

事实上,通常我们使用将 G 无限逼近为无穷小的圆盘,我们由此得到了圆

析函数在圆盘上的展开式.

我们将圆内展开的主要部分和无穷远处称为(解析部分),它在圆盘 G 内部

保持收敛的;将无穷远处称为主要部分,它在圆盘 G 外是绝对收敛的.

问题:证明洛朗展开是唯一的.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-b)^n + \sum_{n=1}^{\infty} a_n' (z-b)^{-n}$$

$$f(z) (z-b)^k dz = 2\pi i A_k = 2\pi i A_k', \quad \Rightarrow A_k = A_k', \quad \text{对于任意 } k \text{ 均成立.}$$

洛朗展开是唯一的.

展开的唯一性告诉我们,不管使用何种方法均能得到正确的洛朗展开结果.

2.2 洛朗展开的求法

问题:求 $\frac{1}{z(z-1)}$ 在 $0 < |z| < 1$ 和 $1 < |z| < \infty$ 中的洛朗展开.

$$0 < |z| < 1, \quad \frac{1}{z(z-1)} = \frac{1}{z} \sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} z^{-n} = \sum_{n=1}^{\infty} z^{n-1}$$

$$1 < |z| < \infty, \quad \frac{1}{z(z-1)} = \frac{1}{z(1-z)} = -\frac{1}{z} \sum_{n=0}^{\infty} z^n = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = -\sum_{n=1}^{\infty} z^{-n}$$

注意到,如果取下展开的中心,将上述函数的奇点.

问题:求 $\frac{1}{z^2}$ 在 $1 < |z| < \infty$ 中的洛朗展开.

$$\frac{1}{z^2} = \frac{1}{z} \cdot \frac{1}{z} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{z^n} = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = \sum_{n=1}^{\infty} z^{-n}$$

注意到此时中心并非奇点,奇点在环域的边界上.

问题:求 $\cos z$ 在 $z=0$ 处的洛朗展开.

注:由泰勒可知 $\cos z$ 的展式中含有 z 的 -1 次项

$$\cos z = \sum_{n=0}^{\infty} \frac{a_n z^n}{n!} = \frac{e^{iz} + e^{-iz}}{2} = \frac{1}{2} \left(\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} + \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!} \right) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

$$1 < |z| < \infty, \quad \frac{1}{z^2} = \frac{1}{z} \cdot \frac{1}{z} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{z^n} = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = \sum_{n=1}^{\infty} z^{-n}$$

$$0 < |z| < 1, \quad \frac{1}{z^2} = \frac{1}{z} \cdot \frac{1}{z} = \frac{1}{z} \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} z^{n-1} = \sum_{n=1}^{\infty} z^{n-1}$$

注: $\cos z = 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \frac{1}{720}z^6 + \dots$

$$\frac{1}{z^2} = (1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \frac{1}{720}z^6 + \dots)^{-1} = \frac{1}{z^2} (1 + \frac{1}{2}z^2 + \frac{1}{24}z^4 + \frac{7}{240}z^6 + \dots)$$

$$= \frac{1}{z^2} (1 + \frac{1}{2}z^2 + \frac{1}{24}z^4 + \frac{7}{240}z^6 + \frac{1}{24}z^4 + \frac{1}{720}z^6 + \dots)$$

$$= \frac{1}{z^2} + \frac{1}{2} + \frac{7}{240}z^2 + \frac{1}{720}z^4 + \dots$$

$$0 < |z| < 1, \quad \frac{1}{z^2} = \frac{1}{z} \cdot \frac{1}{z} = \frac{1}{z} \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} z^{n-1} = \sum_{n=1}^{\infty} z^{n-1}$$

$$\cos z = (1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \frac{1}{720}z^6 + \dots)^{-1} = \frac{1}{z^2} (1 + \frac{1}{2}z^2 + \frac{1}{24}z^4 + \frac{7}{240}z^6 + \dots)$$

$$= \frac{1}{z^2} + \frac{1}{2} + \frac{7}{240}z^2 + \frac{1}{720}z^4 + \dots$$

$$0 < |z| < 1, \quad \frac{1}{z^2} = \frac{1}{z} \cdot \frac{1}{z} = \frac{1}{z} \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} z^{n-1} = \sum_{n=1}^{\infty} z^{n-1}$$

$$= \frac{1}{z^2} + \frac{1}{2} + \frac{7}{240}z^2 + \frac{1}{720}z^4 + \dots$$

$$\Rightarrow \cos z = \frac{1}{z^2} + \frac{1}{2} + \frac{7}{240}z^2 + \frac{1}{720}z^4 + \dots$$

注:利用 $\tan z = z + \frac{1}{3}z^3 + \frac{1}{5}z^5 + \dots$

$$\cos z = \frac{1}{z} (1 + \frac{1}{3}z^2 + \frac{1}{5}z^4 + \frac{1}{7}z^6 + \dots)$$

$$= \frac{1}{z} (1 + \frac{1}{3}z^2 + \frac{1}{5}z^4 + \frac{1}{7}z^6 + \frac{1}{9}z^8 + \dots)$$

$$= \frac{1}{z} (1 + \frac{1}{3}z^2 + \frac{1}{5}z^4 + \frac{1}{7}z^6 + \frac{1}{9}z^8 + \dots)$$

$$= \frac{1}{z} (1 + \frac{1}{3}z^2 + \frac{1}{5}z^4 + \frac{1}{7}z^6 + \frac{1}{9}z^8 + \dots)$$

$$= \frac{1}{z} (1 + \frac{1}{3}z^2 + \frac{1}{5}z^4 + \frac{1}{7}z^6 + \frac{1}{9}z^8 + \dots)$$

$$0 < |z| < 1, \quad \frac{1}{z^2} = \frac{1}{z} \cdot \frac{1}{z} = \frac{1}{z} \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} z^{n-1} = \sum_{n=1}^{\infty} z^{n-1}$$

$$= \frac{1}{z^2} + \frac{1}{2} + \frac{7}{240}z^2 + \frac{1}{720}z^4 + \dots$$

$$0 < |z| < 1, \quad \frac{1}{z^2} = \frac{1}{z} \cdot \frac{1}{z} = \frac{1}{z} \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} z^{n-1} = \sum_{n=1}^{\infty} z^{n-1}$$

$$= \frac{1}{z^2} + \frac{1}{2} + \frac{7}{240}z^2 + \frac{1}{720}z^4 + \dots$$

$$0 < |z| < 1, \quad \frac{1}{z^2} = \frac{1}{z} \cdot \frac{1}{z} = \frac{1}{z} \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} z^{n-1} = \sum_{n=1}^{\infty} z^{n-1}$$

$$= \frac{1}{z^2} + \frac{1}{2} + \frac{7}{240}z^2 + \frac{1}{720}z^4 + \dots$$

$$0 < |z| < 1, \quad \frac{1}{z^2} = \frac{1}{z} \cdot \frac{1}{z} = \frac{1}{z} \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} z^{n-1} = \sum_{n=1}^{\infty} z^{n-1}$$

$$\arg(z-1) - \arg(z+1) = \pi \quad \ln \frac{z-1}{z+1} = \int_0^z \frac{dz}{z^2-1} = \int_0^z \frac{dz}{(z-1)(z+1)}$$

$$= \int_0^z \frac{dz}{z^2-1} = \int_0^z \frac{dz}{(z-1)(z+1)} = \int_0^z \frac{dz}{z^2-1} = \int_0^z \frac{dz}{(z-1)(z+1)}$$

$$= \sum_{n=0}^{\infty} \int_0^z (z^2-1)^{-n-1} dz = \sum_{n=0}^{\infty} (z^2-1)^{-n-1} \cdot \frac{1}{n} z^n = \sum_{n=0}^{\infty} \frac{z^n}{n(z^2-1)^{n+1}}$$

这在 $z=0$ 处收敛.

问题:求 $e^{\frac{1}{z-1}}$ 在环域 $0 < |z-1| < \infty$ 内的洛朗展开

$$e^{\frac{1}{z-1}} = \sum_{n=0}^{\infty} \frac{(z-1)^{-n}}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} (z-1)^{-n}$$

$$e^{\frac{1}{z-1}} = \sum_{n=0}^{\infty} \frac{(z-1)^{-n}}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} (z-1)^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} (z-1)^{-n} = \sum_{n=0}^{\infty} \frac{1}{n!} (z-1)^{-n}$$

$$e^{\frac{1}{z-1}} = \sum_{n=0}^{\infty} \frac{(z-1)^{-n}}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} (z-1)^{-n}$$

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§3 解析函数的奇点

孤立奇点: 解析函数 $f(z)$ 在单连通域 D 内有奇点 z_0 , 如在 z_0 的邻域

D_1 内处处可导, 则称 z_0 为 $f(z)$ 的可去奇点, 反之则为非孤立奇点.

我们考虑函数 $f(z) = \frac{1}{z}$, $z=0$ 是奇点, 而 $z=0$ 也是奇点, 当 n 充分大时,

z 的邻域内有无穷多个奇点, 故 $z=0$ 为非孤立奇点.

对于非孤立奇点 z_0 , 我们总可作一在环域 $0 < |z-z_0| < \delta$ 中作洛朗级数展开

$$f(z) = \sum_{n=0}^{\infty} a_n (z-b)^n$$

可有奇点: 展开式中无限项次, 洛朗展开称为主要部分

特点: 展开式中含有无穷个负幂次项.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-b)^n = (z-b)^{-m} (a_{-m} + a_{-m+1} (z-b) + \dots) = (z-b)^{-m} f_1(z)$$

$f_1(z)$ 在 $z=b$ 的邻域内解析, 则 $z=b$ 为 $f(z)$ 的 m 阶极点.

$\lim_{z \rightarrow b} f(z) = \infty$ $z=b$ 为 $f(z)$ 的极点.

本性奇点: 展开式中含有无穷个负幂次项.

考虑函数 $f(z) = e^{\frac{1}{z}}$ $z=0$ 是奇点, 展开式中含有无穷个负幂次项

当 z 从实轴趋近于 0 时, $z \rightarrow 0^+$, $\frac{1}{z} \rightarrow \infty$, $e^{\frac{1}{z}} \rightarrow \infty$

当 z 从实轴趋近于 0 时, $z \rightarrow 0^-$, $\frac{1}{z} \rightarrow -\infty$, $e^{\frac{1}{z}} \rightarrow 0$

当 z 从虚轴趋近于 0 时, $e^{\frac{1}{z}}$ 趋近于不确定的数

问题: 如果 $z=b$ 是 $f(z)$ 的本性奇点, 则任意 A , 总存在 z_0 使

得 $f(z) \rightarrow A$

令 $G(z) = \frac{1}{f(z)-A}$, 则 $\lim_{z \rightarrow b} G(z)$ 有限

若 $\lim_{z \rightarrow b} G(z) = T$, $\Rightarrow \lim_{z \rightarrow b} f(z) = \frac{1}{T-A}$ 为可去奇点, 与假设矛盾;

若 $\lim_{z \rightarrow b} G(z) = \infty$, 则 $\lim_{z \rightarrow b} f(z) = 0$ 为极点, 与假设矛盾.

故存在 z_0 使得 $\lim_{z \rightarrow b} f(z) = A$

第七节 解析延拓

§1 解析延拓的理论基础

1.1 解析函数零点的孤立性

定理: 不为 0 的函数 $f(z)$ 在 G 内解析, 若 z_0 为 $f(z)$ 的零点, 则一定存在 $\rho > 0$, 使得在 $0 < |z-z_0| < \rho$ 的邻域内 $f(z)$ 无零点.

设 $f(z) = (z-a)^m \phi(z)$, $z=a$ 为 m 阶零点, $\phi(a) \neq 0$, $\lim_{z \rightarrow a} \phi(z) = \phi(a)$

$$\forall \epsilon > 0, \exists \delta, \text{ s.t. } \forall |z-a| < \delta, \text{ 都有 } |\phi(z) - \phi(a)| < \epsilon, \text{ 取 } \epsilon = \frac{|\phi(a)|}{2}$$

$$\Rightarrow |\phi(z)| = |\phi(a) - (\phi(a) - \phi(z))| \geq \frac{|\phi(a)|}{2} \Rightarrow |\phi(z)| > \frac{|\phi(a)|}{2}, \text{ 则在 } |z-a| < \delta \text{ 内无零点.}$$

该定理的逆命题为: G 内的解析函数 $f(z)$, z_0 为其零点, 若 $\forall \rho > 0, \exists z_1 \in G, |z_1 - z_0| = \rho$, 则 $f(z_1) = 0$, 则 $f(z) \equiv 0$.

§2 解析延拓的黎曼定理

对于该命题, 有以下两种更通用的等价描述.

命题1: G 内的解析函数, 若存在无穷个不相同点, 使 $f(z_n) = 0$, 且 $\lim_{n \rightarrow \infty} z_n = a$

则 $z=a$ 为 G 内 $f(z) \equiv 0$.

命题2: G 内的解析函数, 若存在过 a 点的无限弧, 弧上有 a 点的一个区间 I .

在 I 或 G 内 $f(z) \equiv 0$, 则在 G 内 $f(z) \equiv 0$.

命题3: G 内的解析函数 $f(z), g(z)$, 若在 G 的任一弧上或一个区间 I 内 $f(z) = g(z)$,

则在 G 上 $f(z) \equiv g(z)$.

命题4: 复平面上的解析函数 $f(z), g(z)$, 在实轴上满足取偶数式可推知全平面.

1.2 解析函数的唯一性

定理: G 上的解析函数 $f(z), g(z)$, 若存在序列 $\{z_n\}$, $\forall n \in \mathbb{N}, f(z_n) = g(z_n)$

若 $\lim_{n \rightarrow \infty} z_n = a$ 也在 G 内, 则在 G 内 $f(z) \equiv g(z)$.

令 $f(z) = f(z), g(z) = g(z)$, 利用解析函数的零点孤立性证明

§2 解析延拓

2.1 解析延拓的预备知识

$f(z)$ 在 G_1 内解析, $f(z)$ 在 G_2 内解析, 若在 $G_1 \cap G_2$ 内 $f(z) \equiv g(z)$, 则

和 $f(z)$ 为 $f(z)$ 在 G_2 上的解析延拓, $f(z)$ 为 $f(z)$ 在 G_2 上的解析延拓.

2.2 解析延拓的实际

幂级数 $\sum_{n=0}^{\infty} a_n z^n$ 在 $|z| < 1$ 内收敛, 即其和函数 $\frac{1}{1-z}$ 都在 $|z| < 1$ 内的全平面

解析, 求级数和函数相当于一种解析延拓.

事实上, 也可一步步求得解析延拓

考虑 $f(z) = 1 + z + z^2 + z^3 + \dots$ $|z| < 1$.

$$f(z) = 1 + z + z^2 + \dots \quad f(z) = 1 + z + z^2 + \dots \quad f'(z) = \frac{1}{(1-z)^2}$$

则 $f(z)$ 为解析, 如果幂级数收敛, 由此可求出 $f(z)$ 在 $z=1$ 处的展开

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} (z-1)^n = \sum_{n=0}^{\infty} \frac{1}{(1-1)^{n+1}} (z-1)^n = \frac{1}{1-z}$$

利用此方法, 可求得解析函数延拓

$$\text{事实上, 对于级数 } \sum_{n=0}^{\infty} \frac{1}{n!} z^n \text{ 求和, 有 } f(z) = \frac{1}{1-z} \cdot \frac{1}{1-\frac{z}{2}} = \frac{1}{1-z} = \frac{1}{1-z}$$

所以幂级数和函数 $\frac{1}{1-z}$ 在不同 z 处的泰勒展开一致

2.3 解析延拓的应用

解析延拓可应用于特殊函数的定义, 对于幂级数也有所帮助.

问题: $\frac{dW}{dz} = P(z) \frac{dW}{dz} + Q(z) = 0$, $W(z)$ 为某在 G_1 内的函数, $W(z)$ 为 G_2 内的

解析函数. 试说明 $W(z)$ 也是 G_2 的解.

$$g(z) = \frac{dW}{dz} = P(z) \frac{dW}{dz} + Q(z) \text{ 在 } G_2 \text{ 内解析}$$

$$\text{在 } G_1 \cap G_2 \text{ 内, } W(z) = W(z), \quad g(z) = \frac{dW}{dz} = P(z) \frac{dW}{dz} + Q(z) = 0$$

由零点的孤立性, $g(z)$ 在 G_2 内恒为 0 , $g(z) \equiv 0$, 即 $W(z)$ 也是 G_2 的解

事实上, 只要满足变换后的函数 $W(z)$ 为解析函数, 任意函数 $W(z)$ 均可解析延拓

§3 含参量的积分

3.1 含参量的积分的解析性

定理: $f(z)$ 是实数 t 和 z 的连续函数, 且在 $t \in [a, b]$ 时, $f(z)$ 总是 G 上的单值解析函数, 那么 $F(z) = \int_a^b f(z, t) dt$ 也是 G 上的解析函数, 且

$$F'(z) = \int_a^b \frac{\partial f(z, t)}{\partial z} dt$$

$$F(z) = \int_a^b f(z, t) dt = \int_a^b \frac{1}{t} \frac{f(z, t)}{t} dt = \frac{1}{t} \int_a^b \frac{f(z, t)}{t} dt = \frac{1}{t} \int_a^b f(z, t) dt$$

这是柯西型积分, 因此 $F(z)$ 是解析的

$$F'(z) = \frac{d}{dz} \int_a^b \frac{1}{t} \frac{f(z, t)}{t} dt = \int_a^b \frac{1}{t} \frac{d}{dz} \frac{f(z, t)}{t} dt = \int_a^b \frac{1}{t} \frac{\partial f(z, t)}{\partial z} dt$$

$$= \int_a^b \frac{\partial f(z, t)}{\partial z} dt$$

事实上, 我们如果不加限制地直接对号两端对 z 求偏导, 也能通过结果的验证

面求解: $\int_0^{\infty} \frac{x^2}{e^x-1} dx = \pi^2/6 - \int_0^{\infty} \frac{1}{2} x^2 dy = 0$

$\Rightarrow \int_0^{\infty} \frac{x^2}{e^x-1} dx = \frac{1}{6} (2\pi^2 - \frac{1}{3} 8\pi^2) = \frac{\pi^2}{3} (2 - \frac{4}{3}) = \frac{\pi^2}{3} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

例题: $\sum_{n=1}^{\infty} \frac{1}{n^4}$

$\mathcal{L}^{-1}\{\frac{1}{n^4}\} = \mathcal{L}^{-1}\{\frac{1}{s^4}\} = \frac{1}{6} t^3 = \frac{t^3}{6}$

$\sum_{n=1}^{\infty} \frac{1}{n^4} = \int_0^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^4} e^{-nt} dt = \int_0^{\infty} \frac{1}{t^4} \frac{e^{-t}}{e^t-1} dt$ 取同上函数

考虑积分 $\oint \frac{z^2}{e^z-1} dz$:

$\int_0^{\infty} \frac{x^2}{e^x-1} dx + \int_0^{\infty} \frac{(x+2\pi i)^2}{e^{x+2\pi i}-1} dx - \frac{1}{2} 2\pi i (2\pi i)^2 + \int_{2\pi i}^0 \frac{(y)^2}{e^y-1} dy = 0$

$= \int_0^{\infty} \frac{1}{x^2-1} (4\pi^2 x i + 4\pi^2 (x^2+4\pi^2 i x) + 4\pi^2 (x^2+4\pi^2 i x + 4\pi^4)) dx - 8\pi^2 i$

$- \int_0^{\infty} \frac{y^2 (y^2-4\pi^2 i y)}{e^y-1} dy$

取极限: $-\int_0^{\infty} \frac{8\pi^2 i y-32\pi^4}{e^y-1} dy - 8\pi^2 i + \int_0^{\infty} \frac{1}{2} y^4 dy = 0$

$\Rightarrow \int_0^{\infty} \frac{y^4}{e^y-1} dy = -\pi^4 + \frac{\pi^4}{6} = \frac{\pi^4}{6}$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

例题: $\sum_{n=1}^{\infty} \frac{1}{n^2 \cdot 2^n}$

$\mathcal{L}^{-1}\{\frac{1}{n^2 \cdot 2^n}\} = \mathcal{L}^{-1}\{\frac{1}{s^2} \cdot \frac{1}{1-\frac{s}{2}}\} = \frac{1}{2\pi} (e^{2t} - e^{-t})$

$\sum_{n=1}^{\infty} \frac{1}{n^2 \cdot 2^n} = \int_0^{\infty} \sum_{n=1}^{\infty} e^{-nt} \cdot \frac{e^{-2t}}{2^n} dt = \int_0^{\infty} \frac{1}{1-e^t} \cdot \frac{e^{-2t}}{2} dt$

$= \int_0^{\infty} \frac{e^{-t}}{e^t-1} dt = \int_1^{\infty} \frac{dx}{x-1} = \frac{x}{2} - \frac{1}{2\pi} \int_0^1 \frac{x^{-\frac{1}{2}}}{1-x} dx$

$= \frac{1}{2\pi} (\int_0^{\infty} \frac{x^{-\frac{1}{2}}}{x-1} dx + \int_0^1 \frac{x^{-\frac{1}{2}}}{x-1} dx) = \frac{1}{2\pi} (\int_0^{\infty} \frac{x^{-\frac{1}{2}}}{x-1} dx - \frac{1}{2\pi} \int_0^1 x^{-\frac{1}{2}} dx)$

$\oint \frac{z^{\frac{1}{2}}}{z-1} dz = \pi i (1 + e^{i\pi}) = \int_0^{\infty} \frac{z^{\frac{1}{2}}}{z-1} dz + \int_{\infty}^0 \frac{z^{\frac{1}{2}}}{z-1} dz$

$\Rightarrow \int_0^{\infty} \frac{x^{\frac{1}{2}}}{x-1} dx = \pi i \cdot \frac{(1+e^{i\pi})}{1-e^{i\pi}} = \pi i \cdot \frac{1+(-1)}{2-(-1)} = -\pi i \cot(\frac{\pi}{2})$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2 \cdot 2^n} = -\frac{1}{2\pi i} \cdot \frac{\pi}{2} \cot(\frac{\pi}{2})$

§2 傅里叶变换

2.1 傅里叶级数

$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L}))$

$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos(\frac{n\pi x}{L}) dx, b_n = \frac{1}{L} \int_{-L}^L f(x) \sin(\frac{n\pi x}{L}) dx$ (为求周期)

傅里叶展开的性质 (狄利克雷定理):

① 每个周期中只有有限个极值点

② 处处连续或每个周期中只有有限条间断点

引入复数: $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i n \pi x}$, $C_n = \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) e^{-i n \pi x} dx$

$C_n = \frac{1}{L} \int_{-L}^L f(x) e^{-i n \pi x} dx$

2.2 傅里叶变换

$\omega_n = \frac{n\pi}{L}$

$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{L} \int_{-L}^L f(x') e^{-i \omega_n x'} dx' e^{i \omega_n x}$

$= \int_{-\infty}^{\infty} d\omega \cdot \frac{1}{L} \int_{-L}^L f(x') e^{i \omega (x-x')} dx$

令 $F(\omega) = \int_{-\infty}^{\infty} e^{i \omega x} f(x) dx, f(x) = \frac{1}{L} \int_{-\infty}^{\infty} e^{-i \omega x} F(\omega) d\omega$

即为傅里叶变换 / 逆变换, 恒等在于傅里叶的变量换公式

2.3 拉普拉斯变换的普遍推广公式

若 $g(t) = G(s), \mathcal{L}\{f(t)\} = F(p)$

$G(s) = \int_0^{\infty} e^{-st} g(t) dt, g(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} G(s) ds$

$g(t) = f(t) H(t) e^{-\gamma t}$

$G(s) = \int_0^{\infty} f(t) e^{-(s+\gamma)t} dt = F(s+\gamma) = F(p)$

$g(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} F(s+\gamma) ds = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} F(p) dp$

$\Rightarrow f(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt} F(p) dp$



为了计算这个积分, 需要用到如右图所述:

我们希望 $\int_{ABCE} = 0$, 这样就可利用留数定理计算

事实上, 由不同路径可得, $\int_{BCD} = 0$

$\int_{AB} e^{pt} F(p) dp \leq \int_0^{\gamma} |e^{i\gamma t} \sin \theta| \cdot |F(p)| R d\theta \leq e^{\gamma t} \cdot \varepsilon R \rightarrow 0$

5. 因 $e^{i\gamma t}$ 固定大小, $R \rightarrow \infty \rightarrow$ 固定, 则需 $\varepsilon \rightarrow 0$, 即 $|F(p)| \rightarrow 0, p \rightarrow \infty$

1/2 + 1/2 函数

§1 函数的周期性

1.1 函数的定义

对于任意一个 $\gamma, -\infty < \gamma < \infty$ 内任意自变量所取值的函数 $f(x)$, 满足

$\int_{-\infty}^{\infty} \delta(x) dx = 1, \int_{-\infty}^{\infty} f(x) \delta(x) dx = 0$

δ函数是一个广义函数, 常从积分意义上理解, 否则只能看到 $\delta(x) = \int_{-\infty}^{\infty} \delta(x) dx = 0$

我们非常普遍地用 $\delta(x)$ 的表示, 即 $\delta(x) = \lim_{\epsilon \rightarrow 0} \delta_{\epsilon}(x)$, 下图中 $\delta_{\epsilon}(x)$

$\delta_{\epsilon}(x) = \begin{cases} \frac{1}{\epsilon} & -\frac{\epsilon}{2} < x < \frac{\epsilon}{2} \\ 0 & \text{elsewhere} \end{cases} \quad \delta_{\epsilon}(x) = \frac{1}{\sqrt{\pi} \epsilon} \int_{-\infty}^{\infty} e^{-x^2/\epsilon^2} dx$

$\delta_{\epsilon}(x) = \frac{1}{\sqrt{\pi} \epsilon} \cdot \frac{1}{\sqrt{2\pi} \epsilon} \int_{-\infty}^{\infty} e^{-x^2/\epsilon^2} dx$

1.2 δ函数的性质:

(1) $\int_{-\infty}^{\infty} f(x) \delta(x) dx = f(0)$ [δ函数的定义]

(2) δ函数的积分: $\int_0^x \delta(x) dx = H(x)$, $H(x)$ 为海维塞德函数

在数学上也得到 $H(x)$ 的导数, $\frac{dH(x)}{dx} = \delta(x)$

(3) δ函数的微分: $\int_{-\infty}^{\infty} f(x) \delta'(x) dx = -\int_{-\infty}^{\infty} f'(x) \delta(x) dx$

$= f(0) \delta(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x) \delta(x) dx = -f(0)$

推广性质: $\int_{-\infty}^{\infty} f(x) \delta^{(n)}(x) dx = (-1)^n \int_{-\infty}^{\infty} f^{(n)}(x) \delta(x) dx$

(4) δ函数的奇偶性: $\delta(x) = \delta(-x), \delta'(x) = -\delta'(-x)$

进一步有 $\gamma \delta(x) = 0, \delta(x) = \frac{1}{|a|} \delta(x/a)$

(5) $\delta(f(x)) = \sum_k \frac{1}{|f'(x_k)|} \delta(x-x_k)$, 其中 x_k 为单根

$\delta(x)$ 在单根 x_k 附近展开为 $\delta(f(x)) = \delta(x-x_k) + \gamma(x)$

在 x_k 附近, $\delta(f(x)) \approx \delta(f'(x_k)(x-x_k)) = \frac{1}{|f'(x_k)|} \delta(x-x_k)$

由 $\delta(x)$ 从积分的性质, 取两个 x_k 不相符, 即可得到

简单起见: $\delta(x^2-a^2) = \frac{1}{2|a|} (\delta(x-a) + \delta(x+a))$

(6) δ函数的积分变换

(傅里叶变换: $G(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} \delta(t) dt = 1$

$\Rightarrow \delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} \cdot 1 \cdot d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} d\omega = \frac{1}{i} \int_0^{\infty} \cos(\omega t) d\omega$

拉普拉斯变换: $F(p) = \int_0^{\infty} e^{-pt} \delta(t) dt = 1 = \mathcal{L}\{1\}$

$\mathcal{L}\{\delta(t-\tau)\} = e^{-p\tau}, \tau > 0$

三维 δ函数: $\delta^3(r) = \delta(x) \delta(y) \delta(z) = \frac{1}{r^2} \delta(r) \delta(\theta) \delta(\phi) = \frac{1}{r^2 \sin \theta} \delta(r) \delta(\theta) \delta(\phi)$

例题: 证明 $\nabla^2 \frac{1}{r} = -4\pi \delta(r)$

$\nabla^2 \frac{1}{r} = \frac{1}{r^3} \frac{\partial}{\partial r} (r^2 \frac{\partial}{\partial r}) = \frac{1}{r^3} \frac{\partial}{\partial r} (r^2 \cdot (-\frac{1}{r^2})) = \frac{1}{r^3} \frac{\partial}{\partial r} (-1) = 0$

$\Rightarrow r \neq 0$ 时, $\nabla^2 \frac{1}{r} = 0$

$r=0$ 时, $\oint \nabla^2 \frac{1}{r} dV = \oint \nabla \cdot \frac{1}{r^2} dS$

$= \lim_{R \rightarrow 0} \oint \frac{1}{R^2} \frac{\partial}{\partial r} (r^2 \frac{\partial}{\partial r} \frac{1}{r}) dV = -4\pi \lim_{R \rightarrow 0} \frac{1}{R^2} \int_0^R \frac{1}{r^2} dV$

$= -4\pi \int_0^R \frac{1}{r^2} dV = -4\pi \int_0^R \frac{1}{r^2} \cdot 4\pi r^2 dr = -4\pi \int_0^R 1 dr = -4\pi R$

$= -4\pi \int_0^R \sin \theta d\theta = -4\pi$ 因此, $\nabla^2 \frac{1}{r} = -4\pi \delta(r)$

§2 函数的应用

2.1 利用 δ函数求积分

利用 $\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos(\omega t) d\omega$, 可一对像有

而函数积分进行简化

例题: 计算 $\int_{-\infty}^{\infty} \frac{\sin x}{x} dx$

考虑函数 $\int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx = F(x), F(x) = F(x)$

$F(x) = \int_{-\infty}^{\infty} \sin(x) dx = 2\pi \delta(x) \Rightarrow F(x) = 2\pi H(x) + C$

又 $F(x)$ 是奇函数, $\Rightarrow C=0, \Rightarrow F(x) = \pi = \int_{-\infty}^{\infty} \frac{\sin x}{x} dx$

例题: 计算积分 $I = \int_{-\infty}^{\infty} \frac{\sin(2x)}{x^2+1} dx$

$F(x) = \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{x^2+1} dx, F(x) = \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{x^2+1} dx$

$F''(x) = \int_{-\infty}^{\infty} \frac{-x^2 e^{i\lambda x}}{x^2+1} dx \Rightarrow F''(x) - i F'(x) + F(x) = 0$

$-x^2 - x + 1 = 0, x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2} = e^{\pm i\pi/5}, e^{\mp i\pi/5}$

$F(x) = \begin{cases} A e^{e^{i\pi/5} x} + B e^{e^{-i\pi/5} x} & x > 0 \\ C e^{e^{i\pi/5} x} + D e^{e^{-i\pi/5} x} & x < 0 \end{cases}$ 由 $F(x)$ 的收敛性, $A=D=0$

$F(x)$ 在 0 处应该连续, 否则无法满足 $\delta(x)$ 带来的边界条件

$\Rightarrow B=C, F(x) \Big|_{x=0} = -2\pi \Rightarrow -e^{i\pi/5} B + e^{-i\pi/5} B = -2\pi \Rightarrow B=C = \frac{\pi}{2}$

$\Rightarrow I = \mathcal{L}\{F(x)\} = \mathcal{L}\left\{\frac{\pi}{2} (e^{e^{i\pi/5} x} + e^{e^{-i\pi/5} x})\right\} = \frac{\pi}{2} (e^{-\sqrt{5}} \sin 1 + e^{\sqrt{5}} \sin 1) = \pi \frac{\sin(\sqrt{5})}{\sqrt{5}}$

2.2 傅里叶方程初值问题的格林函数

在上节我们看到, 对于非齐次项为 δ函数的二阶常微分方程, 它的

解就是 $f(x)$ 连续但 $f'(x)$ 不连续的性质, 现进行一个更一般化的分析

考虑 $\frac{d}{dx} p(x) \frac{dy(x)}{dx} + q(x) y(x) = 0, y(x_1), y(x_2)$ 为给定的解

对于 $\frac{d}{dx} p(x) \frac{dy(x)}{dx} + q(x) y(x) = \delta(x-x_0)$, 它的解为

$y(x) = \begin{cases} G(x) y(x_0) + G(x_0) y(x) = g_1 & x < x_0 \\ D(x) y(x_0) + D(x_0) y(x) = g_2 & x > x_0 \end{cases} = g_1 + (g_2 - g_1) H(x-x_0)$

$\frac{dG(x)}{dx} = -\frac{dD(x)}{dx} + \frac{dG(x)}{dx} H(x-x_0) + (G(x_0) - G(x)) \delta(x-x_0)$

格林函数为: $p(x) \left(\frac{dG}{dx} + \left(\frac{dD}{dx} - \frac{dG}{dx} \right) H(x-x_0) + (G(x_0) - G(x)) \delta(x-x_0) \right) + q(x) (G(x) - (G(x_0) - G(x)) H(x-x_0)) = \delta(x-x_0)$

即 $\frac{d}{dx} (p(x) \frac{dG}{dx}) + q(x) G = 0, \frac{d}{dx} (p(x) \frac{dG}{dx}) + q = 0$

$\Rightarrow G(x_0) [2p(x_0) \frac{dG}{dx} - \frac{dD}{dx}] + p(x_0) (G(x_0) - 1) = -p(x_0) (G(x_0) - G(x_0)) \delta(x-x_0)$

从物理意义上理解, 对于任意非齐次项 $f(x)$, 有

$(2p(x) \left(\frac{dG}{dx} - \frac{dD}{dx} \right)) \Big|_{x=x_0} + p(x_0) (G(x_0) - 1) f(x) = \frac{d}{dx} (p(x) (G(x_0) - G(x)) f(x)) \Big|_{x=x_0}$

$= p(x_0) (G(x_0) - G(x_0)) f(x) + p(x_0) (G(x_0) - G(x_0)) f(x) + p(x_0) f(x) \left(\frac{dG}{dx} - \frac{dD}{dx} \right) \Big|_{x=x_0}$

$\Rightarrow [p(x) \left(\frac{dG}{dx} - \frac{dD}{dx} \right) \Big|_{x=x_0} - 1] f(x) = p(x_0) (G(x_0) - G(x_0)) f(x)$

由于 $f(x)$ 任意连续, $f(x)$ 与 $f(x_0)$ 无关, 故 $G(x_0) - G(x_0) = 0, \left(\frac{dG}{dx} - \frac{dD}{dx} \right) \Big|_{x=x_0} = \frac{1}{p(x_0)}$

有了上面的讨论, 我们可以看出非齐次项为 δ函数, 导致了解的特殊的

连续性, 但是这样非齐次项为 δ函数的方程并无实际意义, 仍需要从

积分的角度去理解, 有意思的是, 格林函数所满足的, 这意味

我们可以将非齐次项为任意函数的常微分方程的求解问题转化为

一个积分问题, 下面我们先看简单的例子

例题: 求解 $\frac{d^2 y}{dx^2} = \delta(x-1), y|_{x=0} = 0, \frac{dy}{dx} \Big|_{x=0} = 0, x > 0, x > 0$

y 应该满足 $x > 1$ 的函数, 即 $y = g(x-1)$

直接代入积分: $\frac{dy}{dx} = g'(x-1) + u(x)$

$g(x-1) = (x-1) H(x-1) + u(x-1) f(x)$ 代入初值条件, $u(x_1) = 0, p(x_1) = 0$

$\Rightarrow g(x-1) = (x-1) H(x-1)$

进一步求解 $\frac{d^2 y}{dx^2} = f(x), y(x_0) = 0, y'(x_0) = 0$

我们将这个方程理解为 $\frac{d^2 y}{dx^2} = \delta(x-1)$ 关于 x 的积分

右侧 $f(x) = \int_0^x f(t) \delta(t-1) dt$, 右侧 $f(x)$ 为 $f(x)$ 的积分

右侧 $f(x) = \int_0^x f(t) \delta(t-1) dt$, 右侧 $f(x)$ 为 $f(x)$ 的积分

$y(x) = \int_0^x f(t) g(x-1) dt = \int_0^x f(t) (x-1) H(x-1) dt$

$= \int_0^x f(t) (x-1) dt$

对 y 进行求导, $y(x_0) = 0, y'(x) = \int_0^x f(t) dt, y'(x_1) = f(x_1), y(x_0) = 0$

例题: $\frac{d^2 g(x)}{dx^2} + k^2 g(x) = \delta(x-1), x > 0, x > 0, g(x) \Big|_{x=0} = 0$

$\frac{d^2 g(x)}{dx^2} \Big|_{x=0} = 0$

$$t < 0 \text{ 时, } g_1(t) = A_1 \sin kt + B_1 \cos kt$$

$$t > 0 \text{ 时, } g_1(t) = C_1 \sin kt + D_1 \cos kt$$

$$g_1(t)|_{t=0} = 0, \quad \left. \frac{dg_1(t)}{dt} \right|_{t=0} = 0, \quad \text{所以应取 } t < 0 \text{ 的范围内,}$$

$$\Rightarrow A_1 = 0, \quad B_1 = 0, \quad g_1(t) \text{ 在 } t < 0 \text{ 时连续, } \left. \frac{dg_1(t)}{dt} \right|_{t=0^-} = 1$$

$$\Rightarrow \begin{cases} C_1 \sin kt + D_1 \cos kt = 0 \\ C_1 \cos kt - D_1 \sin kt = \frac{1}{k} \end{cases} \Rightarrow C_1 = \frac{1}{k} \cos kt, \quad D_1 = -\frac{1}{k} \sin kt$$

$$t > 0 \text{ 时, } g_1(t) = \frac{1}{k} \sin k(t-\tau) H(t-\tau)$$

$$\Rightarrow g_1(t) = \frac{1}{k} \sin k(t-\tau) H(t-\tau)$$

$$\text{进一步求解 } \frac{d^2 y}{dt^2} + k^2 y = f(t), \quad y(0) = 0, \quad y'(0) = 0$$

$$f(t) = \int_0^{\infty} f(\tau) \delta(t-\tau) d\tau$$

$$y(t) = \int_0^{\infty} f(\tau) \frac{1}{k} \sin k(t-\tau) H(t-\tau) d\tau = \int_0^t \frac{1}{k} f(\tau) \sin k(t-\tau) d\tau$$

$$\text{根据微分方程, } y(t) = \int_0^t f(\tau) \cos k(t-\tau) d\tau$$

$$y'(t) = f(t) - k \int_0^t f(\tau) \sin k(t-\tau) d\tau \quad \text{且 } y(0) = 0, \quad y'(0) = 0$$

总结以上两式, 可以看出常微分方程的拉氏解法步骤分为两步:

1) 解出非齐次项为 δ 函数的特解函数 $g_1(t)$

2) 利用 δ 函数的积分表达式 $g_1(t)$ 表示出方程的解 $y(t)$

例: 求解 $\frac{d}{dt} [p(t) \frac{dg_1(t)}{dt}] + q(t) g_1(t) = \delta(t-\tau), \quad t > 0, \quad \tau > 0,$

$$g_1(t)|_{t=0} = 0, \quad \left. \frac{dg_1(t)}{dt} \right|_{t=0} = 0$$

$$y_1(t) \text{ 和 } y_2(t) \text{ 是 } \frac{d}{dt} [p(t) \frac{dy}{dt}] + q(t) y = 0 \text{ 的线性无关解}$$

$$t < \tau \text{ 时, } g_1(t) = C_1(t) y_1(t) + C_2(t) y_2(t)$$

$$g_1(t)|_{t=0} = 0 = C_1(t) y_1(0) + C_2(t) y_2(0)$$

$$\left. \frac{dg_1(t)}{dt} \right|_{t=0} = 0 = C_1(t) y_1'(0) + C_2(t) y_2'(0)$$

$$\text{由于 } y_1(t), y_2(t) \text{ 线性无关, 故 } \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} \neq 0 = W(y_1(t), y_2(t))$$

$$\Rightarrow C_1(t) = C_2(t) = 0, \quad \text{即 } g_1(t) = 0$$

$$t > \tau \text{ 时, } g_1(t) = C_1(t) y_1(t) + C_2(t) y_2(t), \quad \text{代入边界条件,}$$

$$g_1(t)|_{t=\tau} = 0, \quad \left. \frac{dg_1(t)}{dt} \right|_{t=\tau} = \frac{1}{p(\tau)}$$

$$\Rightarrow C_1(\tau) y_1(\tau) + C_2(\tau) y_2(\tau) = 0$$

$$C_1(\tau) y_1'(\tau) + C_2(\tau) y_2'(\tau) = \frac{1}{p(\tau)}$$

$$\Rightarrow C_1(\tau) = \frac{y_2(\tau)}{p(\tau) W(y_1, y_2)}, \quad \frac{1}{p(\tau)} = -\frac{y_1(\tau)}{p(\tau) W(y_1, y_2)}$$

$$C_2(\tau) = \frac{y_1(\tau)}{p(\tau) W(y_1, y_2)}$$

$$\Rightarrow g_1(t) = \frac{1}{p(\tau) W(y_1, y_2)} H(t-\tau) H(t-\tau)$$

$$\text{更进一步, 可以得出 } \frac{d}{dt} [p(t) \frac{dg_1(t)}{dt}] + q(t) g_1(t) = f(t), \quad t > 0, \quad y(0) = 0, \quad y'(0) = 0$$

$$\text{所以得到此时的 } y(t) \text{ 并和前面定理情形一样简单积分即可, 因而常}$$

$$\text{微分方程的解为}$$

$$\text{我们希望能与多阶导数一般的常微分方程的初值问题:}$$

$$\frac{d^n y}{dt^n} + p_{n-1}(t) \frac{d^{n-1} y}{dt^{n-1}} + \dots + p_1(t) \frac{dy}{dt} + p_0(t) y = f(t), \quad y(0) = A, \quad y'(0) = B$$

$$\text{可以联想到此时的 } y(t) \text{ 并和前面定理情形一样简单积分即可, 因而常}$$

$$\text{微分方程的解为}$$

$$\text{命题: 考虑 } p(t) = p(t-\tau), \quad q(t) = q(t-\tau) \text{ 的条件下, 证明 } g_1(t) = g_1(t-\tau)$$

$$\text{首先将 } t < \tau \text{ 的范围进行延拓, } -\infty < t < \tau < +\infty,$$

$$g_1(t)|_{t=\tau} = 0, \quad \left. \frac{dg_1(t)}{dt} \right|_{t=\tau} = 0$$

$$\text{作变量替换, } t \rightarrow t-\tau, \quad \tau \rightarrow t-\tau$$

$$\frac{d}{dt} [p(t-\tau) \frac{dg_1(t-\tau)}{dt(t-\tau)}] + q(t-\tau) g_1(t-\tau) = \delta(t-\tau)$$

$$\text{即 } \frac{d}{dt} [p(t) \frac{dg_1(t)}{dt}] + q(t) g_1(t) = \delta(t-\tau) \quad \dots \textcircled{1}$$

$$\text{又 } \frac{d}{dt} [p(t) \frac{dg_1(t)}{dt}] + q(t) g_1(t) = \delta(t-\tau) \quad \dots \textcircled{2}$$

$$\textcircled{1} \times g_1(t-\tau) - \textcircled{2} \times g_1(t-\tau)$$

$$g_1(t) \frac{d}{dt} [p(t) \frac{dg_1(t-\tau)}{dt(t-\tau)}] - g_1(t-\tau) \frac{d}{dt} [p(t) \frac{dg_1(t)}{dt}]$$

$$= \frac{d}{dt} [p(t) (g_1(t) \frac{dg_1(t-\tau)}{dt(t-\tau)} - g_1(t-\tau) \frac{dg_1(t)}{dt})]$$

$$= g_1(t) g_1(t-\tau) - g_1(t-\tau) g_1(t)$$

$$\text{对 } t \text{ 积分, } p(t) [g_1(t) \frac{dg_1(t-\tau)}{dt(t-\tau)} - g_1(t-\tau) \frac{dg_1(t)}{dt}] \Big|_{-\infty}^{+\infty}$$

$$= g_1(t) g_1(t-\tau) - g_1(t-\tau) g_1(t)$$

$$t > 0 \text{ 时, } \tau > 0 \text{ 时, } g_1(t) = 0 = \frac{dg_1(t)}{dt}$$

$$t < -\tau \text{ 时, } t < -\tau, \quad t < -\tau, \quad g_1(t-\tau) = \frac{dg_1(t-\tau)}{dt} = 0$$

$$\Rightarrow g_1(t-\tau) - g_1(t-\tau) = 0, \quad \text{即 } g_1(t) = g_1(t-\tau)$$

$$\text{例: 求解 } \frac{d}{dt} [p(t) \frac{dy(t)}{dt}] + q(t) y(t) = f(t), \quad y(0) = A, \quad y'(0) = B,$$

$$\frac{d}{dt} [p(t) \frac{dg_1(t)}{dt}] + q(t) g_1(t) = \delta(t-\tau), \quad g_1(t)|_{t=0} = 0, \quad \left. \frac{dg_1(t)}{dt} \right|_{t=0} = 0$$

$$\text{由 } t < \tau, \quad \tau < t, \quad \Rightarrow \frac{d}{dt} [p(t) \frac{dg_1(t-\tau)}{dt(t-\tau)}] + q(t-\tau) g_1(t-\tau) = \delta(t-\tau)$$

$$\text{利用 } p(t) = p(t-\tau), \quad q(t) = q(t-\tau), \quad g_1(t) = g_1(t-\tau), \text{ 有}$$

$$\frac{d}{dt} [p(t) \frac{dg_1(t)}{dt}] + q(t) g_1(t) = \delta(t-\tau) \quad \dots \textcircled{1}$$

$$\text{又 } \frac{d}{dt} [p(t) \frac{dg_1(t)}{dt}] + q(t) y(t) = f(t) \quad \dots \textcircled{2}, \quad \textcircled{1} \times y(t) - \textcircled{2} \times g_1(t), \text{ 有}$$

$$\frac{d}{dt} [p(t) [y(t) \frac{dg_1(t)}{dt} - g_1(t) \frac{dy(t)}{dt}]] = y(t) \delta(t-\tau) - f(t) g_1(t)$$

$$\text{对 } t \text{ 积分 } p(t) [y(t) \frac{dg_1(t)}{dt} - g_1(t) \frac{dy(t)}{dt}] \Big|_{-\infty}^{+\infty} = y(t) - \int_0^{\infty} f(\tau) g_1(\tau) d\tau$$

$$\tau > 0 \text{ 时, } \left. \frac{dg_1(t)}{dt} \right|_{\tau=0} = \frac{dg_1(t-\tau)}{dt(t-\tau)} \Big|_{-\tau=0} = \frac{dg_1(t)}{dt} \Big|_{t=\tau} = 0$$

$$g_1(t)|_{t=\tau} = 0, \quad \text{再利用 } y(0) = A, \quad y'(0) = B, \text{ 得到:}$$

$$y(t) = \int_0^{\infty} f(\tau) g_1(\tau) d\tau - p(0) (A \frac{dg_1(t)}{dt} - B g_1(t)) \Big|_{t=0}$$

$$\text{以上, 我们得到了由同一个格林函数得到非齐次项为任意}$$

$$\text{函数 } f(t) \text{ 的方法, 从某种意义上来说, 我们成功地解决了求特解时}$$

$$\text{需要“猜”特解的问题, 然而实际上这种做法的实用性并不强...}$$

$$\text{2) 常微分方程初值问题的格林函数}$$

$$\text{上一步中, 我们给出的初值条件为 } y(0) = A, \quad y'(0) = B;$$

$$\text{而在本章中, 我们将处理为某边界条件, 即 } y(a) = A, \quad y(b) = B$$

$$\text{例: 求解 } \frac{d^2 g_1(t)}{dt^2} = \delta(t-\tau), \quad a < t < b, \quad g_1(a) = 0, \quad g_1(b) = 0$$

$$\text{对 } t \text{ 进行积分, } \frac{dg_1(t)}{dt} = H(t-\tau) + \alpha(t)$$

$$\text{再次对 } t \text{ 进行积分, } g_1(t) = (t-\tau) H(t-\tau) + \alpha(t) t + \beta(t)$$

$$g_1(a) = 0 = \alpha(a) \cdot a + \beta(a), \quad g_1(b) = 0 = (b-\tau) + \alpha(b) \cdot b + \beta(b)$$

$$\Rightarrow \alpha(t) = \frac{\tau-b}{b-a}, \quad \beta(t) = -a \cdot \frac{\tau-b}{b-a}$$

$$\Rightarrow g_1(t) = (t-\tau) H(t-\tau) + \frac{\tau-b}{b-a} (t-a)$$

$$\text{例: 求解 } \frac{d^2 g_1(t)}{dt^2} + k^2 g_1(t) = \delta(t-\tau), \quad a < t < b, \quad g_1(a) = 0, \quad g_1(b) = 0$$

$$\text{由上一步例题中可看出, } g_1(t) \text{ 的通解为:}$$

$$g_1(t) = \frac{1}{k} \sin k(t-\tau) H(t-\tau) + C_1 \sin ka + D_1 \cos ka$$

$$\text{代入边界条件, } \begin{cases} 0 = C_1 \sin ka + D_1 \cos ka \\ 0 = \frac{1}{k} \sin k(b-\tau) + C_1 \sin kb + D_1 \cos kb \end{cases}$$

$$\Rightarrow C_1 = \frac{1}{k} \frac{\sin k(b-\tau)}{\sin k(b-a)}, \quad D_1 = \frac{1}{k} \frac{\sin k(b-\tau)}{\sin k(b-a)} \sin k(a)$$

$$\Rightarrow g_1(t) = \frac{1}{k} \sin k(t-\tau) H(t-\tau) + \frac{1}{k} \frac{\sin k(b-\tau)}{\sin k(b-a)} \sin k(a)$$

$$\text{说明: 二阶常微分方程对于 } t < \tau, \quad t > \tau \text{ 内某区间, 会出现四个}$$

$$\text{特征参数 (各二个), 与对应有四个方程 (二个边界条件 二个 } t > \tau \text{ 时连续条}$$

$$\text{件), 上节中所给出的边界条件均在 } t < \tau \text{ 一侧; 在本节中边界条件在}$$

$$\text{在 } t < \tau, \quad t > \tau \text{ 范围内各有一个, 我们经研究会发现, } g_1(t) \text{ 的连}$$

$$\text{续条件与自变量的初值并不相关, 因而所以用一个带有自由项函数}$$

$$\text{的通解形式来匹配这两个条件, 进而只留下初值对应的 } t > \tau \text{ 待定参数.}$$

$$\text{例: 求解 } \frac{d}{dt} [p(t) \frac{dg_1(t)}{dt}] + q(t) g_1(t) = \delta(t-\tau), \quad a < t < b,$$

$$g_1(a) = 0, \quad g_1(b) = 0$$

$$\text{给出符合 } g_1(t) \text{ 特殊的连续性通解:}$$

$$g_1(t) = \frac{1}{p(\tau) W(y_1, y_2)} H(t-\tau) H(t-\tau) + A_1(t) y_1(t) + B_1(t) y_2(t)$$

$$\text{代入边界条件, } \begin{cases} A_1(t) y_1(a) + B_1(t) y_2(a) = 0 \\ \frac{1}{p(\tau) W(y_1, y_2)} \frac{dg_1(t)}{dt} \Big|_{t=\tau} + A_1(t) y_1(b) + B_1(t) y_2(b) = 0 \end{cases}$$

$$\Rightarrow A_1(t) = -\frac{y_1(a) W(y_1, y_2) (y_2(b) - y_1(b))}{p(\tau) W(y_1, y_2) (y_1(b) y_2(a) - y_1(a) y_2(b))}$$

$$B_1(t) = \frac{y_2(a) W(y_1, y_2) (y_1(b) - y_2(b))}{p(\tau) W(y_1, y_2) (y_1(b) y_2(a) - y_1(a) y_2(b))}$$

$$\Rightarrow g_1(t) = \frac{1}{p(\tau) W(y_1, y_2)} H(t-\tau) H(t-\tau) + \frac{1}{p(\tau) W(y_1, y_2)} \frac{y_1(a) y_2(b) - y_2(a) y_1(b)}{y_1(b) y_2(a) - y_1(a) y_2(b)}$$

$$\text{为解法初值并不为 0 的情况, 仍需进行对 } t > \tau \text{ 的讨论.}$$

$$\text{命题: 证明 } g_1(t) = g_1(t-\tau)$$

$$\frac{d}{dt} [p(t) \frac{dg_1(t)}{dt}] + q(t) g_1(t) = \delta(t-\tau) \quad \dots \textcircled{1}$$

$$\text{作替换 } t \rightarrow t-\tau, \quad \frac{d}{dt} [p(t) \frac{dg_1(t-\tau)}{dt(t-\tau)}] + q(t-\tau) g_1(t-\tau) = \delta(t-\tau) \quad \dots \textcircled{2}$$

$$\text{又 } g_1(t-\tau) - \textcircled{2} \times g_1(t-\tau) = \frac{d}{dt} [p(t) (g_1(t) \frac{dg_1(t-\tau)}{dt(t-\tau)} - g_1(t-\tau) \frac{dg_1(t)}{dt})]$$

$$= g_1(t) \delta(t-\tau) - g_1(t-\tau) \delta(t-\tau), \quad \text{对 } t \text{ 进行积分,}$$

$$g_1(t) - g_1(t-\tau) = p(t) [g_1(t) \frac{dg_1(t-\tau)}{dt(t-\tau)} - g_1(t-\tau) \frac{dg_1(t)}{dt}] \Big|_{-\infty}^{+\infty} \quad (*)$$

$$\text{由于 } g_1(a) = g_1(b) = 0, \text{ 故 } (*) \text{ 可化为 } 0, \text{ 从而 } g_1(t) = g_1(t-\tau)$$

$$\text{可得到, 尽管边界条件不同, 但都能保证 } (*) \text{ 式为 } 0$$

$$\text{例: 求解 } \frac{d}{dt} [p(t) \frac{dg_1(t)}{dt}] + q(t) g_1(t) = f(t), \quad a < t < b, \quad y(a) = y(b) = 0,$$

$$\text{列出对应的格林函数满足的方程, } \frac{d}{dt} [p(t) \frac{dg_1(t)}{dt}] + q(t) g_1(t) = \delta(t-\tau)$$

$$\text{交换 } t, \tau, \text{ 并利用 } g_1(t) = g_1(\tau), \Rightarrow \frac{d}{dt} [p(t) \frac{dg_1(t)}{dt}] + q(t) g_1(t) = \delta(t-\tau) \quad \dots \textcircled{1}$$

$$\frac{d}{dt} [p(t) \frac{dg_1(t)}{dt}] + q(t) y(t) = f(t) \quad \dots \textcircled{2} \quad \text{又 } y(t) - \textcircled{2} \times g_1(t)$$

$$\Rightarrow \frac{d}{dt} [p(t) (y(t) \frac{dg_1(t)}{dt} - g_1(t) \frac{dy(t)}{dt})] = y(t) \delta(t-\tau) - f(t) g_1(t)$$

$$\text{而例证定理 } y(t) = \int_a^b f(\tau) g_1(\tau) d\tau = p(t) [y(t) \frac{dg_1(t)}{dt} - g_1(t) \frac{dy(t)}{dt}] \Big|_a^b$$

$$\text{利用 } y(a) = A, \quad y(b) = B \Rightarrow y(t) = \int_a^b f(\tau) g_1(\tau) d\tau + B p(b) \frac{dg_1(t)}{dt} \Big|_{t=b} - A p(a) \frac{dg_1(t)}{dt} \Big|_{t=a}$$

$$\text{注意在利用 } g_1(t) = g_1(\tau) = g_1(a) = g_1(b) = 0$$

$$\text{例: 求解 } \frac{d^2 y}{dx^2} - k^2 y = f(x), \quad k > 0, \quad x > 0 \text{ 时, } y(0) \text{ 有解}$$

$$\text{先写出格林函数满足的方程 } \frac{d^2 g_1(x)}{dx^2} - k^2 g_1(x) = \delta(x-\tau)$$

$$\text{由于 } x > 0 \text{ 时 } y(0) \text{ 有解, 故 } \frac{d^2 g_1}{dx^2} - k^2 g_1 = 0 \text{ 对应的通解为 } g_1 = e^{kx}, \quad g_2 = e^{-kx}$$

$$\text{得到 } g_1(x) = A e^{-k(x-\tau)}, \text{ 再利用 } \frac{dg_1(x)}{dx} \Big|_{x=\tau} = 1$$

$$\Rightarrow g_1(x) = -\frac{1}{k} e^{-k(x-\tau)}, \text{ 格林函数有解形式,}$$

$$y(x) = \frac{1}{k} \int_0^{\infty} f(\tau) e^{-k(x-\tau)} d\tau + p(x) [y(x) \frac{dg_1(x)}{dx} - g_1(x) \frac{dy(x)}{dx}] \Big|_{x=0}^{x=\infty}$$

$$\text{由于 } g_1(x), \quad y(x) \text{ 在 } x > 0 \text{ 时均有解, 故后一项为 } 0$$

$$\Rightarrow y(x) = -\frac{1}{k} \int_0^{\infty} f(\tau) e^{-k(x-\tau)} d\tau$$

$$\text{第十三章 线性方程的拉氏解法}$$

$$\text{§1 拉氏解法的一般性讨论}$$

$$1.1 零点与极点$$

$$\text{二阶常系数线性微分方程: } \frac{d^2 u(x)}{dx^2} + p(x) \frac{du(x)}{dx} + q(x) u(x) = 0$$

$$\text{零点: } p(x), \quad q(x) \text{ 在 } z_0 \text{ 处解析的点}$$

$$\text{极点: } p(x), \quad q(x) \text{ 至少有一个在 } z_0 \text{ 处不解析}$$

$$\text{实际情况中, 我们遇到的微分方程都会化成如下的形式, 下面给出例:}$$

$$\text{超几何方程: } z(1-z) \frac{d^2 u}{dz^2} + [c - (a+b+1)z] \frac{du}{dz} - ab u = 0$$

$$\text{超几何方程: } (1-z) \frac{d^2 u}{dz^2} - 2z \frac{du}{dz} + (1+u) u = 0$$