\eta(t)$$

$$(2) \quad \mathcal{L}\{f(t)\}=\frac{1}{2\pi i}\int_{C-\infty}^{C+\infty} e^{pt}\frac{1}{p^2+\omega^2}dp$$

$$\text{取 } C, \quad p=\omega e^{\frac{2\pi i}{3}}, \quad \omega e^{\frac{4\pi i}{3}}$$

$$\operatorname{res}(f(p))=\frac{e^{pt}}{4p^3}\cdot\frac{1}{\frac{d}{dp}(p^2+\omega^2)}=\frac{e^{pt}}{8\omega^2 p^2}\cdot\frac{1}{2p}=\frac{e^{pt}}{16\omega^2 p^3} \quad \omega=\omega_0, \quad \frac{2\pi i}{3}$$

$$0=2\pi i, \quad \operatorname{res}(f(\omega_0))\operatorname{res}(f(-\omega_0))=\frac{e^{i\omega_0 t}}{8\omega_0^2}\cdot\frac{1}{2\omega_0}=\frac{e^{i\omega_0 t}}{16\omega_0^3} \quad \omega=\omega_0, \quad \frac{2\pi i}{3}$$

$$\theta=\frac{2\pi i}{3}, \quad \operatorname{res}(f(\omega_0))+\operatorname{res}(f(-\omega_0))=\frac{e^{i\omega_0 t}}{8\omega_0^2}\cdot\frac{1}{2\omega_0}=\frac{e^{i\omega_0 t}}{16\omega_0^3} \quad (\cos\omega_0 t+\sin\omega_0 t)$$

$$\Rightarrow f(t)=\frac{1}{4\omega_0^3}(\cos\omega_0 t+\sin\omega_0 t)$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{e^{pt}}{p^2+\omega^2}\right\}=\frac{1}{\omega^2}(\sin\omega t-\tau_1\sin\omega(t-\tau_1)-\sin\omega(t-\tau_1)\sin\omega(t-\tau_1))\eta(t-\tau_1)$$

$$(3) \quad \text{类似, } f(t)=\frac{1}{2\pi i}\int_{C-\infty}^{C+\infty} e^{pt}\frac{1}{p}dp=1 \quad \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{p}\right\}=\eta(t)$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{p}e^{-\omega p}\right\}=\eta(t-\omega)$$

$$(4) \quad f(t)=\frac{1}{2\pi i}\int_{C-\infty}^{C+\infty} \frac{e^{pt}}{p}\cdot\frac{1}{p^2+\omega^2}dp$$

$$\operatorname{res}(0)=1, \quad \operatorname{res}(1/p)=0, \quad \Rightarrow \int_0^{\infty} \frac{1}{p^2+\omega^2}dp=\frac{1}{\omega^2}\cdot\frac{\pi}{2}$$

$$\operatorname{res}(p)=\frac{1}{p}\cdot\frac{1}{\sin(\frac{\pi}{2})}\cdot\frac{1}{\cos(\frac{\pi}{2})}\cdot\frac{1}{\sin(\frac{\pi}{2})}\cdot\frac{1}{\cos(\frac{\pi}{2})}=\frac{1}{\omega^2}\cdot\frac{\pi}{2}$$

$$=\frac{2\omega(\frac{1}{\omega^2}\cdot\frac{\pi}{2})}{(\frac{1}{\omega^2}\cdot\frac{\pi}{2})}=\frac{2\sin(\frac{\pi}{2})}{-\frac{1}{\omega^2}\pi\cos(\frac{\pi}{2})}=\frac{2}{-\frac{1}{\omega^2}\pi\cos(\frac{\pi}{2})}$$

$$=-\frac{4}{\omega^2\pi}\cos(\frac{\pi}{2})\cdot\frac{1}{\omega^2}\cdot\frac{\pi}{2}e^{-\frac{\pi}{\omega^2}\cdot\frac{1}{\omega^2}\cdot\frac{\pi}{2}}$$

$$\Rightarrow \ln x=\left(1-\frac{4}{\omega^2}\cdot\frac{\pi}{2}\right)\cos(\frac{\pi}{2})\cdot\frac{1}{\omega^2}\cdot\frac{\pi}{2}e^{-\frac{\pi}{\omega^2}\cdot\frac{1}{\omega^2}\cdot\frac{\pi}{2}}\eta(t)$$

$$8. \quad \text{取 } \mathcal{L}\{f(t)\}=\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\cdot\frac{1}{n!}$$

$$\text{注意到 } \int_0^{\infty} x^ne^{-x}dx=\frac{1}{n!}, \quad (-1)^n\int_0^{\infty} x^ne^{-x}dx=\frac{(-1)^n}{n!}$$

$$\sum_{n=0}^{\infty} (-1)^n\int_0^{\infty} x^ne^{-x}dx=\int_0^{\infty} x^ne^{-x}dx=\int_0^{\infty} \frac{x^ne^{-x}}{n!}dx=\int_0^{\infty} \frac{x^ne^{-x}}{n!}dx=\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

$$(1) \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}=\int_0^{\infty} \frac{x^ne^{-x}}{n!}dx=\int_0^{\infty} \frac{dx}{x^2+1}=\int_0^{\infty} \left(\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{x^2+1}\right)dx$$

$$y_1=e^{\frac{1}{2}t}, \quad y_2=e^{-\frac{1}{2}t}, \quad A=\frac{1}{2}, \quad B=\frac{1}{2}, \quad C=\frac{1}{2}$$

$$I=\frac{1}{2}\ln 2+\frac{1}{2}e^{-\frac{1}{2}t}\ln\left|\frac{1-e^{-\frac{1}{2}t}}{1+e^{-\frac{1}{2}t}}\right|+\frac{1}{2}e^{\frac{1}{2}t}\ln\left|\frac{1+e^{\frac{1}{2}t}}{1-e^{\frac{1}{2}t}}\right|$$

$$=\frac{1}{2}\ln 2+\frac{1}{2}\ln\left|\frac{1-e^{-\frac{1}{2}t}}{1+e^{-\frac{1}{2}t}}\right|+\frac{1}{2}\ln\left|\frac{1+e^{\frac{1}{2}t}}{1-e^{\frac{1}{2}t}}\right|$$

$$=\frac{1}{2}\ln 2+\frac{1}{2}\ln\left|\frac{1-e^{-\frac{1}{2}t}}{1+e^{-\frac{1}{2}t}}\right|+\frac{1}{2}\ln\left|\frac{1+e^{\frac{1}{2}t}}{1-e^{\frac{1}{2}t}}\right|$$

$$=\frac{1}{2}\ln 2+\frac{1}{2}\ln\left|\frac{1-e^{-\frac{1}{2}t}}{1+e^{-\frac{1}{2}t}}\right|+\frac{1}{2}\ln\left|\frac{1+e^{\frac{1}{2}t}}{1-e^{\frac{1}{2}t}}\right|$$

$$(2) \quad a=1, \quad b=4 \quad \ln x=\int_0^{\infty} \frac{dx}{x^2+1}=\int_0^{\infty} \frac{A}{x-1}+\frac{B}{x+2}+\frac{C}{x-2}+\frac{D}{x^2+4}dx$$

$$x_1, x_2=e^{\pm\frac{1}{2}t}, \quad x_1, x_2=e^{\pm\frac{1}{2}t}, \quad A, B=\frac{1}{4}, \quad C, D=\frac{1}{4}$$

$$e^{-\frac{1}{2}t}\ln\left|\frac{1-e^{-\frac{1}{2}t}}{1+e^{-\frac{1}{2}t}}\right|+e^{\frac{1}{2}t}\ln\left|\frac{1+e^{\frac{1}{2}t}}{1-e^{\frac{1}{2}t}}\right|=e^{-\frac{1}{2}t}\ln\left|\frac{1+e^{\frac{1}{2}t}}{1-e^{\frac{1}{2}t}}\right|-e^{-\frac{1}{2}t}\ln\left|\frac{1+e^{\frac{1}{2}t}}{1-e^{\frac{1}{2}t}}\right|$$

$$=\frac{3}{2}t+\frac{1}{2}\ln\left|\frac{1+e^{\frac{1}{2}t}}{1-e^{\frac{1}{2}t}}\right|+\frac{1}{2}\ln\left|\frac{1+e^{\frac{1}{2}t}}{1-e^{\frac{1}{2}t}}\right|=\frac{3}{2}t+\frac{1}{2}\ln\left|\frac{1+e^{\frac{1}{2}t}}{1-e^{\frac{1}{2}t}}\right|$$

$$e^{-\frac{1}{2}t}\ln\left|\frac{1+e^{\frac{1}{2}t}}{1-e^{\frac{1}{2}t}}\right|+e^{\frac{1}{2}t}\ln\left|\frac{1+e^{\frac{1}{2}t}}{1-e^{\frac{1}{2}t}}\right|=\frac{1}{2}t+\frac{1}{2}\ln\left|\frac{1+e^{\frac{1}{2}t}}{1-e^{\frac{1}{2}t}}\right|$$

$$\text{类似可得, } \frac{1}{2}t+\frac{1}{2}\ln\left|\frac{1+e^{\frac{1}{2}t}}{1-e^{\frac{1}{2}t}}\right|+\frac{1}{2}t+\frac{1}{2}\ln\left|\frac{1+e^{\frac{1}{2}t}}{1-e^{\frac{1}{2}t}}\right|=\frac{1}{2}t+\frac{1}{2}\ln\left|\frac{1+e^{\frac{1}{2}t}}{1-e^{\frac{1}{2}t}}\right|$$

$$\frac{1-e^{\frac{1}{2}t}}{1-e^{-\frac{1}{2}t}}=\frac{1+t}{2-2t}=\frac{1}{2}\cdot\frac{1+t}{1-t}=\frac{1}{2}\cdot\frac{1+t}{1-t}$$

$$\ln x=\frac{1}{2}\left(\frac{1}{x-1}+\$$



$$\Rightarrow A_1 A_2 = 0, \quad A_1 A_2 (2k+1)^{-n} A_2 k = -\left(\frac{A_1}{2k}\right)^{2k+1} \cdot \left(\frac{A_2}{(2k+1)^2}\right)$$

$$A_1 k \cdot (2k)^2 - n^2 A_2 k = -\left(\frac{A_1}{2k}\right)^{2k} \cdot \frac{A_2 k}{(2k)^2}$$

$$A_2 k = \frac{A_1}{2k^2} \cdot A_2 k = \left(\frac{A_1}{2k}\right)^{2k} \cdot \frac{1}{(2k)^2}$$

$$= \left(\frac{A_1}{2k}\right)^{2k} \cdot \frac{1}{(2k)^2} + \frac{A_1^{2k}}{4k^{2k}} \cdot \left(\frac{A_1^{2k}}{4k^{2k}} - A_2 k - 4 - \left(\frac{A_1}{2k}\right)^{2k-2} \cdot \frac{1}{4k^{2k-2}}\right)$$

$$= \left(\frac{A_1}{2k}\right)^{2k} \cdot \frac{1}{(2k)^2} \cdot \left(\frac{1}{2} + \frac{1}{2k}\right) + \frac{A_1^{2k}}{4k^{2k}} \cdot A_2 k$$

$$\Rightarrow A_2 k = -\left(\frac{A_1}{2k}\right)^{2k} \cdot \frac{1}{(2k)^2} \cdot \left(\frac{1}{2} + \frac{1}{2k}\right) + A_0 \cdot \left(\frac{A_1}{2k}\right)^{2k} \cdot \frac{1}{(2k)^2}$$

后设即为后条-前条相同的部分,可求得

$$\text{则 } W_1(t) = \sum_{k=0}^{\infty} \left(\frac{A_1}{2k}\right)^{2k} \cdot \frac{1}{(2k)^2} \cdot \left[\frac{1}{2} \cdot \sum_{i=0}^k \left(\frac{A_1}{2i}\right)^{2i} \cdot \left(\frac{1}{2} + \frac{1}{2i} + \dots + 1\right)\right] \cdot 2^k$$

$$W_1(t) = \sum_{k=0}^{\infty} \left(\frac{A_1}{2k}\right)^{2k} \cdot \frac{1}{(2k)^2}$$

第10章习题

$$1. (1) \int_{-\infty}^{\infty} f(x) \delta(x) dx = f(0)$$

$$\int_{-\infty}^{\infty} f(x) \delta(x-1) dx = \int_{-\infty}^{\infty} f(x) \delta(x) \delta(x-1) dx = \int_{-\infty}^{\infty} f(x) \delta(x) dx \cdot f(0) \Rightarrow \delta(x) = \delta(x-1)$$

$$(2) \int_{-\infty}^{\infty} f(x) \cdot x \delta(x) dx = f(0) \cdot x \Big|_{x=0} = 0, \quad \Rightarrow x \delta(x) = 0$$

$$(3) \int_{-\infty}^{\infty} f(x) g(x) \delta(x) dx = f(0) g(0) \Big|_{x=0} = f(0) g(0) = \int_{-\infty}^{\infty} f(x) g(x) \delta(x) dx$$

$$= g(0) f(0) \Rightarrow g(x) \delta(x) = g(0) \delta(x)$$

$$(4) \int_{-\infty}^{\infty} f(x) \cdot x \delta(x) dx = \int_{-\infty}^{\infty} f(x) \cdot x \delta(x) dx = - \int_{-\infty}^{\infty} \delta(x) dx \cdot f(x)$$

$$= - \int_{-\infty}^{\infty} \delta(x) f(x) dx = - \int_{-\infty}^{\infty} \delta(x) \cdot x f'(x) dx = -f(0) - (x f(x)) \Big|_{x=0} = -f(0)$$

$$\Rightarrow x \delta(x) = -\delta(x)$$

$$(5) \int_{-\infty}^{\infty} f(x) \delta(x) dx = \int_{-\infty}^{\infty} f\left(\frac{x}{2}\right) \delta\left(\frac{x}{2}\right) dx = \frac{1}{2} \int_{-\infty}^{\infty} f\left(\frac{x}{2}\right) \delta(x) dx$$

$$= \frac{1}{2} f(0) \Rightarrow \delta(x) = \frac{1}{2} \delta\left(\frac{x}{2}\right)$$

$$(6) \int_{-\infty}^{\infty} f(x) g(x) \delta(x) dx = \int_{-\infty}^{\infty} f(x) g(x) \delta(x) dx = \int_{-\infty}^{\infty} \delta(x) dx \cdot f(x) g(x)$$

$$= - \int_{-\infty}^{\infty} \delta(x) f(x) g(x) dx = - \int_{-\infty}^{\infty} \delta(x) g(x) f(x) dx = -f'(0) g(0) - g'(0) \int_{-\infty}^{\infty} f(x) \delta(x) dx$$

$$-f'(0) = \int_{-\infty}^{\infty} f(x) \delta(x) dx = \int_{-\infty}^{\infty} f(x) \delta(x) dx = - \int_{-\infty}^{\infty} \delta(x) f'(x) dx$$

$$\Rightarrow \int_{-\infty}^{\infty} g(x) \delta(x) f(x) dx = g(0) \int_{-\infty}^{\infty} f(x) \delta(x) dx - g'(0) \int_{-\infty}^{\infty} f(x) \delta(x) dx$$

$$\Rightarrow g(x) \delta(x) = g(0) \delta(x) - g'(0) \delta(x)$$

$$2. (1) g(x) = A e^{kx} + B e^{-kx} \quad (x > 0)$$

$$g(x) = C e^{kx} + D e^{-kx} \quad (x > 0)$$

$$f(x) g(x) = 0, \quad \frac{dg(x)}{dx} \Big|_{x=0} = 0 \Rightarrow A = B = 0$$

$$2. \quad x < 0 \text{ 时, } g(x) = g_1(x), \quad \frac{dg_1(x)}{dx} = \frac{dg_2(x)}{dx} \Big|_{x=0} = 0$$

$$\Rightarrow C e^{kx} + D e^{-kx} = 0 \quad k C e^{kx} + D e^{-kx} = 1$$

$$\Rightarrow C = \frac{1}{2k} e^{-kx}, \quad D = -\frac{1}{2k} e^{kx}$$

$$g_1(x) = \frac{1}{2k} (e^{-kx} - e^{kx}) = -\frac{1}{k} \sinh(kx)$$

$$\Rightarrow g_1(x) = -\frac{1}{k} \sinh(kx) \quad (x < 0)$$

$$(2) \quad x < 0 \text{ 或 } x > 0 \text{ 时, } \frac{d}{dx} W = -W = 0, \quad \text{该方程为齐次方程,可设 } W = e^{\lambda x}$$

$$\text{代入方程,得特征方程为 } W_1(x), W_2(x)$$

$$W'' - W = 0 \Rightarrow W_1 = e^x, W_2 = e^{-x} \Rightarrow W_1 W_2 = W_1 W_2 = 0$$

$$W_1(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{1}{k!} \cdot x^k \quad W_2(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{1}{k!} \cdot x^k$$

$$\text{不取 } W_1, W_2 = W_1 W_2 = 1 \quad (\text{该方程为齐次方程,可设 } W = e^{\lambda x})$$

$$g_1 = 0, \quad g_2 = A W_1(x) + B W_2(x)$$

$$\text{因 } A W_1(x) + B W_2(x) = 0, \quad A W_1'(x) + B W_2'(x) = 1$$

$$B \left( \frac{W_1(x)}{W_2(x)} \right) W_1(x) + B W_2(x) = 1 \Rightarrow B = \frac{W_1(x)}{W_1(x)W_2(x) - W_2(x)W_1(x)}$$

$$\Rightarrow g = \left( W_1(x) W_2(x) - W_2(x) W_1(x) \right) \cdot g_1(x)$$

$$(2) \quad \text{若解为 } g(x) = \frac{1}{2k} W + 2 \left( \frac{1}{2k} \right) \frac{d}{dx} W + 2 W = 0$$

$$\text{该方程的解法为 } \frac{1}{2k} W + 2 \left( \frac{1}{2k} \right) \frac{d}{dx} W + 2 W = 0 \Rightarrow W_1(x) = \frac{1}{1+x^2}, \quad W_2(x) = \frac{x}{1+x^2}$$

$$g_1 = 0, \quad g_2 = \frac{A+Bx}{1+x^2} \quad \text{因 } g_2 \Big|_{x=0} = 0, \quad \frac{dg_2}{dx} \Big|_{x=0} = \frac{1}{1+x^2}$$

$$\text{因 } \frac{A+Bx}{1+x^2} = 0, \quad \frac{1}{1+x^2} = \frac{1}{1+x^2}$$

$$\Rightarrow A+Bx = 0, \quad B = 1, \quad A = -1$$

$$\Rightarrow g_2 = \frac{1}{1+x^2} \quad (x > 0)$$

$$3. \quad \text{若解为 } g(x) = Y(x), \quad g(x) \Rightarrow p(Y(x) - Y(x)) = p(Y(x) - A)$$

$$Y''(x) = p(p(Y(x) - A) - Y(x)) = p^2(Y(x) - A) - B$$

$$p^2 Y - pA - B + k^2 Y = F \Rightarrow Y = \frac{F}{p^2 k^2} + \frac{pA+B}{p^2 k^2}$$

$$\frac{d}{dx} \left[ F \right] = f \quad \frac{d}{dx} \left[ \frac{F}{p^2 k^2} \right] = \frac{1}{k^2} \sin kx \quad \frac{d}{dx} \left[ \frac{pA+B}{p^2 k^2} \right] = \cos kx$$

$$\Rightarrow y = A \cos kx + \frac{1}{k^2} \sin kx + \frac{1}{k^2} \int_0^x \sin k(x-\tau) f(\tau) d\tau$$

$$\text{若解为 } g(x) = \frac{d}{dx} g_1(x) + k^2 g_2(x) = \delta(x-x)$$

$$\Rightarrow g_1(x) = \frac{1}{k} \sin k(x-\tau) \quad g_2(x) = \delta(x-x)$$

$$\text{初值条件为 } g_1(x) \Big|_{x=0} = 0, \quad \frac{dg_1(x)}{dx} \Big|_{x=0} = 0$$

$$g_1(x) \Big|_{x=0} = 0, \quad \frac{dg_1(x)}{dx} \Big|_{x=0} = 0, \quad \frac{dg_2(x)}{dx} \Big|_{x=0} = 1$$

$$\frac{d}{dx} g_1(x) + k^2 g_2(x) = \delta(x-x) \Rightarrow \frac{d}{dx} g_1(x) + k^2 g_2(x) = f(x)$$

$$\frac{d}{dx} \left( y(x) \frac{dg_1(x)}{dx} - g_1(x) \frac{dy(x)}{dx} \right) = y(x) \delta(x-x) - f(x) g_2(x)$$

$$2 \text{ 若 } f(x) \text{ 有, } y(x) \frac{dg_1(x)}{dx} - g_1(x) \frac{dy(x)}{dx} \Big|_{x=0} = y(x) - \int_0^x f(x) g_2(x) d\tau$$

$$\frac{d}{dx} \left( y(x) \frac{dg_1(x)}{dx} - g_1(x) \frac{dy(x)}{dx} \right) \Big|_{x=0} = y(x) \frac{dg_1(x)}{dx} \Big|_{x=0} - g_1(x) \frac{dy(x)}{dx} \Big|_{x=0} = 0$$

$$g_1(x) \frac{dg_2(x)}{dx} \Big|_{x=0} = 0 \quad y(x) \frac{dg_2(x)}{dx} \Big|_{x=0} = \frac{k}{2} \sin k(x-x) g_2(x) \Big|_{x=0} = -\cos kx A$$

$$g_2(x) \frac{dg_1(x)}{dx} \Big|_{x=0} = \frac{B}{k} \sin kx$$

$$\Rightarrow g_2(x) = A \cos kx + \frac{B}{k^2} \sin kx + \int_0^x f(x) \frac{1}{k} \sin k(x-\tau) d\tau$$

$$\text{若初值条件为 } y(x) = C_1(x) \sin kx + C_2(x) \cos kx$$

$$y(x) = C_1(x) \sin kx + k C_2(x) \cos kx + C_3(x) \sin kx$$

$$\text{令 } C_1(x) \sin kx + k C_2(x) \cos kx = 0, \quad \Rightarrow y_1' = k C_1(x) \cos kx - k C_2(x) \sin kx$$

$$y_1' = k C_1(x) \cos kx - k C_2(x) \sin kx - k C_3(x) \cos kx$$

$$\Rightarrow k C_1(x) \cos kx - k C_2(x) \sin kx = f(x)$$

$$C_1' = \frac{f(x)}{k} \cos kx \quad C_2' = -\frac{f(x)}{k} \sin kx$$

$$C_1 = \frac{1}{k} \int_0^x f(x) \cos kx d\tau + C \quad C_2 = -\frac{1}{k} \int_0^x f(x) \sin kx d\tau + D$$

$$y = C \sin kx + D \cos kx + \frac{1}{k} \int_0^x f(x) \sin k(x-\tau) d\tau$$

$$y \Big|_{x=0} = D = A \quad y' \Big|_{x=0} = k C = B$$

$$\Rightarrow y = A \cos kx + \frac{B}{k^2} \sin kx + \frac{1}{k} \int_0^x f(x) \sin k(x-\tau) d\tau$$

$$A_1(x) = \frac{d}{dx} g_1(x) - k' g_2(x) = \delta(x-x)$$

$$g_1(x) \Big|_{x=0} = 0, \quad \frac{dg_1(x)}{dx} \Big|_{x=0} = 0, \quad g_2(x) \Big|_{x=0} = 0, \quad \frac{dg_2(x)}{dx} \Big|_{x=0} = 1$$

$$g_1(x) = g_1(x) \sin k(x-x) - \frac{1}{k}$$

$$\frac{dg_1(x)}{dx} - k' g_2(x) = f(x) \quad \frac{dg_2(x)}{dx} - k' g_1(x) = \delta(x-x)$$

$$\frac{d}{dx} \left( g_1(x) \frac{dg_2(x)}{dx} - g_2(x) \frac{dg_1(x)}{dx} \right) = f(x) g_2(x) - y(x) \delta(x-x)$$

$$2 \text{ 若 } f(x) \text{ 有, } g_1(x) \frac{dg_2(x)}{dx} - g_2(x) \frac{dg_1(x)}{dx} \Big|_{x=0} = \int_0^x f(x) g_2(x) d\tau - y(x)$$

$$g_1(x) \frac{dg_2(x)}{dx} \Big|_{x=0} = 0 \quad y_1(x) \frac{dg_2(x)}{dx} \Big|_{x=0} = y_1(x) \frac{dg_2(x)}{dx} \Big|_{x=0} = 0$$

$$g_2(x) \frac{dg_1(x)}{dx} \Big|_{x=0} = \frac{B}{k} \sin kx \quad y_1(x) \frac{dg_1(x)}{dx} \Big|_{x=0} = -A \sin kx$$

$$\Rightarrow y_1(x) = A \cos kx + \frac{B}{k^2} \sin kx + \frac{1}{k} \int_0^x f(x) \sin k(x-\tau) d\tau$$

$$(2) \quad \text{若解 } 2. (2), \quad g = (W_1(x) W_2(x) - W_2(x) W_1(x)) \cdot g_1(x) = D_1(x) \cdot g_1(x)$$

$$g_1(x) \frac{dg_2(x)}{dx} - g_2(x) \frac{dg_1(x)}{dx} \Big|_{x=0} = \int_0^x f(x) g_2(x) d\tau - y(x)$$

$$y(x) = \int_0^x f(x) g_2(x) d\tau + (g_1(x) B - A \frac{dg_1(x)}{dx}) \Big|_{x=0}$$

$$\Rightarrow g_2(x) \Big|_{x=0} = W_1(x) W_2(x) - W_2(x) W_1(x) = D_1(x) \cdot g_1(x)$$

$$\frac{d}{dx} g_2(x) \Big|_{x=0} = W_1'(x) W_2(x) - W_2'(x) W_1(x) = D_2(x) \cdot g_1(x)$$

$$\text{令 } D_1(x) = \frac{W_1(x) W_2(x)}{W_1(x) W_2(x)} \quad D_2(x) = \frac{W_1'(x) W_2'(x)}{W_1'(x) W_2'(x)}$$

$$\Rightarrow y_1(x) = A D_2(x) - B D_1(x) + \int_0^x f(x) D_1(x) d\tau$$

$$k_1(x) \cdot g_1' = A \sin kx \quad g_1' = B \sin k(x-x)$$

$$A \sin kx = B \sin k(x-x) \quad k B \sin k(x-x) - k A \sin kx = 0$$

$$B \sin kx \sin k(x-x) = \frac{1}{k} \sin kx \quad B \sin k(x-x) \sin kx$$

$$\Rightarrow B = \frac{1}{k} \frac{\sin kx}{\sin kx} \quad A = \frac{1}{k} \frac{\sin k(x-x)}{\sin kx}$$

$$g_1(x) = \frac{\sin k(x)}{k \sin kx} \sin kx + \frac{\sin k(x) \sin k(x-x) - \sin k(x-x) \sin kx}{k \sin kx} \cdot g_1(x)$$

$$\text{后设 } \sin kx \sin k(x-x) \sin kx - \sin kx \sin k(x-x) \sin kx = \sin kx \sin k(x-x)$$

$$+ \sin kx \sin kx \sin kx = \sin kx \sin k(x-x)$$

$$\Rightarrow g_1(x) = \frac{\sin k(x-x)}{k \sin kx} \sin kx + \sin kx \sin k(x-x) \cdot g_1(x)$$

$$(2) \quad g_1' = A W_1(x) + B W_2(x) \quad g_1' = C W_1(x) + D W_2(x)$$

$$A W_1(x) + B W_2(x) = C W_1(x) + D W_2(x) \quad C W_1(x) + D W_2(x) - A W_1(x) - B W_2(x) = 0$$

$$A W_1(x) + B W_2(x) = 0 \quad C W_1(x) + D W_2(x) = 0$$

$$C W_1(x) W_2(x) - W_2(x) W_1(x) + A W_1(x) W_2(x) + B W_2(x) W_1(x) = A W_1(x) W_2(x) + B W_2(x) W_1(x) - W_2(x) W_1(x)$$

$$\Rightarrow C = A - W_2(x) \quad D = B + W_1(x) \quad \Rightarrow \begin{cases} A W_1(x) + B W_2(x) = W_1(x) W_2(x) \\ A W_1(x) + B W_2(x) = 0 \end{cases}$$

$$A W_1(x) - W_1(x) W_2(x) + B W_2(x) + W_1(x) W_2(x) = 0$$

$$A = W_2(x) \quad \frac{W_1(x) W_2(x) - W_2(x) W_1(x)}{W_1(x) W_2(x) - W_2(x) W_1(x)} = W_2(x) \quad \frac{D_1(x)}{D_1(x)} \quad B = -W_1(x) \quad \frac{D_2(x)}{D_2(x)}$$

$$g = \frac{D_1(x)}{D_1(x)} D_1(x) + g_1(x) D_2(x) \quad \text{若 } g_1(x) = 0, \text{ 则 } g = \frac{D_1(x)}{D_1(x)}$$

$$g = \frac{W_2(x)}{W_2(x)} D_1(x) + D_2(x) \cdot g_1(x)$$

$$(3) \quad g_1 = \frac{A+Bx}{1+x^2}, \quad g_2 = \frac{C+Dx}{1+x^2} \quad A=0, \quad \frac{C+Dx}{1+x^2} = 0 \quad A+Bx = C+Dx$$

$$\left( \frac{D_1(x) g_2(x) - (C+Dx) g_1(x)}{(1+x^2)^2} \right) - \left( \frac{D_2(x) g_1(x) - (A+Bx) g_2(x)}{(1+x^2)^2} \right) \cdot (1+x^2) = 1$$

$$\Rightarrow \begin{cases} D=0 \\ B=C+D \end{cases} \Rightarrow \begin{cases} A=0 \\ C=1 \end{cases} \Rightarrow g = \frac{1}{1+x^2} \cdot \frac{x}{1+x^2} + \frac{x}{1+x^2} \cdot g_1(x)$$

$$b. \quad y(x) = Y(x) \quad \frac{d}{dx} y(x) = p(Y(x) - y(x)) = p(Y(x) - 0) = p(Y(x))$$

$$\frac{d}{dx} y(x) = p(p(Y(x) - y(x)) - p(Y(x) - y(x)) \quad \frac{d}{dx} y(x) = e^{-p}$$

$$p'Y - y'(x) - k'Y = e^{-p}$$

$$\Rightarrow Y = \frac{y'(x)}{p-k} + \frac{e^{-p}}{p-k} = \left( \frac{1}{p-k} - \frac{1}{p-k} \right) \frac{y'(x)}{2k} + \left( \frac{1}{p-k} - \frac{1}{p-k} \right) \frac{e^{-p}}{2k}$$

$$y = \frac{1}{2k} \sin kx + \frac{1}{k} \sin k(x-x) \cdot g_1(x)$$

$$x \rightarrow \infty, \quad \left( \frac{1}{k} \sin k(x-x) - y'(x) \sin kx \right) = 0 \quad y(x) = \sin k(x-x) \sin kx = e^{-kx}$$

$$\Rightarrow g_1(x) = -\frac{1}{k} e^{-kx} \sin kx = -\frac{1}{k} \sin k(x-x) \cdot g_1(x)$$

$$T. (1) \quad g_1' = A e^{kx} - B e^{-kx} \quad g_1' = C e^{kx} - D e^{-kx}$$

$$0 = k(A-B), \quad C=0 \quad A e^{kx} + B e^{-kx} = D e^{kx}$$

$$-k D e^{-kx} - (k A e^{kx} - k B e^{-kx}) = 1 \quad B e^{-kx} - A e^{kx} = \frac{1}{k} + A e^{kx} + B e^{-kx}$$

$$\Rightarrow A = -\frac{1}{2k} e^{-kx} \quad B = -\frac{1}{2k} e^{kx} \quad D = -\frac{1}{2k} e^{-kx} - \frac{1}{2k} e^{kx}$$

$$g = -\frac{1}{2k} e^{-kx} \sin kx + \frac{1}{2k} \sin k(x-x) \cdot g_1(x)$$

$$(2) \quad g_1' = A \sin k(x-x) \quad g_1' = D \cos k(x-x)$$

$$A \sin k(x-x) = 0 = D \cos k(x-x) \quad -k D \sin k(x-x) - k A \cos k(x-x) = 1$$

$$\frac{1}{k} \sin k(x-x) + D \sin k(x-x) \sin k(x-x) + D \cos k(x-x) \cos k(x-x) = 0$$

$$D \cos k(x-x) - \frac{1}{k} \sin k(x-x) \quad D = -\frac{1}{k} \frac{\sin k(x-x)}{\sin k(x-x)} \quad A = -\frac{1}{k} \frac{\cos k(x-x)}{\cos k(x-x)}$$

$$g_1(x) = -\frac{1}{k} \frac{\cos k(x-x)}{\cos k(x-x)} \cos k(x-x) + \frac{1}{k \cos k(x-x)} (\sin k(x-x) \cos k(x-x) - \cos k(x-x) \sin k(x-x)) \cdot g_1(x)$$

$$\text{若 } g_1(x) = \frac{1}{k} \sin k(x-x) \sin k(x-x) + \cos k(x-x) \sin k(x-x) \quad \frac{1}{k} (\sin k(x-x) \sin k(x-x) - \sin k(x-x) \sin k(x-x))$$

$$C_2 \sin k\{x\} \cos k\{y-1\} = \frac{1}{k} \sin k\{x\} + C_2 \sin k\{x\} \cos k\{y\}$$

$$\Rightarrow C_3 \sin k = \left\{ \frac{1}{k} \sin k \right\} \Rightarrow C_3 = \frac{\sin k}{k \sin k} \quad , \quad C_1 = \frac{\sin(k-1)}{k \sin k}$$

$$g(x) = \frac{\sin k(x-1)}{k \sin c} \sin kx + \frac{1}{k} \sin c(x-1) \eta(x-1)$$

$$\frac{d^2 y(\cdot)}{d\tau^2} + k^2 y(\cdot) = f(\cdot) \qquad \frac{d^2 g(x, \cdot)}{d\tau^2} + k^2 g(x, \cdot) = \delta(x - \cdot)$$

$$\frac{d}{dx} \left[ g(x) \cdot \frac{dy}{dx} - y(x) \frac{d}{dx} g(x) \right] = f(x) g(x) - y(x) \delta(x - \xi)$$

$$\text{2) 7.2.6. } \left( g(x) \frac{dy(x)}{dx} - y(x) \frac{d}{dx} g(x) \right) \Big|_0^1 = \int_0^1 f(x, g(x), y(x)) dx - y(x)$$

$$\Rightarrow y(x) = \int_0^1 f(\xi) g(x;\xi) d\xi + y(1) \frac{d}{d\xi} g(x;\xi) \Big|_{\xi=1} - y(0) \frac{d}{d\xi} g(x;\xi) \Big|_{\xi=0}$$

$$\xi=0: g(x; \xi) = \frac{\sin kx}{k \sin k} \sin k(x-1) \quad \left. \frac{d}{d\xi} g(x; \xi) \right|_{\xi=0} = \frac{1}{\sin k} \sin k(x-1)$$

$$y = 1, g(x; y) = \frac{\sin(x-1)}{\sin k} \sin kx \quad \left. \frac{d}{dx} g(x; y) \right|_{y=1} = \frac{\cos(x-1)}{\sin k} \sin kx = \frac{1}{\sin k} \sin kx$$

$$y(x) = \int_0^1 f(\xi) \left( \frac{\sin k(x-\xi)}{k \sin k} \sin kx + \frac{1}{k} \sin k(x-\xi) \eta(x-\xi) \right) d\xi + B \frac{1}{\sin k} \sin kx - A \frac{1}{\sin k} \sin k(x-1)$$

$$= B \frac{\sin kx}{\sin k} - A \frac{\sin k(x-1)}{\sin k} + \frac{\sin kx}{k \sin k} \left( \int_0^1 f(\xi) \sin k(\xi-1) d\xi + \frac{1}{k} \int_0^x f(\xi) \sin k(x-\xi) d\xi \right)$$

(2) 旧章习题 5.11, 临界点数为:

(2) 由 (1) 可得  $\sin(x-k\pi) = (-1)^k \sin x$ ，于是有：

$$g_1(x, y) = \frac{\sin k(x-y)-1}{k \sin k} \sin kx + \frac{1}{k} \sin k(x-y) y'(x-y) = \begin{cases} \frac{\sin k(x-y)-1}{k \sin k} \sin kx & x < y \\ \frac{-\sin k(x-y)}{k \sin k} \sin k(x-y) & x > y \end{cases}$$

$$\left. \frac{d}{dz} g(x; z) \right|_{z=1} = \frac{\sinh kx}{\sinh k} \quad \left. \frac{d}{dz} g(x; z) \right|_{z=0} = \frac{\sinh k(x-1)}{\sinh k}$$

$$y(x) = B \frac{\sinh kx}{\sinh k} - A \frac{\sinh k(x-1)}{\sinh k} + \frac{\sinh kx}{k \sinh k} \int_0^1 f(\xi) \sinh k(\xi-1) d\xi + \frac{1}{k} \int_1^x f(\xi) \sinh k(x-\xi) d\xi$$

(3) 由本章习题 5.12, 转移函数为:

$$g = \frac{w_2(x)}{w_2(1)} D_L(1; \gamma) + D_L(\gamma; \pi) \gamma(X - \gamma) = \begin{cases} \frac{w_2(x)}{w_2(1)} D_L(1; \gamma) & x < \gamma \\ \frac{w_2(x)}{w_2(1)} D_L(1; \gamma) + D_L(\gamma; \pi) & x > \gamma \end{cases}$$

$$\left. \frac{dg}{dz} \right|_{z=1} = \frac{w_2(x)}{w_2(1)} \quad \left. \frac{dg}{dz} \right|_{z=0} = \frac{w_2(0)}{w_2(1)} D_1(1;x)$$

$$\Rightarrow y(x) = B \frac{W(x)}{W(x_1)} + A \frac{1}{W(x)} D_1(x) + \frac{W(x)}{W(x_1)} \int_0^1 f_1(z) D_1(z) dz + \int_0^x f_2(z) D_2(z) dz$$