

北京大学  
PEKING UNIVERSITY

理  
论  
力  
学

Theoretical Mechanics

PKU 2023 - fall

陈、弦、

地址：中国 北京市海淀区颐和园路5号

邮编：100871

[www.pku.edu.cn](http://www.pku.edu.cn)

略:

I. 课堂笔记

II. (期中前)整理

III. (期中前)应试编

注: 由于期中前笔记均为当堂直录,

略显潦草.

故附上 II、III 块以辅助学习.

2023. 9. 11



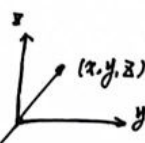
北京大学  
PEKING UNIVERSITY

In-Class Notes

理力 → “流” → 流至同一数学框架内。

[所有“关于时间变化”的过程背后同一套框架]

第一节课：力学内容回顾。

① 定量描述“时空中的位置”： $t, \vec{r}$ “速度”： $\vec{v} = \frac{d\vec{r}}{dt}$  (x)  $\frac{\Delta t}{\Delta \vec{r}} \rightarrow$  隐含的条件：A, B 的时间相同(可进行伽利略变换  $\vec{r}' = \vec{r} + \Delta \vec{r}$ )笛卡尔坐标：任一  $\vec{r}$  用空间中正交的坐标系来描述。∴ (x) 的定量分解： $\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}(x,y,z)}{dt} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$ 当然，加速度  $\vec{a} = \frac{d\vec{v}}{dt} = (\ddot{x}, \ddot{y}, \ddot{z})$ 

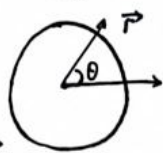
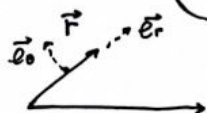
$$\vec{r} = \vec{r}(x, y, z) = x\hat{i} + y\hat{j} + z\hat{k}$$

# Remark: 为何采取这样的右手系?

考虑矢量力学的叉乘 (x) 符号:

 $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$  定义所示, 取符号更便Q. 对称性的描述复杂?  $\Rightarrow$  引入 极坐标

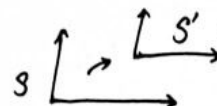
极坐标:

原定义:  $\sqrt{x^2 + y^2} = r$ 新定义:  $(r, \theta)$ , 一般规定  $\theta$  朝向纸外由  $\hat{e}_r, \hat{e}_\theta$  定义,  $\vec{v} = \dot{\vec{r}} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$ (另推:  $\vec{r} = r\hat{e}_r$ ,  $\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{e}_r + r\frac{d\hat{e}_r}{dt} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$ )# 极坐标下的加速度,  $\vec{a} = \frac{d\vec{v}}{dt} = \ddot{r}\hat{e}_r + \dot{r}\dot{\theta}\hat{e}_\theta + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}(-\dot{\theta}\hat{e}_r)$   
 $= \hat{e}_r(\ddot{r} - r\dot{\theta}^2) + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta$ # Remark: 统一性体现在, 只要记住  $\vec{r} = r\hat{e}_r$  (和  $\hat{e}_r, \hat{e}_\theta$  变换关系?) 便可推导。

球坐标:

 $\hat{e}_\phi = \hat{e}_r \times \hat{e}_\theta$ ,  $\phi, \theta$  可视作“经纬”

推理作课后思考

一切的基础, 均为  $\vec{r} = r\hat{e}_r$ # 狭义相对论: 直接对  $\vec{r}$  进行挑战. 基础:  $\begin{pmatrix} cdt' \\ dx' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} cdt \\ dx \end{pmatrix}$  (x)

地址: 中国北京市海淀区颐和园路5号

邮编: 100871 www.pku.edu.cn

# Notes: 关于(Δ)变换

为何称所有公式都在此变换中?

例: 长度  $ds^2 = dx^2 + dy^2 = (dx, dy) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$

2 在狭义相对论中  $ds^2 = (cdt', dx') \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} cdt' \\ dx' \end{pmatrix} = -c^2 dt'^2 + dx'^2$

$$= \begin{pmatrix} cdt' & dx' \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} cdt' \\ dx' \end{pmatrix} = -c^2 dt'^2 + dx'^2$$

(其中  $\lambda = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}$ )

因此衍生“固有时” ~ (proper time)

例: 时间的相对性:  $dx \neq 0, dt = 0$

代入  $\Rightarrow dt' = \gamma dt - \frac{\gamma\beta}{c} dx = -\frac{\gamma\beta}{c} dx \neq 0$

例: 动钟变慢.  $dx' = 0, dt' \neq 0$ .

$$dt' = \gamma dt - \frac{\gamma\beta}{c} dx = dt (\gamma - \frac{\gamma\beta}{c} v) = dt \gamma (1 - \beta^2) = dt \cdot \frac{1}{\gamma} \Rightarrow \frac{dt'}{dt} = \frac{1}{\gamma}$$

回顾: Dynamics

# Newton's Laws

#1. “States that” 力是改变物体运动状态的原因 (definition of Force)

#2.  $\vec{F} = m\vec{a}$  (定量解释, 描述力)

#3. 力的相互作用 (使体系完备).


1 # Questions.

#1. 只要有“两个物体间”才能产生力? 单个物体呢?

Ans: 考虑非惯性参考系中的惯性力.

[ # 推广:  $\vec{F} = m\vec{a} = \frac{d}{dt}(m\vec{v})$ , 假设为“质量与时间无关”  
 $= \frac{d}{dt}(\vec{p})$ , 定义  $\vec{p}$  为动量.

#2. 有没有系统没有动量却在运动?

Ans: 考虑匀速旋转的圆盘 

[ # 推广: 定义  $\vec{L} = \vec{r} \times \vec{p}$ , 这样,  $\vec{L} = \vec{r} \times \vec{p}$ ,  $\vec{v}$  恒定时  $\vec{L}$  不变  
 $\frac{d\vec{L}}{dt} = \vec{\dot{r}} \times \vec{p} + \vec{r} \times \vec{\dot{p}} = \vec{v} \times \vec{p} + \vec{r} \times \vec{F}$   
 $= \vec{v} \times (m\vec{v}) = 0$   
 $= \vec{r} \times \vec{F} \stackrel{\text{def}}{=} \vec{\tau}$ , 称为“力矩”.





2023.9.11

43. 有3个量和角动量就能描述运动了吗?

Ans. 考虑两个同速匀速、转向相反的两个圆

$$\left[ \begin{aligned} \text{推广: 引入能量 } T = \frac{1}{2}mv^2, \quad \frac{dT}{dt} = ? \quad \left( \frac{dT}{dt} = \frac{m}{2} \frac{d}{dt} \vec{v} \cdot \vec{v} = \frac{1}{2} \frac{d}{dt} \vec{v} \cdot \vec{p} \right) \\ dT = \frac{1}{2} d(\vec{v} \cdot \vec{p}) = \vec{v} \cdot d\vec{p} = dt \cdot \vec{v} \cdot \frac{d\vec{p}}{dt} = d\vec{r} \cdot \vec{F} \quad (*) \end{aligned} \right]$$

Def: 保守力

考虑重力场中.



$$\int_A^B \vec{F} \cdot d\vec{r} = \int_A^B m\vec{g} \cdot d\vec{r} \quad \text{不随路径而变化!}$$

 故有:  $\oint \vec{F} \cdot d\vec{r} = 0$  即保守力环路定理 (做功为0)

$$(\oint \vec{F} \cdot d\vec{r} = \oint (\nabla \times \vec{F}) \cdot d\vec{S})$$

$$\# \int_A^B \vec{F} \cdot d\vec{r} = V(B) - V(A) \stackrel{U=-V}{=} U(A) - U(B)$$

$$\Rightarrow -\nabla U = \vec{F}, \quad \text{此即“势”的定义.}$$

$$(*) : d\vec{r} \cdot \vec{F} = d\vec{r} \cdot (-\nabla U) = -dU$$

$$\Rightarrow d(T+U) = 0, \quad \text{守恒量出现!}$$

回顾及推广: 质点系的运动

考虑  $m_i$  与  $\vec{r}_i \Rightarrow$ 

$$\text{Def: } \frac{\sum m_i \vec{r}_i}{\sum m_i} = \vec{r}_c, \quad \text{称为质心的位置. 记 } \sum m_i = M$$

$$\text{显然, } \vec{p} = \sum m_i \vec{v}_i = \sum \vec{p}_i, \quad \frac{d}{dt} \vec{p} = \sum \frac{d}{dt} \vec{p}_i = \frac{d}{dt} (\sum m_i \vec{v}_i)$$

$$\vec{r}_c = \frac{\sum m_i \vec{r}_i}{\sum m_i} \Rightarrow \frac{d\vec{p}}{dt} = \frac{d}{dt} (M \vec{v}_c)$$

$$\vec{p} = \sum \vec{p}_i = \sum \vec{f}_{\text{int } i} + \sum \vec{f}_{\text{ext } i} = \sum \vec{f}_{\text{ext } i} = \vec{F}_{\text{ext}} \quad (\text{因为 } \sum \vec{f}_{\text{int } i} = 0)$$

2023.9.14

Class 2

$$\text{推广上述系流的角动量, } \vec{L} = \vec{L}_c + \sum \vec{L}' = \vec{r}_c \times (M \vec{v}_c) + \sum (\vec{r}_i - \vec{r}_c) \times m_i (\vec{v}_i - \vec{v}_c)$$

地址: 中国北京市海淀区颐和园路5号

邮编: 100871 www.pku.edu.cn

$$\vec{L} = \sum_i \vec{r}_i \times (m_i \vec{v}_i)$$

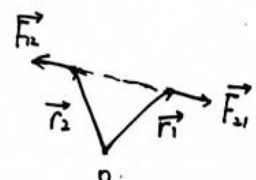
注意上式又是对质心分解才成立. 若选取其他点则上式不成立.

# 考虑能量:  $T = \sum_i \frac{1}{2} m_i (\vec{v}_i)^2 = \frac{1}{2} M v_c^2 + \sum_i \frac{m_i}{2} (\vec{v}_i - \vec{v}_c)^2$ .

$$\begin{aligned} \# \# \frac{d\vec{L}}{dt} &= \frac{d}{dt} \left( \sum_i \vec{r}_i \times \vec{p}_i \right) = \sum_i \frac{d}{dt} (\vec{r}_i \times \vec{p}_i) \\ &= \sum_i \underbrace{\dot{\vec{r}}_i \times \vec{p}_i}_{\vec{v}_i \times (m_i \vec{v}_i) \equiv 0} + \sum_i \vec{r}_i \times \dot{\vec{p}}_i = \sum_i \vec{r}_i \times \vec{F}_i = \vec{M}_S \end{aligned}$$

考虑  $\vec{F}_i = \vec{F}_i^{\text{外}} + \vec{F}_i^{\text{内}}$ , 分解上式得.

$$\sum_i \vec{r}_i \times \vec{F}_i = \sum_i \vec{r}_i \times \vec{F}_i^{\text{外}} + \sum_i \vec{r}_i \times \vec{F}_i^{\text{内}}$$

$$\left[ \begin{array}{l} \text{考虑任一对质点} \end{array} \right. \begin{array}{c} \vec{F}_2 \\ \vec{r}_2 \\ \vec{F}_1 \\ \vec{r}_1 \end{array} \begin{array}{c} \vec{r}_1 \times \vec{F}_2 + \vec{r}_2 \times \vec{F}_1 = (\vec{r}_1 - \vec{r}_2) \times \vec{F}_1 \equiv 0 \end{array} \left. \right]$$


故  $\sum_i \vec{r}_i \times \vec{F}_i^{\text{内}} = 0$ . 即  $\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i^{\text{外}} = \vec{M}_{\text{合外}}$

# 思考. 为何有  $\frac{d\vec{p}}{dt} = \vec{F}_{\text{合外}}$  而  $\frac{d\vec{L}}{dt} \neq \vec{r}_c \times \vec{F}_{\text{合外}}$ ? (例:



下考虑  $\frac{dT}{dt}$ : 
$$\begin{aligned} \frac{dT}{dt} &= \sum_i \frac{\vec{F}_i \cdot d\vec{r}_i}{dt} = \sum_i \vec{F}_i \cdot \vec{v}_i \\ &= \sum_i (\vec{F}_i^{\text{外}} + \vec{F}_i^{\text{内}}) \cdot \vec{v}_i = W_{\text{合外}} + W_{\text{内}} \end{aligned}$$

以上, 矢量力学分析结束.

# Remark & Questions:



## Constraints:

对于三维空间中的一个自由粒子，须用坐标描述其运动  
若给足约束  $z = \sqrt{x^2 + y^2}$ ，则运动退化为二维，自由度 -1。

例：单摆的运动



首先存在： $l = \sqrt{x^2 + y^2 + z^2}$

其次有： $z = 0$  (不可跳出平面)  $\Rightarrow$  区别系统的唯一位形，只需一个变量  $\theta$ 。

\* 推广：考虑  $N$  个粒子，存在  $k$  个约束，有可能存在  $f = 3N - k$ 。

一般意义上，定义约束  $\phi_k(t, \vec{r}_1, \dots, \vec{r}_N, \dot{\vec{r}}_1, \dots, \dot{\vec{r}}_N) = 0$

假如这  $k$  个约束使得  $f = 3N - k$  成立，则称之为完整约束。

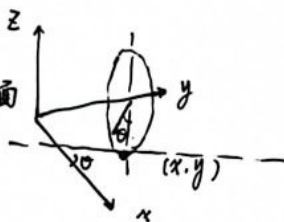
Def: 完整约束的形式可以是： $\phi_k(t, \vec{r}_1, \dots, \vec{r}_N) = 0$ 。

\* 非完整约束的最简单形式，

如果取约束  $\phi: \dot{x} = 3$ ，则无法使系统定形。

例：纯滚的圆盘。

已知圆盘半径为  $a$ ，垂直于  $xy$  平面



倘若知道接触点  $(x, y)$ 。

圆盘投影与  $x$  轴夹角  $\varphi$ ，转动角度  $\theta$ 。

$$\text{纯滚} \begin{cases} \sqrt{\dot{x}^2 + \dot{y}^2} = a\dot{\theta} \\ \tan \varphi = \frac{\dot{y}}{\dot{x}} \end{cases} \Rightarrow \begin{cases} \dot{x} = a\dot{\theta} \cos \varphi \\ \dot{y} = a\dot{\theta} \sin \varphi \end{cases}$$

我们有： $dx = (a \cos \varphi) d\theta$  而  $\varphi$  可能是  $t$  或  $\theta$  的函数，故无法写成几何约束。

\* 究竟什么样的微分约束，才可转化成几何约束。(仅讨论线性微分约束)。

$$A(x, y, t) \dot{x} + B(x, y, t) \dot{y} + C(x, y, t) = 0. \text{ 若 } A dx + B dy + C dt = 0 \text{ 能写成某函数全微分。}$$

地址：中国北京市海淀区颐和园路5号

邮编：100871 www.pku.edu.cn



(接上) 则该线性约束可表示为  $0 = dF(x, y, t) \Leftrightarrow F(x, y, t) - F(x_0, y_0, t_0) = 0$ .

Def:  $\phi_k$  不显含时间, 则称为定常约束;

一些  $\phi_k$  称为可解约束, 例如单摆中  $l = \sqrt{g/y}$ ; 若改成  $l < \sqrt{g/y}$  则为单面的约束(可解约束)

# 广义坐标

$$\vec{r}_1, \dots, \vec{r}_n \rightarrow q_1, q_2, \dots, q_s$$

或  $f = 3N - k$  中,  $k$  为完整约束的个数.

$\therefore s = f$ , 一般写为  $q_i, (i=1 \sim f)$

Def: 称  $\begin{cases} \vec{r}_1(q_1, \dots, q_s, t) \\ \vdots \\ \vec{r}_n(q_1, \dots, q_s, t) \end{cases}$

为变换方程. 微分形式为

$$\dot{\vec{r}}_i = \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \vec{r}_i}{\partial t}$$

$$(\text{或: } d\vec{r}_i = \sum_a \frac{\partial \vec{r}_i}{\partial q_a} dq_a + \frac{\partial \vec{r}_i}{\partial t} dt)$$

$$(\text{或: } \frac{d\vec{r}_i}{dt} = \sum_a \frac{\partial \vec{r}_i}{\partial q_a} \frac{dq_a}{dt} + \frac{\partial \vec{r}_i}{\partial t})$$

称  $\frac{dq}{dt}$  为广义速度

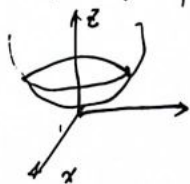
# 一些定义.

虚位移:  $\delta \vec{r}$  表示与  $d\vec{r}$  不同, 是假想量;

$$\delta \vec{r} = \sum_a \frac{\partial \vec{r}}{\partial q_a} \delta q_a$$

广义坐标: 已考虑约束条件而能表出所有坐标的坐标组.

如:  $z = x^2 + y^2$  可用  $(r, \theta)$  这组广义坐标表出.



$$\text{应用: } \begin{cases} \delta x = \cos\theta \delta r - r \sin\theta \delta\theta; \\ \delta z = 2r \delta r \\ \delta y = \sin\theta \delta r + r \cos\theta \delta\theta. \end{cases}$$

虚位移和广义坐标的关系一定反映了约束关系.

由于支持力一定垂直于运动面, 而  $\delta r$  一定沿着运动方向, 处于该面中

$$\therefore \vec{F}_{\text{约束}} \cdot \delta \vec{r} = 0, \text{ 称此类为理想约束}$$

Preview: 一个小证明.  $\frac{d}{dt} \frac{\partial \vec{r}}{\partial q_a} = \frac{\partial \dot{\vec{r}}}{\partial q_a}$

$$\text{证: } \frac{d}{dt} \left( \frac{\partial \vec{r}}{\partial q_k} \right) = \frac{\partial}{\partial t} \left( \frac{\partial \vec{r}}{\partial q_k} \right) + \sum_j \frac{\partial}{\partial q_j} \left( \frac{\partial \vec{r}}{\partial q_k} \right) \cdot \frac{dq_j}{dt}$$

$$= \frac{\partial}{\partial q_k} \left( \frac{\partial \vec{r}}{\partial t} \right) + \sum_j \frac{\partial}{\partial q_k} \sum_j \left( \frac{\partial \vec{r}}{\partial q_j} \right) \frac{dq_j}{dt}$$

$$= \frac{\partial}{\partial q_k} \left[ \dot{\vec{r}} \right] \checkmark$$



9.18

虚位移  $\delta \vec{r}$  可视为一种 瞬时变化

$$\delta \vec{r} = \sum_{\alpha} \frac{\partial \vec{r}}{\partial q_{\alpha}} \delta q_{\alpha}, \quad \vec{r} = \vec{r}(t, q)$$

同一时刻的虚位移可以有多个。

如果  $\vec{r} = \vec{r}(q)$ , 则实位移  $d\vec{r}$  一定是  $\delta \vec{r}$  之中的某一个。  $\Rightarrow$  \* 稳定约束,  $d\vec{r} \in \delta \vec{r}$

$$\text{故有: } d\vec{r} = \frac{\partial \vec{r}}{\partial t} dt + \sum_{\alpha} \frac{\partial \vec{r}}{\partial q_{\alpha}} dq_{\alpha} \quad \leftarrow \Rightarrow d\vec{r} = \frac{\partial \vec{r}}{\partial t} dt + \frac{\partial \vec{r}}{\partial q_{\alpha}} dq_{\alpha}$$

爱因斯坦求和规则: 2个重复指标默以求和

在前半学期, 3个指标 (不太懂) 也可视为一种求和, 如  $T = \frac{1}{2} m_i (\vec{v}_i \cdot \vec{v}_i)$

\* 考虑  $f = f(t, \vec{r}, \dot{\vec{r}}, \ddot{\vec{r}}, \dots)$ , 讨论  $f$  对  $t$  取高阶导。

fig. 1  $\vec{r} \propto -\vec{v}$  如左图, 确实会跟二阶甚至高阶时间导数有关。

技巧: 将其视为某种“主动”, 移出约束方程讨论。

\* 考虑  $\vec{r}(t, q_1, \dots, q_s, \dot{q}_s)$

例: 狭义相对论  $\begin{pmatrix} cdt' \\ dx' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} cdt \\ dx \end{pmatrix}$ , 变换过程中含速度。  
 $\rightarrow$  存在一定任意性。

以下: 开始新内容。

$$dW = \vec{F} \cdot d\vec{r}$$

$$\delta W \stackrel{!}{=} \vec{F} \cdot \delta \vec{r} \text{ 称为虚功。}$$

$$\text{记 } \delta W = (\vec{F}^{\pm} + \vec{R}) \cdot \delta \vec{r}$$

理想约束、平衡态下。

$$\vec{R} \cdot \delta \vec{r} = 0 \Rightarrow \delta W = \vec{F}^{\pm} \cdot \delta \vec{r} = 0.$$

Q. 理想、稳定约束下, 能否用  $\delta W = \vec{F}^{\pm} \cdot \delta \vec{r} \approx 0$  推出平衡状态?

Ans: 可以证明如下:



$$\delta W = 0 = (\vec{F}_i + \vec{R}) \cdot \delta \vec{r} \Rightarrow \delta W = \vec{F}_i \cdot \delta \vec{r} = 0.$$

反过来说，假如下一时刻不平衡  $\Rightarrow dW = \vec{F}_i \cdot d\vec{r} > 0$

而稳定的条件下  $d\vec{r} \in \delta \vec{r}$ ,  $\vec{F}_i \cdot \delta \vec{r} = 0$  矛盾!

$\therefore$  系统必然处于平衡状态.

\* 广义力.

$$-\nabla V = \vec{F} \quad \checkmark \text{ 保守}$$

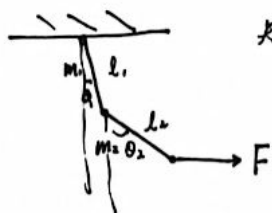
$$\delta W = \vec{F}_i \cdot \delta \vec{r}_i = \vec{F}_i \left( \frac{\partial \vec{r}_i}{\partial q_\alpha} \delta q_\alpha \right) = \left( \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha} \right) \delta q_\alpha = Q_\alpha \delta q_\alpha \quad \swarrow \text{广义力.}$$

$$\text{另一种思考: } \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha} = -\nabla V \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha}$$

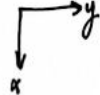
$$\text{或} = \left( -\frac{\partial V}{\partial x} \frac{\partial x}{\partial q_\alpha} - \frac{\partial V}{\partial y} \frac{\partial y}{\partial q_\alpha} - \frac{\partial V}{\partial z} \frac{\partial z}{\partial q_\alpha} \right) \delta q_\alpha = -\frac{\partial V}{\partial q_\alpha} \delta q_\alpha$$

平衡条件下,  $\frac{\partial V}{\partial q} = 0$ . 在某时刻为 0  $\Rightarrow V$  处于极值

例: 复合摆



规定



$$\begin{cases} \vec{F}_1 = m_1 \vec{g} \\ \vec{F}_2 = m_2 \vec{g} \\ \vec{F}_3 = \vec{F} \end{cases}$$

$$\vec{r}_3 = (l_1 \cos \theta_1 + l_2 \cos \theta_2) \hat{x} + (l_1 \sin \theta_1 + l_2 \sin \theta_2) \hat{y}$$

$$\vec{r} = \frac{l_1}{2} \cos \theta_1 \hat{x} + \frac{l_2}{2} \cos \theta_2 \hat{y}$$

$$\vec{r} = \dots \hat{x} + \dots \hat{y}$$

$$\text{同理试计算: } Q_{\theta_1} = \vec{F}_1 \cdot \frac{\partial \vec{r}_1}{\partial \theta_1} + \vec{F}_2 \cdot \frac{\partial \vec{r}_2}{\partial \theta_1} + \vec{F}_3 \cdot \frac{\partial \vec{r}_3}{\partial \theta_1}$$

$$= m_1 g \cdot \left( -\frac{l_1}{2} \sin \theta_1 \right) + m_2 g \cdot (-l_1 \sin \theta_1) + F l_1 \cos \theta_1 = 0$$

$$\Rightarrow \tan \theta_1 = \frac{2F}{m_1 g + 2m_2 g}$$

$$\text{对 } Q_{\theta_2} \text{ 求得 } \tan \theta_2 = \frac{2F}{m_1 g}$$

同!

广义势

\* 用 ~~虚功原理~~ 计算

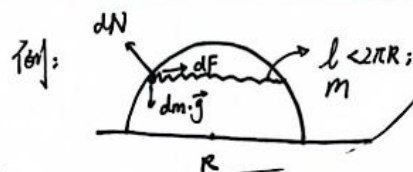
$$V = -m_1 g \frac{l_1}{2} \cos \theta_1 - m_2 g (l_1 \cos \theta_1 + \frac{l_2}{2} \cos \theta_2) - F (l_1 \sin \theta_1 + l_2 \sin \theta_2)$$

$$Q_{\theta_1} = -\frac{\partial V}{\partial \theta_1} = m_1 g \frac{l_1}{2} \sin \theta_1 + m_2 g l_1 \sin \theta_1 - F l_1 \cos \theta_1 = 0$$



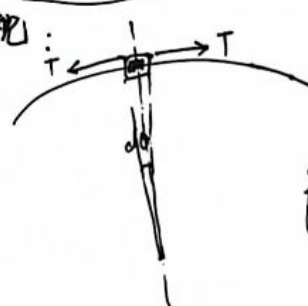
9.18

向量力学法



设  $x$  为平衡长度。

俯视:



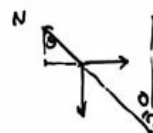
$$\begin{aligned} dm &= \frac{dx}{l} m \\ dx &= d\theta \cdot \frac{x}{2\pi} \Rightarrow dF = 2T \sin \frac{d\theta}{2} \stackrel{\theta \rightarrow 0}{\approx} T d\theta \end{aligned}$$

$$\begin{cases} dF \cdot \tan \theta = dm g = \left( \frac{dx}{x} m \cdot g \right) \\ \cos \theta = \frac{x/2\pi}{R} \end{cases} \Rightarrow k(x-l) \cdot \frac{2R}{x} \cdot \frac{\sqrt{R^2 - \frac{x^2}{4\pi^2}}}{\frac{x}{2\pi}} = \frac{mg}{x}$$

用势能求解: 仅一个自由度  $x$

$$V = mg \sqrt{R^2 - \frac{x^2}{4\pi^2}} + \frac{k}{2} (x-l)^2$$

$$Q_x = - \frac{\partial V}{\partial x} = 0 \quad \text{求解即可!}$$



$$\begin{aligned} N \cos \theta &= mg \\ N \sin \theta &= F \end{aligned}$$

$$\frac{F}{mg} = \tan \theta$$



# Review:

$$\delta W^A = 0 = Q_\alpha^A \delta q_\alpha$$

假如是完整体系, 即  $q_\alpha$  独立,

则  $Q_\alpha^A = 0$  (完整约束下的条件).

\* d'Alembert 原理:

$$\vec{F}_i = m_i \ddot{\vec{r}}_i$$

由于  $\vec{F}_i \delta \vec{r}_i = 0$  (平衡静力学条件下)

$$\therefore \text{推广得 } (\vec{F}_i - m_i \ddot{\vec{r}}_i) \delta \vec{r}_i = 0$$

$$\Leftrightarrow (\vec{F}_i^A - m_i \ddot{\vec{r}}_i) \cdot \delta \vec{r}_i = 0.$$

$$m_i \ddot{\vec{r}}_i \cdot \delta \vec{r}_i = m_i \left( \frac{d}{dt} \dot{\vec{r}}_i \right) \cdot \left( \frac{\partial \vec{r}_i}{\partial q_\alpha} \delta q_\alpha \right)$$

$$= \left( m_i \frac{d}{dt} \dot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_\alpha} \right) \delta q_\alpha$$

$$= m_i \left( \frac{d}{dt} \left( \dot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_\alpha} \right) - \dot{\vec{r}}_i \frac{d}{dt} \frac{\partial \vec{r}_i}{\partial q_\alpha} \right) \delta q_\alpha$$

$$[ \text{由之前推导, } \frac{\partial \dot{\vec{r}}_i}{\partial q_\alpha} = \frac{\partial \vec{r}_i}{\partial q_\alpha}, \quad \frac{d}{dt} \frac{\partial \vec{r}_i}{\partial q_\alpha} = \frac{\partial \dot{\vec{r}}_i}{\partial q_\alpha} ]$$

$$= m_i \left[ \frac{d}{dt} \left( \dot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_\alpha} \right) - \dot{\vec{r}}_i \frac{\partial \dot{\vec{r}}_i}{\partial q_\alpha} \right] \delta q_\alpha$$

$$= m_i \left[ \frac{d}{dt} \left( \frac{\dot{\vec{r}}_i \cdot \dot{\vec{r}}_i}{2} \right) - \frac{\dot{\vec{r}}_i \cdot \dot{\vec{r}}_i}{2} \right] \delta q_\alpha$$

$$= \left[ \frac{d}{dt} \frac{\partial T}{\partial q_\alpha} - \frac{\partial T}{\partial q_\alpha} \right] \delta q_\alpha$$

$$\text{而 } \vec{F}_i \cdot \delta \vec{r}_i = Q_\alpha \delta q_\alpha$$

$$\text{综上: } \left( \frac{d}{dt} \frac{\partial T}{\partial q_\alpha} - \frac{\partial T}{\partial q_\alpha} - Q_\alpha \right) \delta q_\alpha = 0. \Leftrightarrow \frac{d}{dt} \frac{\partial T}{\partial q_\alpha} - \frac{\partial T}{\partial q_\alpha} = Q_\alpha$$

(Lagrangian equation)

\* 自由度  $f \Rightarrow f$  个方程.

如果是保守力, 则  $\vec{F} = -\nabla V$

$$\Rightarrow Q_\alpha = -\frac{\partial V}{\partial q_\alpha}, \quad V = V(t, q), \quad \text{且 } \frac{\partial V}{\partial q_\alpha} = 0 \quad (V \text{ 与 } \dot{q} \text{ 无关})$$

化简上述公式:  $\frac{d}{dt} \frac{\partial (T-V)}{\partial \dot{q}_\alpha} - \frac{\partial (T-V)}{\partial q_\alpha} = 0.$

记  $L = T - V \Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\alpha} = \frac{\partial L}{\partial q_\alpha}, \quad L = L(t, q, \dot{q})$

\* 变形: 记  $L' = L + \frac{d}{dt} U, \quad U = U(t, q).$

方程不变.

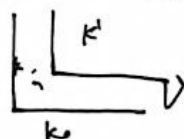
证:  $\frac{d}{dt} \frac{\partial L'}{\partial \dot{q}_\alpha} (L) + \frac{d}{dt} \frac{\partial}{\partial \dot{q}_\alpha} \left( \frac{d}{dt} U \right) = \frac{\partial L}{\partial q_\alpha} + \frac{\partial}{\partial q_\alpha} \frac{d}{dt} U$

即  $\frac{d}{dt} \frac{\partial}{\partial \dot{q}_\alpha} U = \frac{\partial}{\partial q_\alpha} \frac{d}{dt} U$

$$\frac{dU}{dt} = \frac{\partial U}{\partial t} + \frac{\partial U}{\partial q_\alpha} \dot{q}_\alpha$$

由于  $t, q_\alpha$  独立  $\Rightarrow \frac{d}{dt} \frac{\partial}{\partial \dot{q}_\alpha} L' = \frac{\partial L'}{\partial q_\alpha}$

\* 例: 非惯性系.

  $\rightarrow L_0 = \frac{m}{2} V^2 - \Phi(q)$

$$\left[ \frac{\partial U}{\partial q_\alpha} \dot{q}_\alpha \right]$$

K' 系中,  $L' = \frac{1}{2} (m) \cdot (\vec{v} + \vec{V})^2 - \Phi(q').$

考虑

$$= \frac{m}{2} v'^2 + \frac{m}{2} V^2 + m \vec{v}' \cdot \vec{V} - \Phi(q')$$

$$m \frac{d\vec{r}}{dt} = \frac{d}{dt} \left( m \vec{r}' + m \vec{V} \right) \Rightarrow \frac{m}{2} v'^2 + \frac{d}{dt} G(t) + \frac{d}{dt} (m \vec{r}' \cdot \vec{V}) - m \vec{r}' \cdot \dot{\vec{V}} - \Phi(q')$$

(全微分)

取  $L'' = \frac{m}{2} v'^2 - m \vec{r}' \cdot \dot{\vec{V}} - \Phi(q').$

$$\frac{\partial L}{\partial \vec{v}'} = m \vec{v}'; \quad \frac{d}{dt} \frac{\partial L}{\partial \vec{v}'} = m \dot{\vec{v}}', \quad \frac{\partial L}{\partial \vec{r}'} = -m \dot{\vec{V}} - \nabla \Phi$$

$$\underline{m \dot{\vec{v}}' = -m \dot{\vec{V}} - \nabla \Phi}$$





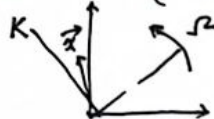
# 北京大学

## PEKING UNIVERSITY

(有能亦可有平功)

$$\nabla(\vec{r} \times \vec{x}) = -2\vec{x} \times (\vec{r} \times \vec{x}).$$

例:



$$\vec{v}' = \vec{v} + \vec{r} \times \vec{x}, \quad \dot{\vec{v}} = m\vec{x} \cdot (\vec{v} \times \vec{r})$$

$$\mathcal{L} = \frac{m}{2} v'^2 + m\vec{v} \cdot (\vec{r} \times \vec{x}) + \frac{m}{2} (\vec{r} \times \vec{x})^2 - m\vec{x} \cdot \vec{v} - \Phi(\vec{x}).$$

$$\frac{d}{dt} [m\vec{v} + m(\vec{r} \times \vec{x})] = m\vec{v} \times \vec{r} - m\vec{r} \times (\vec{r} \times \vec{x}) - m\dot{\vec{v}} - \nabla\Phi(\vec{x})$$

$$\frac{d}{dt} m(\vec{r} \times \vec{x}) = m\vec{x} \times \vec{r} + m\vec{v} \times \vec{r} \quad (\text{叉乘反交换})$$

$$\Rightarrow m\vec{a} = -\nabla\Phi - m\dot{\vec{v}} + 2m\vec{v} \times \vec{r} + m\vec{r} \times (\vec{x} \times \vec{r}) - m\vec{r} \times \vec{x}$$

例: (广又有势体系). 即  $U = U(t, q, \dot{q})$

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial T}{\partial q_\alpha} = Q_\alpha$$

$$\text{若 } Q_\alpha = -\frac{\partial U}{\partial q_\alpha} + \frac{d}{dt} \frac{\partial U}{\partial \dot{q}_\alpha}$$

$$\text{则仍令 } \mathcal{L} = T - U, \text{ 有 } \frac{d}{dt} \frac{\partial}{\partial \dot{q}_\alpha} \mathcal{L} = \frac{\partial \mathcal{L}}{\partial q_\alpha}.$$

$$\vec{r} \cdot (\vec{r} \times \vec{v})$$

$$\vec{r} \cdot (\vec{v} \times \vec{r})$$

例: 电磁场中.  $\vec{F} = e\vec{E} + e\vec{v} \times \vec{B}$ .

$$\text{可定义 } \vec{E} = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \nabla \times \vec{A}.$$

$$\vec{F} = -e\nabla\varphi - e\frac{\partial \vec{A}}{\partial t} + e\vec{v} \times (\nabla \times \vec{A}).$$

$$\text{由 } \vec{A} \times (\nabla \times \vec{B}) = \nabla(\vec{A} \cdot \vec{B}) - \vec{A} \cdot \nabla \vec{B} \Rightarrow \vec{F} = -e\nabla\varphi - e\frac{\partial \vec{A}}{\partial t} + \nabla(\vec{v} \cdot \vec{A}) - \vec{v} \cdot (\nabla \vec{A}).$$

$$\therefore \mathcal{L} = \frac{m}{2} v^2 - e(\varphi - \vec{v} \cdot \vec{A}).$$

$$= -e\nabla(\varphi - \vec{v} \cdot \vec{A}) - e\left(\frac{\partial \vec{A}}{\partial t} + \vec{v} \cdot \nabla \vec{A}\right).$$

$$= -e\nabla(\varphi - \vec{v} \cdot \vec{A}) - e\frac{d\vec{A}}{dt}$$

$$= -\nabla[e(\varphi - \vec{v} \cdot \vec{A})] + \frac{d}{dt} [-e\vec{A}]$$

改写成 ~~形式~~ 形式!

$$-\nabla U + \frac{d}{dt} \frac{\partial U}{\partial \dot{\vec{v}}}.$$

$$\left[ \text{硬凑. } \nabla(e\varphi - e\vec{v} \cdot \vec{A}). \right]$$

$\nearrow$   $\varphi$  不含  $\vec{v}$        $\searrow$  凑出  $\vec{A}$



附: 在所有平衡态中  $Q_\alpha = -\frac{\partial V}{\partial Q_\alpha} = 0$ .



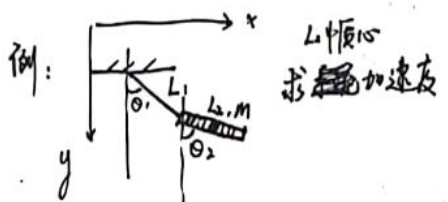
9.25.

回顾.

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial T}{\partial q_\alpha} = Q_\alpha$$

||  $Q_\alpha$  处于保守力场中

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\alpha} - \frac{\partial L}{\partial q_\alpha} = 0.$$



$$x = l_1 \cos \theta_1 + \frac{l_2}{2} \cos \theta_2; \quad y = l_1 \sin \theta_1 + \frac{l_2}{2} \sin \theta_2.$$

$$L = T - V = \frac{m}{2} (\dot{x}^2 + \dot{y}^2) + \frac{I}{2} \omega^2 + mgy, \quad (\omega_2 = \dot{\theta}_2)$$

$$\text{分别有} \begin{cases} \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} = 0 \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} = 0. \end{cases}$$

$$\text{后即得} \begin{cases} \ddot{\theta}_1 = - \frac{2g[\sin \theta_1 + 3\cos \theta_2 \sin(\theta_1 - \theta_2)] + 3l_1 \dot{\theta}_1^2 \sin(2\theta_1 - \theta_2) + 4l_2 \omega_2^2 \sin(\theta_1 - \theta_2)}{l_1[5 - 3\cos(2\theta_1 - 2\theta_2)]} \\ \ddot{\theta}_2 = \dots \end{cases}$$

以下五讲新内容:

 $L = T - V$  条件下又有势函数

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q_\alpha} \dot{q}_\alpha + \frac{\partial L}{\partial \dot{q}_\alpha} \ddot{q}_\alpha$$

$$\frac{d}{dt} \left( \dot{q}_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} \right) = \ddot{q}_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} + \dot{q}_\alpha \frac{\partial L}{\partial q_\alpha}$$

$$\Rightarrow \frac{d}{dt} \left( \dot{q}_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} - L \right) = \frac{\partial L}{\partial t}$$

$$\vec{F}_i = \frac{\partial \vec{F}_i}{\partial q_\alpha} \dot{q}_\alpha + \frac{\partial \vec{F}_i}{\partial t}$$

$$T = \frac{m}{2} v_i^2 = T_2(\dot{q}_\alpha^2) + T_1 + T_0$$

$$T_2 = \frac{m}{2} \left( \frac{\partial \vec{F}_i}{\partial q_\alpha} \cdot \frac{\partial \vec{F}_i}{\partial q_\beta} \right) \dot{q}_\alpha \dot{q}_\beta$$

$$\begin{cases} f(q_1, q_2, \dots, q_n) = \lambda^k f(q_1, \dots, q_n) \\ \text{为 } k \text{ 次齐次函数} \end{cases}$$

$$q_\alpha \frac{\partial f}{\partial q_\alpha} = k f$$

$$\text{故: } q_\alpha \frac{\partial T_2}{\partial q_\alpha} = 2T_2$$

$$\text{故: } \dot{q}_\alpha \frac{\partial T}{\partial \dot{q}_\alpha} = \dot{q}_\alpha \frac{\partial (T_2 + T_1 + T_0)}{\partial \dot{q}_\alpha} = 2T_2 + T_1$$

$$\therefore \dot{q}_\alpha \frac{\partial T}{\partial \dot{q}_\alpha} - L = T_2 - T_0 + V \Rightarrow \text{哈密顿量}$$

若无下项,  $T+V = T+V$  守恒才有“能量守恒”.

$\frac{\partial L}{\partial t}$  / 变换时不显含  $t \Rightarrow E = T+V$  守恒  
 \ / 显含  $t \Rightarrow H = T+V$  守恒.

初积分:  $\frac{d}{dt}(\sim) = 0$

$\downarrow$   
 $L = \frac{1}{2}v^2 - mgz, \frac{\partial L}{\partial t} = 0$   
 ~~$E = \frac{1}{2}v^2 - mgz = 0$~~   
 $E = \frac{1}{2}v^2 + mgz = 0$

如果  $L$  不显含  $q_\alpha$  (不依赖于某个  $q_\alpha$ ).

$\vec{n}$   
 $\vec{r}$   
 $d\vec{r} = (dq_\alpha \vec{n}) \times \vec{r}$   
 $= d\vec{q} \times \vec{r}$

$\frac{\partial L}{\partial q_\alpha} = m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha}$   
 $= m_i \vec{v}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha}$

$= m_i \vec{v}_i \cdot (\vec{n} \times \vec{r})$

$= m_i \vec{n} \cdot (\vec{r} \times \vec{v}_i)$

$= \vec{n} \cdot (\vec{r} \times \vec{p}) = \vec{n} \cdot \vec{J}$

$Q_\alpha = \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha}$

$= \vec{F}_i \cdot (\vec{n} \times \vec{r}_i)$

$= \vec{n} \cdot (\vec{r}_i \times \vec{F}_i)$

$= \vec{n} \cdot \vec{M}$

联立以上式:  $\vec{n} \cdot \frac{d\vec{J}}{dt} = \vec{n} \cdot \vec{M}$

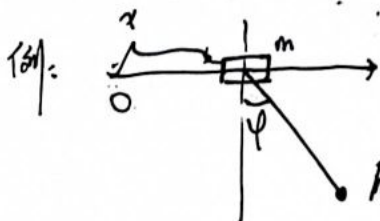


$L = \frac{1}{2} m (r\dot{\theta})^2$

$\frac{\partial L}{\partial \dot{\theta}} = mr^2 \dot{\theta}$  为角动量.

例: 电磁场  $L = \frac{mv^2}{2} - e(\varphi - \vec{v} \cdot \vec{A})$

$\frac{\partial L}{\partial \vec{v}} = m\vec{v} + e\vec{A}$



$L = T - V$

$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} M (\dot{X}^2 + \dot{Y}^2) + Mgl \cos \varphi$

$(V = -Mgl \cos \varphi)$

$X = x + l \sin \varphi, Y = -l \cos \varphi$



由于  $\frac{\partial L}{\partial t} = 0 \Rightarrow E = T + V = \frac{m+M}{2} \dot{x}^2 + \frac{M}{2} l^2 \dot{\varphi}^2 + M l \dot{x} \dot{\varphi} \cos \varphi - M g l \sin \varphi = E_0$

$$\frac{\partial L}{\partial \dot{x}} = 0 \Rightarrow p_x = \frac{\partial L}{\partial \dot{x}} = (m+M) \dot{x} + M l \dot{\varphi} \cos \varphi$$

上述为降阶问题

下例: 降维法

#(不考) 罗斯函数.

$$L(t, q_1, \dots, q_f, \dot{q}_1, \dots, \dot{q}_f) (t, q, \dot{p}), \dots, \dot{q}_f)$$

$$\frac{\partial L}{\partial q_\alpha} L(t, \dots, q_\alpha) = L(\dots, q_\alpha + \Delta q_\alpha)$$

若  $\exists r$  使得  $L = L(t, q_1, \dots, q_{f-r}, \dot{q}_1, \dots, \dot{q}_f)$

$\therefore$  有  $r$  个  $p_\alpha$  守恒.

构造  $R = \sum_{s=f-r+1}^f \dot{q}_s p_s - L$

$$\left. \begin{aligned} \frac{\partial R}{\partial \dot{q}_k} &= -\frac{\partial L}{\partial \dot{q}_k} \\ \frac{\partial R}{\partial q_k} &= -\frac{\partial L}{\partial q_k} \end{aligned} \right\} \frac{d}{dt} \frac{\partial R}{\partial \dot{q}_k} = \frac{\partial R}{\partial q_k}$$



回顾: 拉格朗日方程  $\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial T}{\partial q_\alpha} = Q_\alpha$

原过程:  $(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial T}{\partial q_\alpha} - Q_\alpha) \delta q_\alpha = 0$  一共  $3N-3$  个方程.

为何称 " $\delta q_\alpha$ " 彼此独立?

\* 考虑  $N$  个粒子的坐标,

需要  $3N$  个坐标来表述、定型;  $\delta$  个完整约束,

记  $f_r(t, \vec{r}_1, \dots, \vec{r}_N) = 0, r=1, 2, \dots, \delta$ .

$$\Leftrightarrow f_r(t, q_1, \dots, q_{3N}) = 0.$$

再加上  $(3N-\delta)$  个坐标, 即可唯一确定系统位形.

$$\delta \vec{r}_i = \frac{\partial \vec{r}_i}{\partial q_\alpha} \delta q_\alpha.$$

下面考虑非完整约束:

$$f_1(t, q, \dot{q}) = 0.$$

$$f_k(t, q, \dot{q}) = 0$$

$\delta$  个完整约束

$k$  个非完整约束.

倘若上述取  $r$  个约束后  $k$  个是非完整约束  $\Rightarrow$  则无法唯一求解.

事实上, 需要  $(3N-\delta+k)$  个  $q$  才可唯一确定系统位形.

考虑此时  $(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial T}{\partial q_\alpha} - Q_\alpha) \delta q_\alpha = 0, \alpha=1, 2, \dots, (3N-\delta+k)$

\* 考虑线性非完整约束下:

$$A_r \dot{q}_\alpha + B_r = 0, r=1, 2, \dots, k.$$

取其与  $dt$  乘积:  $A_r dq_\alpha + B_r dt = 0.$

对  $\delta$  符号 (等时变分):  $A_r \delta q_\alpha = 0$

Intro. Lagrange 乘子法

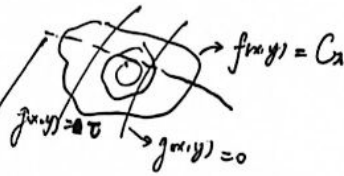
$$\begin{cases} z = f(x, y) \\ g(x, y) = 0 \end{cases}$$



$\Rightarrow$  构造  $F = f + \lambda g \Rightarrow \begin{cases} \frac{\partial F}{\partial x} = 0 \\ \frac{\partial F}{\partial y} = 0 \\ \frac{\partial F}{\partial \lambda} = 0 \end{cases}$  解出的  $x, y$  即为极值条件.



"为何如此"?  $\Rightarrow$  画一系列等势面  
即沿着  $g(x,y)=0$  去找"最大值"  
当两者梯度重合即找到极值.



$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 0 \end{cases} \Leftrightarrow \frac{\partial F}{\partial \lambda} = 0$$

两梯度重合线

\*应用:

$$\begin{cases} A_{ra} \delta q_a = 0 \\ \left( \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_a} - \frac{\partial T}{\partial q_a} - Q_a \right) \delta q_a = 0. \end{cases} \quad k \uparrow \lambda A \text{ 在求和.}$$

$$\Rightarrow \begin{cases} \left( \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_a} - \frac{\partial T}{\partial q_a} - Q_a \right) = \lambda A_{ra} \end{cases}$$

$$A_{ra} \dot{q}_a + B_r = 0 \quad \leftarrow \text{给定约束}$$

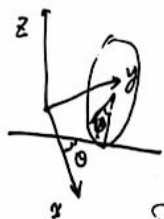
$\downarrow (kT)$

$(3N-s+k) \text{ 方程}$

\* 为何不把  $\lambda_r A_{ra}$  分开写?  $\Rightarrow$  可能导致无解. 如  $\begin{cases} \nabla f = \lambda_1 \nabla g \\ \nabla f = \lambda_2 \nabla h \end{cases}$



例: 陀螺的圆盘



$$\begin{cases} \dot{x} \sin \theta - \dot{y} \cos \theta = 0 \\ \dot{x} \cos \theta + \dot{y} \sin \theta = r \dot{\psi} \end{cases}$$

$$\begin{cases} A_{11} \delta x - A_{12} \delta y = 0 \\ A_{21} \delta x + A_{22} \delta y = -r \delta \psi \end{cases}$$

$$T = \frac{m}{2} (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I_1 \dot{\psi}^2 + \frac{1}{2} I_2 \dot{\theta}^2$$

$$\left( I_1 = \frac{mr^2}{4}, I_2 = \frac{mr^2}{2} \right).$$

$$\begin{cases} -\nabla V = Q \\ Q = \vec{F} \cdot \frac{\partial \vec{r}}{\partial \lambda} \end{cases}$$

$$\begin{cases} m\ddot{x} = \lambda_1 A_{11} + \lambda_2 A_{21} = \lambda_1 \sin \theta + \lambda_2 \cos \theta \quad (1) \\ m\ddot{y} = \lambda_1 A_{12} + \lambda_2 A_{22} = -\lambda_1 \cos \theta + \lambda_2 \sin \theta \quad (2) \\ I_1 \ddot{\psi} = 0 \quad (\text{无 } A_{13} \text{ 项}) \quad (3) \\ I_2 \ddot{\theta} = -\lambda_2 r \quad (4) \end{cases}$$

$$\begin{cases} \dot{x} \sin \theta - \dot{y} \cos \theta = 0 \quad (5) \\ \dot{x} \cos \theta + \dot{y} \sin \theta = r \dot{\psi} \quad (6) \end{cases}$$



北京大学  
PEKING UNIVERSITY

求解, 由 ③  $\Rightarrow \dot{\theta} = \omega \Rightarrow \theta = \omega t + \theta_0$

$$\textcircled{1} \times \sin\theta \quad \textcircled{2} \times \cos\theta \Rightarrow \begin{cases} m(\ddot{x} \sin\theta - \ddot{y} \cos\theta) = \lambda_1 \\ m \frac{d}{dt} (\dot{x} \sin\theta - \dot{y} \cos\theta) - m(\dot{x} \cos\theta + \dot{y} \sin\theta) \dot{\theta} \end{cases}$$

由 ③  $\Rightarrow \lambda_1 = -m \cdot r \dot{\theta} \cdot \dot{\theta}$

类似地,  $\lambda_2 = m r \dot{\theta}$

代入 ④, 得  $(I_z - m r^2) \ddot{\varphi} = 0 \Rightarrow \ddot{\varphi} = 0, \dot{\varphi} = C, \varphi = Ct + \varphi_0$

$$\dot{x} = (r \dot{\varphi}) \cos\theta = r C \cos(\omega t + \theta_0)$$

$$\Rightarrow \begin{cases} x = \frac{rC}{\omega} \sin(\omega t + \theta_0) + x_0 \\ y = -\frac{rC}{\omega} \cos(\omega t + \theta_0) + y_0 \end{cases}$$

$$(x - x_0)^2 + (y - y_0)^2 = \frac{r^2 C^2}{\omega^2}$$

※ (Extension) 耗散函数.

$\vec{F} = k \cdot \vec{v}$ , (记  $\vec{F} = -\nabla U$  形式)

记  $\mathcal{F} = -\frac{\partial}{\partial \vec{v}} \cdot \vec{F}$  (假设  $T$  是位置齐次函数).

称  $\mathcal{F} = \frac{k}{2} v^2$

$$Q' = \vec{F} \cdot \frac{\partial \vec{F}}{\partial \vec{q}} = \vec{F} \cdot \frac{\partial \vec{F}}{\partial \dot{\vec{q}}} = \vec{F} \cdot \frac{\partial \vec{v}}{\partial \dot{\vec{q}}} = -\frac{\partial \mathcal{F}}{\partial \dot{\vec{q}}}$$

$$\dot{\vec{q}} \left( \frac{d}{dt} \frac{\partial T}{\partial \dot{\vec{q}}} - \frac{\partial T}{\partial \vec{q}} \right) = \left( -\frac{\partial V}{\partial \vec{q}} - \frac{\partial \mathcal{F}}{\partial \dot{\vec{q}}} \right) \dot{\vec{q}}$$

(两边同乘  $\dot{\vec{q}}$ )  $LHS = \frac{d}{dt} \left( \dot{\vec{q}} \frac{\partial T}{\partial \dot{\vec{q}}} \right) - \frac{\partial T}{\partial \vec{q}} \dot{\vec{q}} = \frac{d}{dt} (2T) - \frac{\partial T}{\partial t} = \frac{dT}{dt}$

麦克斯韦方程

$$RHS \stackrel{\frac{\partial \vec{v}}{\partial \dot{\vec{q}}} = 1}{=} \frac{dV}{dt} = -2\mathcal{F}$$

$$\Rightarrow \frac{d(T+V)}{dt} = -2\mathcal{F}$$

$$= -k v^2 = -(k \vec{v}) \cdot \vec{v}$$

地址: 中国 北京市海淀区颐和园路5号

邮编: 100871 www.pku.edu.cn

回顾 & 拓展: 拉氏量相关.

拉氏量的性质:

① 可以差一个常数因子

$$L \rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_a} = \frac{\partial L}{\partial q_a}$$

可代入  $L' = kL$

例: 倘若所有项  $\propto m$ , 可令  $k=1$  消去之.


② 可叠加性:

已知有  $L_A(t, q_A, \dot{q}_A)$ ;  $L_B(t, q_B, \dot{q}_B)$

定义  $L = L_A + L_B = L(t, q_A, \dot{q}_A, q_B, \dot{q}_B)$

则  $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$  可描述 A, B 运动

而若  $L = L_A + L_B + U(\frac{q_A, q_B}{l_0})$  引入相互作用. 将其用  $(q_A, q_B, \dot{q}_A, \dot{q}_B)$  表示

例:   $L = L_A + L_B - \frac{k}{2}(l - l_0)^2$

反之: 若  $L = L_A + L_B$ , 则可视为两个独立的运动.

例: 两体运动



$$L = \frac{M+m}{2} (\dot{R}_{cm})^2 + \frac{M}{2} (\dot{r})^2 + \frac{GMm}{r}$$

只与质心有关                      只与两者相对位置有关.

③ 拉氏量不可观测测量.

$(L + \frac{df(t, q)}{dt})$ , 加上一个全微分

与不加全微分时的运动方程相同.

# 自由质点的拉氏量.

$L(t, q, \dot{q})$

如何从  $L$  导出其运动方程?

# 习题课补充:

$T_r$  与  $T_0$  (82-4).

$$L_{\text{rot}} = m r^2 \dot{\theta} = C$$

$$\delta L_{\text{rot}} = 2m r \dot{\theta} \delta r + m r^2 \delta \dot{\theta} = 0$$

$$\Leftrightarrow \delta r \propto \delta \dot{\theta} \quad (\text{取驻值条件})$$

$$\therefore \text{周期一致. } T_r = T_0$$

轨道周期下: 平方反比力  $= T_r$

首先假设以

① 空间平移不变性:  $L(t, x, \vec{v}) = L(t, x + \Delta x, \vec{v})$ ,

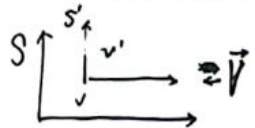
$$\therefore \frac{\partial L}{\partial x} = 0 \Rightarrow L = L(t, \vec{v})$$

② 时空是各向同性的:  $\Rightarrow L = L(t, v) = L(t, |\vec{v}|) = L(t, v^2)$ . ③ 时间平移不变性  $\Rightarrow L = L(v)$

$$\text{由 } \frac{\partial L}{\partial t} = 0 \Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \vec{v}} = \frac{d}{dt} \left( \frac{\partial^2 L}{\partial \vec{v} \partial v^2} \right) = \frac{d}{dt} (2\vec{v} \cdot \frac{\partial L}{\partial v^2}) = 0$$

$$\therefore \vec{v} \frac{\partial L}{\partial v^2} = \text{const}$$

$\vec{v} = \vec{p}/m$  均不变.

④   $\vec{v} = \vec{v}' - \vec{V}, t' = t$  (伽利略变换).

$$L(v') \rightarrow L((v + \vec{V})^2)$$

即 两者只能差一个全微分

$$L((v + \vec{V})^2) = L(v^2) + \frac{df(\vec{v}, t)}{dt}$$

$$L((v + \vec{V})^2) - L(v^2) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \cdot \left( \frac{dx}{dt} \right) = \vec{v} \cdot \vec{V}$$

$\vec{V} \rightarrow 0$  Taylor 展开

$$[L(v^2) + \frac{\partial L}{\partial v^2} \cdot (2\vec{v} \cdot \vec{V} + \vec{V}^2)] - L(v^2) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \cdot \vec{v}$$

$$\frac{\partial L}{\partial v^2} (2\vec{v} \cdot \vec{V}) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \cdot \vec{v}$$

$$\Rightarrow \frac{\partial f}{\partial x} \cdot \vec{v} = \frac{\partial L}{\partial v^2} (2\vec{v} \cdot \vec{V}) = \left( \frac{\partial L}{\partial v^2} \cdot 2\vec{V} \right) \cdot \vec{v}$$

$$\therefore \frac{\partial f}{\partial x} = \frac{\partial L}{\partial v^2} \cdot 2\vec{V} \Rightarrow \underline{L \propto v^2} \text{ 且 } \frac{\partial L}{\partial v^2} = k$$

不显含 v

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = 2k\vec{v} \stackrel{\text{实验}}{=} m\vec{v} \Rightarrow k = \frac{m}{2} \quad \checkmark$$

例:  $L_{\text{相对论}} = \frac{1}{2} mc^2 \sqrt{1 - \frac{v^2}{c^2}}$  在闵氏时空中的形式.

与最小作用量原理  $S = \int L dt$  对应.

$$\vec{p}_{\text{相对}} = \frac{\partial L_{\text{相对}}}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\text{能量: } \vec{p} \cdot \vec{v} - L = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma mc^2 \Rightarrow E = mc^2 \text{ 或 } pc^2$$



2200011630

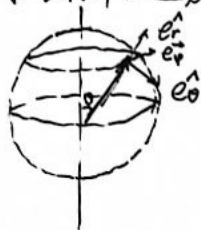
10.9.

北京大学

PEKING UNIVERSITY

$$\left( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} \right) = 0$$

例：球坐标中加速度的分解。



$$\text{记 } L = L(r, \theta, \phi, \dot{r}, \dot{\theta}, \dot{\phi})$$

$$\text{令 } L = \frac{mv^2}{2} - V(r, \theta, \phi)$$

 $\Rightarrow$  用 Lagrangian Equations 求解  $\frac{\partial V}{\partial r}$ 

$$\begin{cases} m a_r = - \frac{\partial V}{\partial r} \\ m a_\theta = - \frac{\partial V}{\partial \theta} \\ m a_\phi = - \frac{\partial V}{\partial \phi} \end{cases}$$

$$L = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) - V$$

$$v^2 = \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \Rightarrow$$

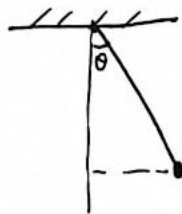
$$r \text{ 方向: } \frac{d}{dt} (m \dot{r}) = m \ddot{r} = m r \ddot{\theta} + m r \sin^2 \theta \dot{\phi}^2 - \frac{\partial V}{\partial r}$$

$$\theta \text{ 方向: } m r^2 \ddot{\theta} + 2 m r \dot{r} \dot{\theta} = m r^2 \sin \theta \cos \theta \dot{\phi}^2 - \frac{\partial V}{\partial \theta}$$

$$\phi \text{ 方向: } 2 m r \dot{r} \sin^2 \theta \dot{\phi} + 2 m r^2 \sin \theta \cos \theta \dot{\theta} \dot{\phi} + m r^2 \sin^2 \theta \ddot{\phi} = - \frac{\partial V}{\partial \phi}$$

$$\begin{cases} a_r = \ddot{r} - r \dot{\theta}^2 - r \sin^2 \theta \dot{\phi}^2 \\ a_\theta = \ddot{\theta} + 2 \dot{r} \dot{\theta} - r \sin \theta \cos \theta \dot{\phi}^2 \\ a_\phi = \ddot{\phi} + 2 \sin \theta \dot{\theta} \dot{\phi} + 2 r \cos \theta \dot{\theta} \dot{\phi} \end{cases}$$

例：圆摆



$$L = \frac{m}{2} l^2 \dot{\theta}^2 - m g l \cos \theta$$

$$\text{可使 } L = \frac{m}{2} l^2 \dot{\theta}^2 + m g l \cos \theta$$

$$\text{拉氏方程: } m l^2 \ddot{\theta} + m g l \sin \theta = 0$$

$$\text{设 } \omega^2 = \frac{g}{l}, \Rightarrow \ddot{\theta} + \omega^2 \sin \theta = 0$$

$$\text{选能量守恒: } E = p \dot{q} - L \Rightarrow \frac{\dot{\theta}^2}{2} - \omega^2 \cos \theta = \text{const}$$

$$\text{记 } \theta_0 \text{ 为振幅, } 0 \text{ 时 } \text{const} = -\omega^2 \cos \theta_0$$

$$\dot{\theta}^2 = 4 \omega^2 \left( \sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2} \right)$$

$$= \frac{d\theta}{\sqrt{\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2}}} = \omega dt \quad u = \frac{\sin \frac{\theta}{2}}{\sin \frac{\theta_0}{2}} \quad \int_0^u \frac{du'}{\sqrt{(1-u'^2)(1-k^2 u'^2)}} = \omega t + \delta \quad (k = \sin \frac{\theta_0}{2})$$

$$\text{若 } k \ll 1: \quad \frac{t}{2\pi\sqrt{l/g}} = \left( 1 + \frac{k^2}{4} + \frac{9k^4}{64} + \dots \right)$$

圆摆摆:

$$\begin{cases} x = b(\theta + \sin \theta) \\ y = b(1 - \cos \theta) \end{cases}$$



① 此点为最低点

T 与摆长位置无关

地址：中国北京市海淀区颐和园路5号

邮编：100871

www.pku.edu.cn



10.12

produced by Xunchang

第2章 有心运动

\* 考虑 \*

$$L = T - V, \quad V = V(\vec{r} - \vec{r}_2), \quad \text{定义 } \vec{r} = \vec{r}_2 - \vec{r}_1$$

$$L = \frac{\frac{m_1+m_2}{2} \dot{\vec{r}}^2}{L_0(t, \dot{\vec{r}})} + \frac{\frac{\mu}{2} (\dot{\vec{r}})^2}{L_2(t, \vec{r}, \dot{\vec{r}})} - V(\vec{r}), \quad \mu \text{ 为约化质量 } \frac{m_1 m_2}{m_1 + m_2}$$

在球坐标  $(r, \theta, \varphi)$  下改写拉氏量  $L_2$ :

$$L_2 = \frac{\mu}{2} (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\varphi}^2) - V(r)$$

$$\text{首先 } \frac{\partial L_2}{\partial \varphi} = 0 = \frac{d}{dt} \left[ \frac{\partial L_2}{\partial \dot{\varphi}} \right] = \frac{d}{dt} (\mu r^2 \sin^2 \theta \dot{\varphi}) = 2\mu r \sin^2 \theta \dot{\varphi} \cdot \dot{r} + 2\mu r^2 \sin \theta \cos \theta \dot{\varphi} \cdot \dot{\theta} + \mu r^2 \sin^2 \theta \ddot{\varphi}$$

$$\text{易知 } p_r = \mu \dot{r}, \quad p_\theta = \mu r^2 \dot{\theta}, \quad E = \frac{\mu}{2} (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\varphi}^2).$$

$$\text{观察 } p_\theta: \frac{d}{dt} p_\theta = \frac{\partial L}{\partial \theta} = \mu r^2 \sin \theta \cos \theta \cdot \dot{\varphi}^2$$

$$\text{选取特殊坐标方向使 } \theta = \frac{\pi}{2} \Rightarrow \left. \begin{array}{l} \frac{d}{dt} p_\theta|_{t=0} = 0 \\ \dot{\theta}(t=0) = 0 \\ p_\theta(t=0) = 0 \end{array} \right\} \Rightarrow \underline{p_\theta \equiv 0, \theta \equiv \frac{\pi}{2}}.$$

因此有心运动被限制于平面内:

$$\text{选取柱坐标 } \rho, \varphi \text{ 再改写 } L_2, \quad L_2 = \frac{\mu}{2} (\dot{\rho}^2 + \rho^2 \dot{\varphi}^2) - V(\rho).$$

$$\frac{\partial L}{\partial \varphi} = \frac{d}{dt} \left[ \frac{\partial L}{\partial \dot{\varphi}} \right] = 0, \quad \underline{\mu \rho^2 \dot{\varphi} = p_\varphi \equiv C}, \quad \Rightarrow \text{有心运动角动量守恒.}$$

$$\text{同理 } E = \frac{\mu}{2} (\dot{\rho}^2 + \rho^2 \dot{\varphi}^2) + V(\rho).$$

可依此解出  $\dot{\varphi}, \dot{\rho}$ , 解出方程.

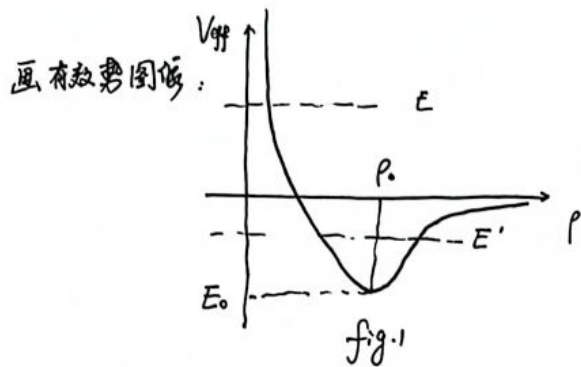
$$\text{当然也可用 Lagrangian Equation } \left\{ \begin{array}{l} \mu \ddot{\rho} = \mu \rho \dot{\varphi}^2 - \frac{\partial V}{\partial \rho} \\ \mu (\rho \ddot{\varphi} + 2\dot{\rho} \dot{\varphi}) = 0 \end{array} \right.$$

$$\text{记 } \underline{\mu \rho^2 \dot{\varphi} = J} \text{ 代入 } E \text{ 表达式: } \frac{\mu}{2} \dot{\rho}^2 + \frac{\mu}{2} \rho^2 \left( \frac{J}{\mu \rho^2} \right)^2 = E \Rightarrow \frac{\mu}{2} \dot{\rho}^2 + \frac{J^2}{2\mu \rho^2} = E - V$$

$$\Rightarrow \frac{\mu}{2} \dot{\rho}^2 + \frac{J^2}{2\mu \rho^2} + V(\rho) = E \quad \text{投影至“一维”, 仅考虑 } \rho.$$

effective potential.

$$V_{\text{eff}} = \frac{J^2}{2\mu \rho^2} + V \Rightarrow \frac{\mu}{2} \dot{\rho}^2 = E - V_{\text{eff}}.$$



称  $E_0$  为基态.

此时  $\dot{r} = 0$ , 有  $\varphi$  方向运动无  $r$  向运动.

$E = 0$  称为 Critical Energy (折射线轨道)

$E < 0$  折射线轨道;  $E > 0$  双曲线轨道.

如果从  $r(\varphi)$  处着手研究.

# 比奈 (Binet) 方程:

$$\frac{d}{dt} = \frac{d\varphi}{dt} \frac{d}{d\varphi} = \dot{\varphi} \frac{d}{d\varphi} \stackrel{\dot{\varphi}=C}{=} \frac{C}{r^2} \frac{d}{d\varphi}$$

$$\begin{cases} \dot{r} = \frac{C}{r^2} \frac{dr}{d\varphi} = -C \frac{d(\frac{1}{r})}{d\varphi} \stackrel{r=1/u}{=} -C \frac{du}{d\varphi} \\ \ddot{r} = \frac{C}{r^2} \frac{d}{d\varphi} (-C \frac{du}{d\varphi}) = -C^2 u^2 \frac{d^2 u}{d\varphi^2} \end{cases}$$

代入得:  $-\mu C^2 u^2 (\frac{d^2 u}{d\varphi^2} + u) = -f(\frac{1}{u})$  ①

$$\frac{\mu}{2} C^2 [(\frac{du}{d\varphi})^2 + u^2] + V(\frac{1}{u}) = E$$
 ②

$$\Rightarrow \frac{du}{\sqrt{(E-V)\frac{2}{\mu C^2} - u^2}} = d\varphi \quad \text{由于 } \varphi \in [0, 2\pi)$$

大多数情况下轨道不是闭合的



由  $V(r)$  决定

# Extensions (不考)  $\rightarrow V(r)$  相关

伯兰特 (Bertrand) 的工作

从第①个比奈方程出发:

$$\frac{d^2 u}{d\varphi^2} + u = -\frac{\mu}{r^2 u^2} f(\frac{1}{u}) = J(u)$$

在圆轨道上, ( $E = E_0$  见 fig. 1) 设  $u_0 = u_0$ .

取  $u' = u_0 + \xi$ ,  $\xi(\varphi) \ll u_0$  为微扰.

$$J(u_0 + \xi) = J(u_0) + J'(u_0) \xi + \frac{1}{2} J''(u_0) \xi^2 + \frac{1}{6} J'''(u_0) \xi^3 + o(\xi^4)$$

圆轨道  $\frac{d^2 u_0}{d\varphi^2} + u_0 = J(u_0)$ ,  $J(u_0) = u_0$

$$\frac{d^2 \xi}{d\varphi^2} + (1 - J'(u_0)) \xi = \frac{J''(u_0)}{2} \xi^2 + \frac{J'''(u_0)}{6} \xi^3 + o(\xi^4)$$

首先,  $\frac{d^2\xi}{d\varphi^2} + (1-J_0')\xi = 0$  需有稳定解.

记  $\beta^2 = 1 - J_0' \Rightarrow \frac{d^2\xi}{d\varphi^2} + \beta^2\xi = 0$ . 当  $\beta^2 > 0$  时取特解  $e^{\pm i\beta\varphi}$  (非正即负), 稳定  
 $\therefore$  需要  $1 - J_0' > 0$ ,  $J_0' < 1$ .

# 闭合轨道  $\Rightarrow \xi \sim \cos(\beta\varphi)$ .  $\varphi$  从  $0 \sim 2\pi$  时,  $\beta\varphi$  可变化  $2k\pi$   
 故  $\beta \in \mathbb{Q}$ , 取  $\beta = \frac{m}{n}$

$$J'(u) = -\frac{2J}{u} + \frac{J}{f} \frac{df}{du} \stackrel{u=u_0}{=} -2 + \frac{J}{f} \frac{df}{du}$$

$$\Rightarrow \beta^2 = 1 - J_0' = 3 - \frac{u_0}{f(u_0)} \cdot \left. \frac{df}{du} \right|_{u=u_0}$$

$$\Rightarrow \frac{df}{dr} = (\beta^2 - 3) \frac{f}{r}, \text{ 假设 } f = -kr^\gamma \Rightarrow f = -kr^{\beta^2-3}$$

$$\beta^2 > 0 \Rightarrow \gamma > -3.$$

下面再将  $J_0''$  /  $J_0'''$  分别加入方程, 看是否有稳定解:

$$\Rightarrow \text{结论 } \beta^2(1-\beta^2)(4-\beta^2) = 0 \quad \therefore \beta = 1 \text{ 或 } 2$$

只有  $\beta=1$  或  $\beta=2$  时, 轨道闭合且稳定.

$$\begin{array}{cc} \swarrow & \downarrow \\ f \propto r^{-2} & f \propto r \text{ (弹簧?)}. \end{array}$$



# 北京大学

设  $M = m_1 + m_2$ ,  $\mu = \frac{m_1 m_2}{m_1 + m_2}$ ;  $u = \frac{1}{r}$

$$V = -\frac{Gm_1 m_2}{r} = -GM\mu \cdot u$$

$$L = \frac{\mu}{2} v^2 + \mu GM u, \quad (\frac{C}{2} = p^2 \dot{\varphi} \text{ 守恒})$$

$$\frac{\mu c^2}{2} \left[ \left( \frac{du}{d\varphi} \right)^2 + u^2 \right] - GM\mu \cdot u = E$$

$$\text{化简为: } \left( \frac{du}{d\varphi} \right)^2 = A^2 - (B - u)^2 \quad \begin{cases} A = \left( \frac{2E}{\mu c^2} + \frac{G^2 M^2}{c^4} \right)^{\frac{1}{2}} \\ B = \frac{GM}{c^2} \end{cases}$$

$$\star \frac{du}{\sqrt{A^2 - (B - u)^2}} = \pm d\varphi \Leftrightarrow \arccos\left(\frac{u - B}{A}\right) = \pm \varphi + \alpha$$

$$\Rightarrow u = B + A \cos(\varphi \pm \alpha)$$

$$\Rightarrow r = \frac{1/B}{1 + \frac{A}{B} \cos(\varphi - \beta)}$$

$$\text{记 } e = \frac{A}{B}, p = \frac{1}{B} \quad e < 1 \text{ 时} \Rightarrow \begin{cases} p = \frac{c^2}{GM} \\ e = \sqrt{1 + \frac{2Ec^2}{G^2 M^2 \mu}} \end{cases}$$

$$\text{半长轴 } a = \frac{p}{1 - e^2}; \quad C^2 = \left( \frac{J}{\mu} \right)^2 = GMp = GMa(1 - e^2)$$

$$E = -\frac{GM\mu}{2a}$$

$$\text{Fob: } y' = p \sin \varphi' = \frac{p \sin \varphi'}{1 + e \cos \varphi'}$$

$$\frac{dy'}{d\varphi'} = 0 \Rightarrow x' = p \cos \varphi' + ae, \quad \cos \varphi' = -e$$

$$p(\varphi') = a, \quad b = \sqrt{a^2 - (ae)^2} = a\sqrt{1 - e^2}$$

$$\Leftrightarrow [b] \text{ 理有 } 1 + \frac{2Ec^2}{G^2 M^2 \mu} > 0 \Rightarrow C_{\max} = \sqrt{GMa} \mu$$

$$E = -\frac{GM}{2a} = -\frac{v^2}{2} - \frac{GM}{a} \Rightarrow v = \sqrt{\frac{GM}{a}}$$

$$F \Rightarrow U, \quad U = -\frac{GM}{r}$$



$$\oint \vec{F} \cdot d\vec{s} = - \oint \nabla u \cdot d\vec{s}$$

$$- \frac{GM}{r^2} \cdot 4\pi r^2 = - \iiint \nabla \cdot (\nabla u) \cdot dV$$

$$= - \nabla^2 u dV$$

$$\Leftrightarrow \nabla^2 u = 4\pi G \rho \quad (\text{Poisson 方程})$$

$$\int_0^{r'} \rho \cdot 4\pi r^2 dr = \iiint \frac{\nabla(\nabla u)}{4\pi G} dV = \frac{\oint \nabla u \cdot d\vec{s}}{4\pi G} = \frac{\oint (-\vec{F} \cdot d\vec{s})}{4\pi G} = -F \cdot \frac{4\pi r^2}{4\pi G}$$

$$\text{即 } \vec{F} = F \hat{n} \Rightarrow F = \frac{-GM(r)}{r^2}$$

$$\textcircled{1} e=1 \Rightarrow \vec{E}=0, \quad \rho = \frac{P}{1+\cos\psi'}$$

$$\textcircled{3} e>1 \Rightarrow E \neq 0, \quad \rho = \frac{P}{1+e\cos\psi'}$$

Laplace - Runge - Lenz vector:

$$(\vec{v} \times \vec{L} - GM\hat{e}_r)$$

$$\frac{d}{dt} (\vec{v} \times \vec{L}) \neq 0$$

$$\frac{d}{dt} (\vec{v} \times \vec{L}) = \frac{d\vec{v}}{dt} \times \vec{L} = [g(r)\hat{e}_r] \times (r \times \vec{v})$$

$$= g(r) [(\hat{e}_r \cdot \vec{v}) \vec{r} - r \vec{v}] \quad \textcircled{1}$$

$$\vec{r} \cdot \frac{d}{dt} \hat{e} = \frac{d}{dt} \left( \frac{\vec{r}}{r} \right) = \frac{\vec{v}}{r} - \frac{\vec{v} \cdot \vec{e}_r}{r^2} \vec{r} = \frac{1}{r^2} (r \vec{v} - (\hat{e}_r \cdot \vec{v}) \vec{r}) \quad \textcircled{2}$$

$$\textcircled{1} = GM \cdot \textcircled{2}$$

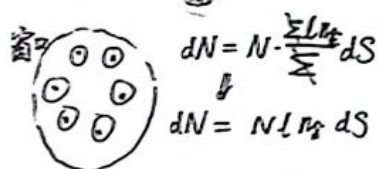
$$\star \text{ 记 } \hat{e} = \vec{v} \times \vec{L} - GM\hat{e}_r$$

$$\vec{L} \cdot \hat{e} = 0$$

$$\vec{r} \cdot \hat{e} = \vec{r} \cdot \vec{v} \times \vec{L} - GM\vec{r} \cdot \hat{e}_r = L^2 - GMr \Rightarrow r = \frac{L^2/GM}{1 + \hat{e} \cdot \hat{e}_r}$$

$$\begin{cases} |\hat{e}| = e \\ (\frac{\hat{e}}{e}) \cdot \hat{e}_r = \cos\psi' \end{cases}$$

\* Review. 最简表面

修时最简表面  $\sigma = \frac{dS}{dL}$  $\theta \rightarrow \theta + d\theta \Leftrightarrow$ 

$$\therefore \sigma = \frac{dS}{2\pi R \sin \theta d\theta}$$

以下进入新内容:

普通保守势下  $L = T - V$ 

考虑一维振动的平衡位置附近)

$$L = T - V(x).$$

当  $\left. \frac{\partial V}{\partial x} \right|_{x=x_0} = 0$ . 即不受力.对  $V(x)$  在  $x_0$  处 Taylor 展开

$$V(x) = V(x_0) + \left. \frac{\partial V}{\partial x} \right|_{x_0} (x - x_0) + \frac{1}{2} \left. \frac{\partial^2 V}{\partial x^2} \right|_{x_0} (x - x_0)^2 + o((x - x_0)^3).$$

回顾: Lagrange 量可差常数.

$$\text{而 } \left. \frac{\partial V}{\partial x} \right|_{x_0} = 0 \Rightarrow \frac{1}{2} \left. \frac{\partial^2 V}{\partial x^2} \right|_{x_0} (x - x_0)^2 \text{ 进入拉氏方程.}$$

$$L = \frac{m}{2} (\Delta \dot{x})^2 - \frac{1}{2} \left. \frac{\partial^2 V}{\partial x^2} \right|_{x_0} \Delta x^2$$

$$\frac{\Delta}{\Delta t \Delta x} \frac{m}{2} \dot{x}^2 - \frac{1}{2} \left. \frac{\partial^2 V}{\partial x^2} \right|_{x_0} x^2$$

$$= \frac{m}{2} \dot{x}^2 - \frac{1}{2} k x^2$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x} \Leftrightarrow m \ddot{x} = -kx$$

地址: 中国 北京市海淀区颐和园路5号

$$\Rightarrow m \ddot{x} + kx = 0. \text{ 记 } \frac{k}{m} = \omega^2$$

$$\Leftrightarrow \ddot{x} + \omega^2 x = 0.$$

简谐振动 (Simple Harmonic Oscillation)

写出两个通解  $A \cos(\omega t + \varphi_1)$ ;  $B \sin(\omega t + \varphi_2)$ .

$$\Rightarrow x = A \cos(\omega t + \varphi_1) + B \sin(\omega t + \varphi_2).$$

$$(= K_1 \cos \omega t + K_2 \sin \omega t).$$

通解反写为  $x = A e^{i\omega t}$ . ( $A$  也是复数)所有初始条件都会在  $A$  中.

$$A = c + di = \sqrt{c^2 + d^2} \cdot e^{i\theta} \quad (a)$$

只要定下  $a, b$  便可解得  $A, x$ 受迫振下:  $\ddot{x} + \omega_0^2 x = f(t)$ .对  $f(t)$  作 Fourier 展开:先探讨  $\ddot{x} + \omega_0^2 x = f e^{i\omega t}$  (单频)

$$\text{记 } x = A e^{i\omega t} + B e^{i\kappa t} \rightarrow \text{试探解}$$

代入得  $\kappa = \omega$ .

$$(-\omega^2) B e^{i\omega t} + \omega_0^2 B e^{i\omega t} = f e^{i\omega t}$$

$$\Leftrightarrow B = \frac{f}{\omega_0^2 - \omega^2}$$

$$\text{即 } x = A e^{i\omega_0 t} + \frac{f}{\omega_0^2 - \omega^2} e^{i\omega t}$$

从  $B$  中可看出共振为特殊情况共振时  $\omega = \omega_0$ , 取  $x = A e^{i\omega t} + B t e^{i\omega t}$  作试探.

$$\text{代入: } B \frac{d}{dt} (i e^{i\omega t} + t i e^{i\omega t}) + \omega_0^2 B t e^{i\omega t} = f e^{i\omega t}$$

$$= B (2i\omega_0 e^{i\omega t} - t \omega_0^2 e^{i\omega t}) + \omega_0^2 B t e^{i\omega t}$$

$$= 2B\omega_0 i e^{i\omega t} \stackrel{\text{等于}}{=} f e^{i\omega t}$$

$$\Rightarrow B = \frac{f}{2\omega_0 i}$$

$$\text{得 } x = Ae^{i\omega t} + \frac{f}{2i\omega_0} e^{i\omega t}$$

(右项振幅随时间单调递增)

回到共振情况:

$$x = A_1 e^{i\omega t} + A_2 e^{i\omega t}$$

$\swarrow$  本征频率       $\nwarrow$  驱动频率

形成“拍”



当  $|A_1| = |A_2| = A$  时

$$\begin{aligned} \text{Re}(x) &= A[\cos(\omega_1 t + \varphi_1) + \cos(\omega_2 t + \varphi_2)] \\ &= 2A \left[ \cos\left[\frac{\omega_1 - \omega_2}{2} t + a\right] \times \cos\left[\frac{\omega_1 + \omega_2}{2} t + b\right] \right] \end{aligned}$$

// 阻尼振动:

$$\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = 0$$

$$\text{记 } x = Ae^{\lambda t}$$

$$\Rightarrow \lambda^2 + 2\gamma\lambda + \omega_0^2 = 0 \Rightarrow$$

$$\lambda = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

①  $\gamma^2 < \omega_0^2$ : 低阻尼

$$\lambda = -\gamma \pm i\sqrt{\omega_0^2 - \gamma^2}$$

代入试探解:

$$x = A_1 e^{-\gamma t} e^{i\sqrt{\omega_0^2 - \gamma^2} t}$$

$$+ A_2 e^{-\gamma t} e^{-i\sqrt{\omega_0^2 - \gamma^2} t}$$

$$\text{Re}(x) = E e^{-\gamma t} \cos(\sqrt{\omega_0^2 - \gamma^2} t + \theta_0)$$

②  $\gamma^2 > \omega_0^2$ : 过阻尼

$$x = A_1 e^{-\gamma t} e^{\sqrt{\gamma^2 - \omega_0^2} t} + A_2 e^{-\gamma t} e^{-\sqrt{\gamma^2 - \omega_0^2} t}$$

③  $\gamma^2 = \omega_0^2$ : 临界阻尼

$$\text{试探解} \Rightarrow x = A e^{-\gamma t} + B t e^{-\gamma t}$$

代入方程:

$$\begin{cases} 2\lambda + \lambda^2 t + 2\gamma \lambda t + 2\gamma + \gamma^2 t = 0 \\ \Leftrightarrow (\lambda^2 + 2\gamma\lambda + \gamma^2)t + 2\gamma(\lambda + 1) = 0 \\ \lambda = -\gamma \text{ 时才为 } 0 \quad \checkmark \end{cases}$$

$$\Rightarrow x = A e^{-\gamma t} + B t e^{-\gamma t}$$

// 受迫阻尼振动:

$$\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = f e^{i\omega t}$$

$$x_{\text{特}} = A_{\text{特}} e^{i\omega t}$$

$$\text{后得 } A_{\text{特}} = \frac{f}{(\omega_0^2 - \omega^2) + 2\gamma i \omega}$$

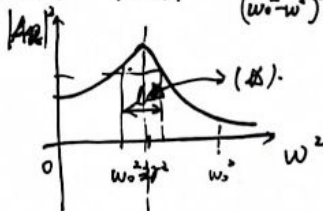
$$\therefore x = A e^{\lambda t} + A_{\text{特}} e^{i\omega t}$$

例:  $\gamma^2 < \omega_0^2$

$$x = A e^{-\gamma t} e^{i\sqrt{\omega_0^2 - \gamma^2} t} + A_{\text{特}} e^{i\omega t}$$

\* 暂态解  $T \sim \frac{1}{\gamma}$

$$|A_{\text{特}}|^2 = |A_{\text{特}}|^2 = \frac{f^2}{(\omega_0^2 - \omega^2)^2 + 4\gamma^2 \omega^2} = \frac{f^2}{(\omega_0^2 - (\omega^2 - 2\gamma^2))^2 + 4\gamma^2(\omega_0^2 - \gamma^2)}$$



$$|A_m|^2 = \frac{f^2}{4\gamma^2(\omega_0^2 - \gamma^2)}$$

$\gamma^2 \ll \omega_0^2$  时

$$A_{\text{特}}^2 \approx \frac{f^2}{(\omega_0^2 - \omega^2)^2 + 4\gamma^2 \omega_0^2}$$

要使其变为  $\frac{A^2}{2} \Leftrightarrow$

$$\omega^2 - \omega_0^2 = \pm 2\gamma \omega_0 \Leftrightarrow (\omega \approx \omega_0)$$

$$\Delta \omega = \omega - \omega_0 = \gamma$$

$$\therefore \ell = 2\Delta \omega = 2\gamma \quad (\text{此处})$$

$$\text{又 } Q = \frac{\omega_0}{2\gamma}, \quad Q \uparrow \text{ 则 } \omega \text{ 峰尖}$$



上课时间: 10.28

小振动 (第2课时)

开多维小振动:

假设  $N$  个粒子,  $f$  个广义坐标  $q_1 \sim q_f$ 可知  $\left. \frac{\partial V}{\partial q_\alpha} \right|_0 = 0$  ( $\alpha=1 \sim f$ ) 定义了零点与平衡位置.

$$V(q) = V(0) + \left. \frac{\partial V}{\partial q_\alpha} \right|_0 (q_\alpha) + \frac{1}{2} \frac{\partial^2 V}{\partial q_\alpha \partial q_\beta} q_\alpha q_\beta + o(q^3)$$

$$V(q) \stackrel{\text{为书方便+约去常数}}{=} \frac{1}{2} V_{\alpha\beta} q_\alpha q_\beta = \frac{1}{2} (q_1, \dots, q_f) (\vec{V}) \begin{pmatrix} q_1 \\ \vdots \\ q_f \end{pmatrix}$$

$$(\vec{q}^T \vec{V} \vec{q})$$

$$T = T_2 = \frac{1}{2} a_{\alpha\beta}^{(0)} \dot{q}_\alpha \dot{q}_\beta$$

(不显含  $t$ , 所有二阶导数)

$$a_{\alpha\beta}^{(0)} = m_i \left. \frac{\partial x_i}{\partial q_\alpha} \right|_0 \left. \frac{\partial x_i}{\partial q_\beta} \right|_0 \triangleq T_{\alpha\beta}$$

$$\mathcal{L} = T_2 - V$$

$$= \frac{1}{2} T_{\alpha\beta} \dot{q}_\alpha \dot{q}_\beta - \frac{1}{2} V_{\alpha\beta} q_\alpha q_\beta$$

根据 Lagrangian Equations:

$$\text{对 } \alpha: T_{\alpha\beta} \ddot{q}_\beta + V_{\alpha\beta} q_\beta = 0$$

$$\square(\ddot{\phantom{x}}) + \square(\phantom{x}) = 0$$

如果存在解  $q_\beta$  满足  $(\vec{q}) = (\vec{b}) e^{i\omega t}$ 

$$\vec{T} \vec{b} (-\omega^2) + \vec{V} \vec{b} = 0$$

$$(\vec{V} - \omega^2 \vec{T}) \vec{b} = 0$$

(显然我们不考虑  $\vec{b} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ )要有非零解, 显然  $|\vec{V} - \omega^2 \vec{T}| = 0$ .故只有特定的  $\omega$  才存在振动式为  $\omega$  的一元  $f$  次方程:考虑第  $r$  个可能解  $\omega_r$ :

$$(\vec{V} - \omega_r^2 \vec{T}) \vec{b}_r = 0$$

$[c \vec{b}_r + d \vec{b}_r - \text{均可求解 (考虑重根)}]$

 $\omega_r$  与  $\vec{b}_r$  (在无量纲时) 一一对应

$$\text{考虑 } \vec{q} = \sum_{k=1}^f \vec{b}_k e^{i\omega_k t}$$

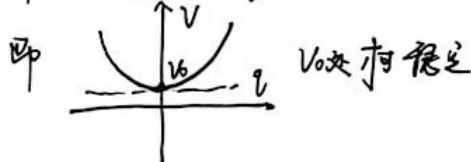
下考察  $\omega$  的分类:

- $\omega_r > 0$ , 有振荡解
- $\omega_r = 0$ , 不振动 (在某些情况下)
- $\omega_r < 0$ , 系统不稳定 (发散)

$$\vec{b}_r^T \vec{V} \vec{b}_r = \omega_r^2 (\vec{b}_r^T \vec{T} \vec{b}_r) \geq 0$$

动能系数矩阵满足  $(T)_{20}$

$$\text{即要求 } \vec{b}_r^T \vec{V} \vec{b}_r \geq 0$$



下分析解:

$$\omega_r > 0, \vec{q}_r = \vec{b}_r (c_1 e^{i\omega_r t} + c_2 e^{-i\omega_r t})$$

将  $\vec{b}_r$  化为实振幅, 将复数因子并入  $c_1, c_2$  中.

$$\text{再化为 } \vec{b}_r (|c_1| e^{i(\omega_r t + \theta_1)} + |c_2| e^{-i(\omega_r t + \theta_2)})$$

$$\begin{aligned} \text{Re}(\vec{q}_r) &= \vec{b}_r [(|c_1| \cos(\omega_r t + \theta_1) + |c_2| \cos(\omega_r t + \theta_2))] \\ &= \vec{b}_r (\square \cos \omega_r t + \Delta \sin \omega_r t) \\ &\triangleq \vec{f}_r \cos(\omega_r t + \varphi) \end{aligned}$$



写成矩阵形式:

$$\begin{aligned} \vec{r} &= (c_1 \vec{f}_1, c_2 \vec{f}_2, \dots, c_f \vec{f}_f) \begin{pmatrix} \cos(\omega_1 t + \varphi_1) \\ \vdots \\ \cos(\omega_f t + \varphi_f) \end{pmatrix} \\ &= \vec{B} \begin{pmatrix} c_1 \cos(\omega_1 t + \varphi_1) \\ \vdots \\ c_f \cos(\omega_f t + \varphi_f) \end{pmatrix} \quad (*) \end{aligned}$$

$\vec{B}$  代表一系列方向

$$\vec{r} = \begin{pmatrix} B_{11} c_1 \cos(\omega_1 t + \varphi_1) & \dots & + B_{1f} c_f \cos(\omega_f t + \varphi_f) \\ \vdots & & \\ B_{f1} c_1 & \dots & + B_{ff} c_f \end{pmatrix}$$

下研究 (\*) 的特征:

$$\vec{b}_r = \hat{b}_r \cos(\omega_r t + \varphi_r)$$

$$\Rightarrow (\vec{V} - \omega_r^2 \vec{T}) \hat{b}_r = 0$$

称  $\hat{b}_r$  为本征向量

$$\begin{cases} \vec{V} \hat{b}_1 = \omega_1^2 \vec{T} \hat{b}_1 \\ \vec{V} \hat{b}_2 = \omega_2^2 \vec{T} \hat{b}_2 \\ \vdots \end{cases} \quad \rightarrow \text{作积}$$

$$\begin{aligned} (\hat{b}_1^T \vec{V}) &= \omega_1^2 (\hat{b}_1^T \vec{T}) \\ (\vec{V}^T = \vec{V}) & \quad (\vec{T}^T = \vec{T}) \end{aligned}$$

$$\begin{aligned} (\hat{b}_1^T \vec{V}) \hat{b}_2 &= \omega_2^2 (\hat{b}_1^T \vec{T}) \hat{b}_2 \\ \Rightarrow \hat{b}_1^T \vec{V} \hat{b}_2 &= \omega_2^2 (\hat{b}_1^T \vec{T}) \hat{b}_2 \\ \omega_2^2 (\hat{b}_1^T \vec{T}) \hat{b}_2 &= \omega_1^2 (\hat{b}_1^T \vec{T}) \hat{b}_2 \end{aligned}$$

$$\Leftrightarrow (\omega_1^2 - \omega_2^2) \hat{b}_1^T \vec{T} \hat{b}_2 = 0 \Leftrightarrow \omega_1 \neq \omega_2 \text{ 时 } \hat{b}_1^T \vec{T} \hat{b}_2 = 0$$

若  $\omega_1 = \omega_2 \Rightarrow \hat{b}_1^T \vec{T} \hat{b}_1$  可以不为 0.

即  $\vec{B}$  中每个矢量关于  $\vec{T}$  正交.

同理  $\vec{B}$  中每个矢量关于  $\vec{V}$  正交

$$\begin{aligned} (\vec{V} - \omega_r^2 \vec{T}) \hat{b}_r &= 0 \\ \hat{b}_\alpha^T (\vec{V} - \omega_r^2 \vec{T}) \hat{b}_r &= 0 \end{aligned}$$

$$\alpha \neq r \text{ 时 } \hat{b}_\alpha^T \vec{V} \hat{b}_r = 0$$

$\omega_2^2 = \omega_1^2$  时用 Gram-Schmidt 方法



## 小振动(2)

 Review.  $T a_p \ddot{q}_p + V a_p q_p = 0$ 

 试探  $q = \vec{b} e^{i\omega t}$ 

$$\Rightarrow (V - \omega^2 T) \vec{b} = 0$$

 有非平凡解  $\Rightarrow |V - \omega^2 T| = 0$ 

 称  $\omega_1^2 \sim \omega_r^2$  为简正频率

 $\omega_g^2 \sim \vec{b}_r$  (实数)

$$q = \sum_r \vec{b}_r \cos(\omega_r t + \varphi_r), \quad \vec{b}_r = \begin{pmatrix} b_{1r} \\ b_{2r} \\ \vdots \\ b_{nr} \end{pmatrix}$$

$$(A_1) \vec{q} = (b_1, b_2, \dots, b_f) \begin{pmatrix} \cos(\omega_1 t + \varphi_1) \\ \vdots \\ \cos(\omega_f t + \varphi_f) \end{pmatrix}$$

简正坐标

$$(A_2) \vec{q} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \cdot \begin{pmatrix} b_{11} \cos(\omega_1 t + \varphi_1) + \dots + b_{f1} \cos(\omega_f t + \varphi_f) \\ \vdots \\ b_{1f} \cos(\omega_1 t + \varphi_1) + \dots + b_{ff} \cos(\omega_f t + \varphi_f) \end{pmatrix}$$

(A1) 比 (A2) 简洁得多, 是基底取简正频率对应的  $b$  的结果。

 称  $B$  为模态矩阵

 讨论正交性:  $\vec{b}_\alpha^T T \vec{b}_\beta = 0$ , 如若  $\alpha \neq \beta$ .

$$V \vec{b} = \omega^2 T \vec{b} \Leftrightarrow T^{-1} V \vec{b} = \omega^2 \vec{b}$$

 Q: 为何可判断  $\vec{b}_\alpha^T \vec{b}_\beta = 0$ ?

 理由:  $T^{-1}, V$  为对称矩阵, 而

$$T^{-1} V \text{ 不是对称 } \left( \text{如 } \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \right)$$

 记  $\vec{b}_\alpha^T T \vec{b}_\alpha = k_\alpha$ 

$$\Rightarrow B^T T B = \begin{pmatrix} k_1 & & 0 \\ & \ddots & \\ 0 & & k_f \end{pmatrix} = \sum_{i=1}^f k_i \cdot I$$

 同理:  $B^T V B = \sum_{i=1}^f d_i \cdot I$  (记  $\vec{b}_\alpha^T V \vec{b}_\alpha = d_\alpha$ ).

$$|B^T T B| = \prod_i k_i = |B|^2 |T|$$

 而已知  $T$  的所有本征值,  $|T| \neq 0 \Rightarrow |B| \neq 0$ 

 即  $B^{-1}$  存在.

 $\Rightarrow T \ddot{q} + V q \approx$  变为  $B^T T B (B^{-1} \ddot{q}) + B^T V B (B^{-1} q) = 0$ 

$$\text{而 } B^{-1} q = \begin{pmatrix} \cos(\omega_1 t + \varphi_1) \\ \vdots \\ \cos(\omega_f t + \varphi_f) \end{pmatrix} = u$$

$$\Rightarrow (B^T T B) \ddot{u} + (B^T V B) u = 0$$

$$\text{即 } \begin{pmatrix} k_1 & & \\ & \ddots & \\ & & k_f \end{pmatrix} \ddot{u} + \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_f \end{pmatrix} u = 0.$$

$$\text{变化: } \begin{cases} k_1 \ddot{u} + d_1 u = 0 \\ \vdots \\ k_f \ddot{u} + d_f u = 0 \end{cases} \Rightarrow \omega_g^2 = \frac{d_r}{k_r}.$$

$$[\text{数学基础}] \quad B^{-1} = \frac{1}{|B|} \begin{pmatrix} C_{11} & \dots & C_{1f} \\ \vdots & & \vdots \\ C_{f1} & \dots & C_{ff} \end{pmatrix}$$

$$C_{\alpha\beta} = (-1)^{\alpha+\beta} M_{\alpha\beta}$$

余式

\* 简正坐标:

$$u = B^{-1} q = \begin{pmatrix} b_{11} q_1 & \dots & b_{1f} q_f \\ \vdots & & \vdots \\ b_{f1} q_1 & \dots & b_{ff} q_f \end{pmatrix}$$

从这开始  
不要直接与  $\cos$

理力→求小振动解的一般形式.

① 系数矩阵  $T, V$

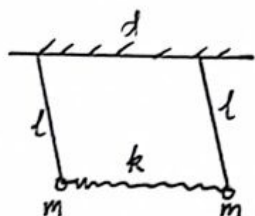
② 固有频率  $\omega$

③ 本征向量 / 简正模 / 模态矩阵

④ 合成运动解

⑤ 由初条件确定系数.

例: 耦合摆.



取广义坐标  $x_1, x_2$  (偏离平衡位置的水平量)

$$①: V = mgy_1 + y_2 + \frac{k}{2} [\sqrt{(d+x_1)^2 + y_2^2} - d]^2 + V_0$$

$$T = \frac{m}{2} (\dot{x}_1^2 + \dot{y}_1^2) + \frac{m}{2} (\dot{x}_2^2 + \dot{y}_2^2) = \frac{m}{2} (\dot{x}_1^2 + \dot{x}_2^2)$$

$$\text{下展开: } y_i = l - \sqrt{l^2 - x_i^2} \stackrel{(\text{Taylor})}{\approx} \frac{l}{2} \left( \frac{x_i}{l} \right)^2$$

可看出  $y$  为二阶小量  $\Rightarrow$  展至二阶时, 舍去  $y \cdot x / y^2$  项.

$$V = V_0 + \frac{mgy_1 + y_2}{2l} + \frac{k}{2} [ |d+x_1| - d ]^2$$

$$= V_0 + \frac{1}{2} \left( k + \frac{mg}{l} \right) (x_1^2 + x_2^2) - kx_1x_2$$

$$\Rightarrow \vec{T} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

$$\vec{V} = \begin{pmatrix} k + \frac{mg}{l} & -k \\ -k & k + \frac{mg}{l} \end{pmatrix}$$

$$②: |V - \omega^2 T| = 0$$

$$\Rightarrow \begin{vmatrix} k + \frac{mg}{l} - \omega^2 m & -k \\ -k & k + \frac{mg}{l} - \omega^2 m \end{vmatrix} = 0$$

$$\Rightarrow \left( k + m \left( \frac{g}{l} - \omega^2 \right) \right)^2 - k^2 = 0$$

$$\Rightarrow 2km \left( \frac{g}{l} - \omega^2 \right) + m^2 \left( \frac{g}{l} - \omega^2 \right)^2 = 0$$

$$\omega_1^2 = \frac{g}{l}; \quad \omega_2^2 = \frac{g}{l} + 2\frac{k}{m}$$

$$③ (V - \omega^2 T) b_i = 0$$

$$\text{代入 } \omega_1^2: \begin{pmatrix} k & -k \\ -k & k \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} = 0$$

$$\Leftrightarrow \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} = 0$$

$$\Rightarrow \text{取 } b_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\text{代入 } \omega_2^2 \Rightarrow b_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$④ q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} f_1 \cos(\omega_1 t + \phi_1) \\ f_2 \cos(\omega_2 t + \phi_2) \end{pmatrix}$$

$$⑤ \begin{cases} x_1(0) \\ x_2(0) \end{cases} \begin{cases} \dot{x}_1(0) \\ \dot{x}_2(0) \end{cases} \text{ 代入 } \checkmark$$

即求解小振动的标准步骤





# Review: 耦合摆

得到  $\vec{q} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} A_1 \cos(\omega_1 t + \varphi_1) \\ A_2 \cos(\omega_2 t + \varphi_2) \end{pmatrix}$

$\vec{q} \approx \begin{pmatrix} 1 \\ 1 \end{pmatrix} f_1 \cos(\omega_1 t + \varphi_1) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} f_2 \cos(\omega_2 t + \varphi_2)$

简正坐标  $B^T \vec{q} = \begin{pmatrix} f_1 \cos(\omega_1 t + \varphi_1) \\ f_2 \cos(\omega_2 t + \varphi_2) \end{pmatrix}$

$B^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$\therefore \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \frac{q_1 + q_2}{2} \\ \frac{q_1 - q_2}{2} \end{pmatrix}$

例: N 个小球、弹簧 (m, k)

在环上.



取位移量  $x_1 \sim x_N$

$(\ddot{x}_n = \frac{k}{m}(x_{n+1} + x_{n-1} - 2x_n))$  (\*)

① 写出拉氏力.  $L = \sum \frac{1}{2} m \dot{x}_n^2 - \sum \frac{k}{2} (x_n - x_{n+1})^2$

$\vec{r} = \begin{pmatrix} m & & \\ & m & \\ & & \ddots \\ & & & m \end{pmatrix};$

$\vec{V} = \begin{pmatrix} 2k & -k & 0 & \dots & -k \\ -k & 2k & -k & & 0 \\ 0 & -k & 2k & -k & \\ \vdots & & \ddots & \ddots & \vdots \\ -k & 0 & \dots & -k & 2k \end{pmatrix}$

记  $\frac{k}{m} = \omega_0^2$

(\*) 处可先“猜”解:  $x_n = A e^{i\omega t + ipn}$

$\Rightarrow -\omega^2 = \omega_0^2 (e^{-ip} + e^{ip} - 2)$

$x_{n+N} = x_n \Rightarrow e^{ipN} = 1,$

$\rightarrow pN = 2\pi l \Rightarrow p = \frac{2\pi l}{N}, l = 0 \sim N-1$

$\Rightarrow \omega^2 = 2\omega_0^2 (1 - \cos p)$

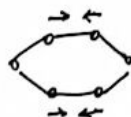
(应用) 苯环



N=6



$\rightarrow \omega^2 = 0 / \omega_0^2 / 3\omega_0^2 / 4\omega_0^2$



$F = 3 \cdot k a x, \omega^2 = 3\omega_0^2 (= \frac{3k}{m})$

-共2种模式(独立振动模式)



$\omega^2 = 4\omega_0^2$   
(1种)

Extension: 受迫 & 阻尼,

$\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = f e^{i\omega t}$

$\gamma > 0.$

$x_{\text{特}} = A_{\text{特}} e^{i\omega t};$

$A_{\text{特}}^2 = \frac{f^2}{(\omega^2 - (\omega_0^2 - \gamma^2))^2 + 4\gamma^2 \omega^2}.$

共振时  $\omega = \omega_0,$

$A_{\text{特}} = \frac{f}{2\gamma \omega_0}$

$\vec{f} = 2\gamma \dot{x} = -2\gamma \omega_0 A_{\text{特}} e^{i\omega t} = -f e^{i\omega t}$

非共振时  $\omega \neq \omega_0,$

$A_{\text{特}} = |A_{\text{特}}| e^{i\theta_0}$

$|A_{\text{特}}| = \frac{1}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\gamma^2 \omega^2}}, \quad \cos \theta_0 = \frac{\omega_0^2 - \omega^2}{\sqrt{\dots}};$

$\sin \theta_0 = \frac{-2\gamma \omega}{\sqrt{\dots}}$

$W_{\text{振}} = \int_0^{2\pi} f_{\text{振}} \dot{x} dt = -2\gamma \omega \pi |A_{\text{特}}|^2$



$$W_{\text{rot}} = \int_0^{2\pi} \dot{\varphi} dt = 2\pi\omega \left| A_{\varphi} \right|^2$$

期中考后：刚体


[先导课]

矢量(对于刚体?)：分量  $x_i$ ,  $i=1, 2, 3$

基矢： $\vec{r} = x_i \hat{e}_i = r \hat{e}_r$

矩阵： $\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$   $\vec{r}^T = (x, y, z)$ .

二维中：

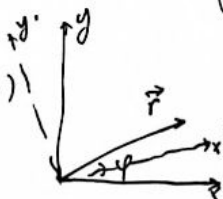


$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

旋转(绕 z 轴)：



$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

分量式： $x'_i = O_{ij} x_j$  (求和).

$$\text{即 } \vec{r}' = \vec{O} \vec{r}$$

张量  $X^{\mu}$

二阶张量  $T_{ij}' = O_{i\alpha} O_{j\beta} T_{\alpha\beta} = O_{i\alpha} T_{\alpha\beta} O_{j\beta}^T$

$$\vec{T}' = \vec{O} \vec{T} \vec{O}^T$$

$$\vec{T} = T_{ij} \hat{e}_i \hat{e}_j \quad T_{ij} = e_i \vec{T} e_j$$


 11.9  
刚体

刚体运动的数学基础,

 标量:  $m$ 

 矢量: 分量形式  $x_i$ 

 基矢形式  $x_i \hat{e}_i$ 

 矩阵形式  $\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ 

考虑被动观点下的坐标变换

$$x'_i = O_{ij} x_j$$

显然, 坐标变换后矢量长度不变

$$\text{故有: } x'_i x'_i = x_i x_i$$

$$x'_i x'_i = (O_{ia} x_a)(O_{ib} x_b) = (O_{ia} O_{ib}) x_a x_b$$

$$\Rightarrow \underline{O_{ia} O_{ib} = I}$$

$$(\text{或写作 } O_{ai} O_{ib} = \delta_{ab}).$$

下考虑基矢变换:

$$x'_i \hat{e}'_i = O_{ij} x_j \hat{e}'_i = x_j (O_{ij} \hat{e}'_i)$$

$$\underline{\text{题设}} \quad x_i (O_{ji} \hat{e}'_j)$$

$$\text{而 } x'_i \hat{e}'_i = x_i \hat{e}_i \text{ (已知),}$$

$$O_{ji} \hat{e}'_j = \hat{e}_i = O_{ij}^T \hat{e}'_j$$

$$\text{乘以 } O_{ij}: O_{ij} \hat{e}_i = O_{ij} O_{ij}^T \hat{e}'_j$$

$$\Leftrightarrow \underline{O_{ij} \hat{e}_i = \hat{e}'_j}$$

$$\text{即 } \underline{x'_j = O_{ji} x_i} \text{ 的基矢表述}$$

下考虑矩阵表述:

$$\text{矩阵形式 } \vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \vec{r}' = \vec{O} \vec{r} \text{ (写作 } r' = Or)$$

$$\underline{r'^T r' = (Or)^T (Or) = r^T O^T O r = r^T r}$$

$$\Rightarrow \underline{O^T O = I}$$

 专题: 二阶张量  $T_{ij} = A_i B_j$  类似于两个一阶张量之积

$$\star \underline{T'_{ij} = O_{ia} O_{ib} T_{ab}} \text{ 即证明:}$$

$$A'_i B'_j = (O_{ia} A_a)(O_{jb} B_b)$$

$$= (O_{ia} O_{jb}) A_a B_b$$

考虑基矢变换:

$$T_{ij} \hat{e}_i \hat{e}_j = (A_i \hat{e}_i)(B_j \hat{e}_j) = (A_i B_j) \hat{e}_i \hat{e}_j$$

$$\searrow T_{ij} (O_{ai} \hat{e}'_a)(O_{bj} \hat{e}'_b)$$

$$= (O_{ai} O_{bj}) T_{ij} \hat{e}'_a \hat{e}'_b$$

$$= T_{ab} \hat{e}'_a \hat{e}'_b$$

$$\text{矩阵表述: } \left. \begin{aligned} T'_{ij} &= O_{ia} T_{ab} O_{jb} \\ O_{ia} T_{ab} &= x'_{ib} \end{aligned} \right\} T' = (O)(T)O$$

张量运算的规则:

$$/ \text{加: } T_{ij} + U_{ij} = V_{ij}$$

$$/ \text{数乘: } s \cdot T_{ij} = V_{ij} \quad \swarrow \text{每个元素乘以 } s.$$

$$/ \text{外积: } \underline{T_i U_{jk} = A_i (B_j C_k) = V_{ijk}}$$

/ 内积:  $T_{ij} U_i = V_j$ ;  $T_{ijr} = V_i$

(所数 -2)

例:  $\vec{r} = x_i \hat{e}_i$ ,  $\vec{s} = y_j \hat{e}_j$

$$\begin{aligned} \vec{r} \cdot \vec{s} &= (x_i \hat{e}_i) \cdot (y_j \hat{e}_j) \\ &= x_i y_j \hat{e}_i \cdot \hat{e}_j = x_i y_j \delta_{ij} \\ &= x_i y_i \end{aligned}$$

// 证明  $\delta_{ab}$  是张量:

即证  $O_{ia} O_{jb} \delta_{ab} = \delta_{ij}$

$\Leftarrow O_{ia} O_{ja} = \delta_{ij}$

先前已得  $O_{ia} O_{ja} = I_{ij} \Rightarrow$  得证!

/ 矢量与张量的内积:

1 左乘:  $\vec{r} \cdot \vec{T}$

$$\begin{aligned} &= (x_i \hat{e}_i) \cdot (T_{jk} \hat{e}_j \hat{e}_k) \\ &= x_i T_{jk} \hat{e}_i \cdot \hat{e}_j \hat{e}_k \\ &= x_i T_{jk} \delta_{ij} \hat{e}_k \\ &= x_i T_{ik} \hat{e}_k (= r^T \vec{T}) \end{aligned}$$

1 右乘:  $\vec{T} \cdot \vec{r}$

$$\begin{aligned} &= (T_{jk} \hat{e}_j \hat{e}_k) \cdot (x_i \hat{e}_i) \\ &= T_{jk} x_i \hat{e}_j \hat{e}_k \cdot \hat{e}_i \\ &= T_{jk} x_i \delta_{ji} \hat{e}_k \\ &= T_{ji} x_i \hat{e}_j \end{aligned}$$

/ 两次内积:

$$T_{ij} U_j = \vec{T} : \vec{U}$$

$$\begin{aligned} &= (T_{ij} \hat{e}_i \hat{e}_j) : (U_{ab} \hat{e}_a \hat{e}_b) \\ &= T_{ij} U_{ab} (\hat{e}_i \hat{e}_j) : (\hat{e}_a \hat{e}_b) \\ &= T_{ij} U_{ab} (\hat{e}_j \cdot \hat{e}_a) (\hat{e}_i \cdot \hat{e}_b) \\ &= T_{ij} U_{ab} \delta_{ja} \delta_{ib} \\ &= (T_{ij} U_{jb}) \delta_{ib} \\ &= \text{Trace } (T U) \end{aligned}$$

则得的基本性质:

任意两质点间距始终不变.

约束方程个数  $\frac{n(n-1)}{2} = N_s$

$n \gg 1$  时,  $N_s \gg 3n$

可见:  $N_s$  中的约束并非彼此独立.





北京大学  
PEKING UNIVERSITY

网络

11-13

≡ 讲课回顾：基矢与坐标的变换。

一般地，设变换矩阵  $C^T \neq C^T$ ，即非正交变换。

由  $x'_i = C_{ij} x_j$  (后指标求和)。

$$\Rightarrow C_{ki}^{-1} x'_i = C_{ki} C_{ij} x_j = \delta_{kj} x_j = x_k \quad (\text{为逆变换})$$

$$\text{对基矢, } x_i \hat{e}_i \equiv C_{ij} x'_j \hat{e}_i = x'_j (C_{ij} \hat{e}_i)$$

$$\text{而 } x_i \hat{e}_i = x'_j \hat{e}'_j \Leftarrow C_{ij} \hat{e}_i = \hat{e}'_j \quad (\text{前指标求和})$$

$$\text{变换基矢为: } \hat{e}'_i = C_{ji}^{-1} \hat{e}_j = (C^{-1})_{ij} \hat{e}_j$$

$$\text{特殊地, 正交变换 } O^T = O^T \Rightarrow \hat{e}'_i = O_{ij} \hat{e}_j$$

思考以下过程：正交坐标下。

$$\hat{e}'_i = (C^{-1})_{in} \hat{e}_n + (C^{-1})_{in} \hat{e}_n$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = C \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{cases} x_1 = C_{11}^{-1} \\ x_2 = C_{21}^{-1} \end{cases}$$

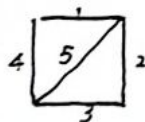
$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = C \left[ C_{11}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_{21}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \quad ((C^{-1})_{in} \sim C_{in}^{-1})$$

思考形式上，C变换含义。

刚体，6个自由度， $\frac{N(N-1)}{2} \gg 3N$  个约束方程！

$\Rightarrow$  这些约束方程 不相互独立。

例：2维正方形。

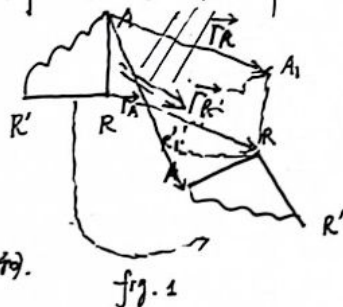


坐标  $4 \times 2 = 8$

约束  $\geq 5$  个即可足够。

刚体运动 = 平动 + 转动

例：(怎么这么像跳舞的洗衣机？ $dm$  为  $\frac{Cue}{x}$ )



下证：只要质点选在刚体上以“看起来像刚体的一部分”，各部分转动参量  $\varphi, \omega$  一致。

证：以 R 为参考点。

$$d\vec{r}_A = d\vec{r}_R + d\vec{\varphi} \times (\vec{R}A_1)$$

$$d\vec{r}_{A'} = d\vec{r}_R + d\vec{\varphi} \times (\vec{R}R_{A'})$$

$$\text{以 } R' \text{ 参考: } d\vec{r}_A = d\vec{r}_{A'} + d\vec{\varphi}' \times (\vec{R}'A_1)$$

代入  $d\vec{r}_A$  和  $d\vec{r}_{A'}$   $\Rightarrow$

$$d\vec{\varphi} \times (\vec{R}A_1) = d\vec{\varphi}' \times (\vec{R}R_{A'}) + d\vec{\varphi}' \times (\vec{R}'A_1)$$

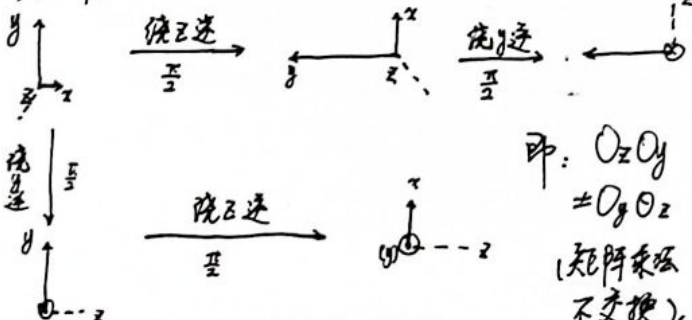
$$\text{移项} \Rightarrow d\vec{\varphi} \times \vec{R}'A_1 = d\vec{\varphi}' \times \vec{R}'A_1$$

由于  $R', A_1$  任取  $\Rightarrow d\vec{\varphi} = d\vec{\varphi}'$   
[思路：两不同参考点]

转动能否交换？

⊗ 小角度可。

反例：



即： $O_z O_y \neq O_y O_z$   
(矩阵乘法不交换)。

地址：中国北京市海淀区颐和园路5号

邮编：100871 www.pku.edu.cn



而对于无限小转动: 下证  $d\vec{\sigma} = d\varphi \hat{n}$  为矢量.

设  $\vec{\sigma} = \vec{I} + d\vec{\sigma}$  (无限小转动)

$$\vec{\sigma}_1 \cdot \vec{\sigma}_2 = \vec{I} + d\vec{\sigma}_1 + d\vec{\sigma}_2 + \frac{d\vec{\sigma}_1 \cdot d\vec{\sigma}_2}{2}$$

$\vec{\sigma}_2 \cdot \vec{\sigma}_1$  舍去  $d\vec{\sigma}_2 \cdot d\vec{\sigma}_1$  后与上式同! = 阶小量舍!

$\therefore \vec{\sigma}_1 \cdot \vec{\sigma}_2 = \vec{\sigma}_2 \cdot \vec{\sigma}_1$  满足交换律!

1 欧拉定理: 定点转动  $\Leftrightarrow$  定轴转动

即必存在  $\vec{\sigma} \cdot \vec{x} = x$  的解.

即证:  $\vec{\sigma}$  存在本征值 1:

$$\vec{\sigma}\vec{\sigma}^T = \vec{I} \Leftrightarrow \det(\vec{\sigma}) \det(\vec{\sigma}^T) = 1$$

$\Rightarrow$  讨论  $\det(\vec{\sigma}) = 1$  时.

$$\vec{\sigma}\vec{\sigma}^T - \vec{I} = \vec{I} - \vec{I} = \vec{0}$$

$$\vec{\sigma}(\vec{\sigma}^T - \vec{I}) = \vec{I} - \vec{I} = \vec{0}$$

$$\Rightarrow \det(\vec{\sigma}) \det(\vec{\sigma}^T - \vec{I}) = \det(\vec{I} - \vec{I})$$

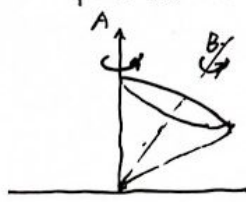
$$\Leftrightarrow \det(\vec{I} - \vec{\sigma}^T) = \det(\vec{I} - \vec{\sigma})$$

注意到  $\det(\vec{I} - \vec{\sigma}) = (-1)^3 \det(\vec{I} - \vec{\sigma}) = \det(\vec{I} - \vec{\sigma})$

$$\Leftrightarrow \det(\vec{I} - \vec{\sigma}) = 0$$

$\Rightarrow$  存在本征值 1 (Trivially)

几种复杂情况:



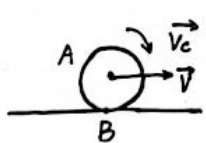
取平动参考系.

~~并不存在参考系与  $\vec{v}$  平行~~

~~使整个运动化为定轴转动.~~

~~不可以把 B 看成转轴~~

- mys



平面平行运动 (运动限制在平面内)

B 瞬时转轴  $\Rightarrow$  瞬心

给质心 A 一个速度  $\vec{v}$  可化为定轴转动.

1 (4) 欧拉推论:

$$\vec{\sigma}^T = \vec{I} - d\vec{\sigma}$$

$$\Leftrightarrow d\vec{\sigma}^T = -d\vec{\sigma}$$

刚体动力学:

对单个质点  $\vec{J} = m\vec{r} \times \vec{v}$ ,  $\vec{\omega} \times \vec{J}$

刚体  $\vec{J} = \iiint \vec{r} \times \vec{v} dm = \iiint \vec{r} \times (\vec{\omega} \times \vec{r}) dm$

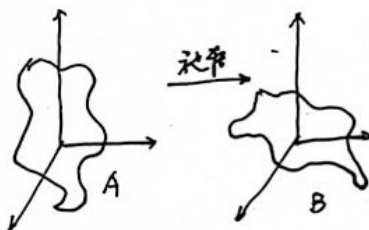
$$= \iiint [\vec{\omega} r^2 - \vec{r}(\vec{r} \cdot \vec{\omega})] dm$$

$$= \left[ \iiint (r^2 \vec{I} - \vec{r}\vec{r}) dm \right] \cdot \vec{\omega}$$

定义  $\vec{I} = \iiint (r^2 \vec{I} - \vec{r}\vec{r}) dm$

为刚体本身性质  $\Rightarrow$  惯量张量.

例:



$$\vec{I}_B = \vec{\sigma} \vec{I}_A \vec{\sigma}^T, \vec{I}_B \text{ 是 } \vec{I} \text{ 的对称张量}$$

例:  $\vec{I} = I_{\mu\nu} \hat{e}_\mu \hat{e}_\nu$

$$I_{\mu\nu} = \hat{e}_\mu \cdot \vec{I} \cdot \hat{e}_\nu$$

而  $\hat{e}_\mu \cdot \iiint (r^2 \vec{I} - \vec{r}\vec{r}) dm \cdot \hat{e}_\nu$

$$= \iiint (r^2 \delta_{\mu\nu} - x_\mu x_\nu) dm$$

互换  $\mu, \nu$ ,  $\delta_{\mu\nu} = \delta_{\nu\mu}$ ,  $x_\mu x_\nu = x_\nu x_\mu$  一致!

得证!

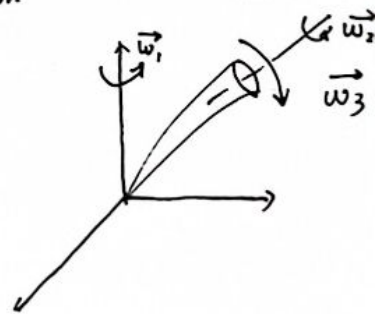
例:  $I_{xx} = \iiint (r^2 - x^2) dm = \iiint (y^2 + z^2) dm$

$$I_{xy} = - \iiint xy dm$$

$$\vec{J} = \vec{I} \cdot \vec{\omega}$$

以欧拉角

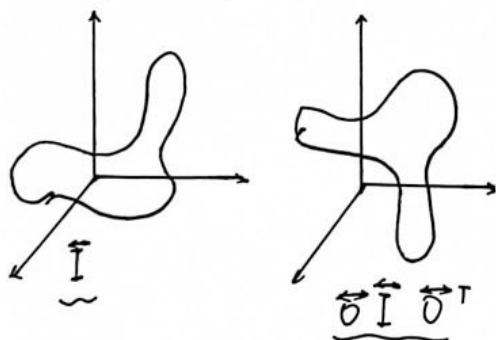
描述刚体运动.



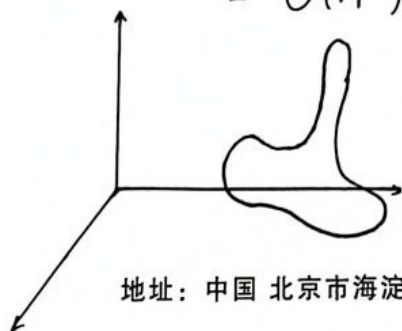
回顾:  $\vec{I} = \iiint (r^2 \vec{I} - \vec{r} \vec{r}) dm$  $\vec{J} = \vec{I} \cdot \vec{\omega}$  一般与  $\vec{\omega}$  不平行.// 寻找平行条件: 定义刚体主矩, 使得  $\vec{I} \cdot \vec{\omega} \parallel \vec{\omega}$ .即,  $\vec{I} \cdot \vec{\omega} = \lambda \vec{\omega} \Leftrightarrow \vec{\omega} = \omega \hat{n}$ , $(\vec{I} - \lambda \vec{I}) \cdot \hat{n} = 0$  有非平凡解// 线性结论: 必然存在正交矩阵  $O$ s.t.  $OIO^T = \text{diag}\{I_1, I_2, I_3\} \triangleq D$  $\Leftrightarrow IO^T = O^T D$ ,  $O^T = (\hat{n}_1, \hat{n}_2, \hat{n}_3)$ 则  $\vec{I}' = \begin{pmatrix} I_1 & & \\ & I_2 & \\ & & I_3 \end{pmatrix}$ 讨论:  $I_1, I_2, I_3$  不同:  $\hat{n}_1, \hat{n}_2, \hat{n}_3$  取正交归一向量即可两者同:  $\hat{n}_i, \hat{n}_j$  进行 Schmidt 正交化:

物理上代表某种对称性.

三者同: 球对称性.



\* 证明转动矩阵的合理性.

 $\vec{I}$  表达式中,  $(Or)(Or)^T$   
 $= O(r r^T) O^T \quad \checkmark$  $\vec{I}$  变化:  
由于参考点选  
取导致  $\vec{r}$  变化.

地址: 中国 北京市海淀区颐和园路5号

主轴坐标系下  $\vec{I} = \begin{pmatrix} I_1 & & \\ & I_2 & \\ & & I_3 \end{pmatrix}$ 在任意转轴  $\hat{n}$  下:

$$I_n = \iiint \rho^2 dm = \iiint [r^2 - (\vec{r} \cdot \hat{n})^2] dm$$

$$\Leftrightarrow r^2 \sim \vec{n} \cdot \vec{r}^2 \vec{I} \cdot \vec{n}$$

$$\therefore I_n = \hat{n} \cdot \iiint (r^2 \vec{I} - \vec{r} \vec{r}) dm \cdot \hat{n} \\ = \hat{n} \cdot \vec{I} \cdot \hat{n}$$

设  $\hat{n}$  在主轴  $\hat{n}_1, \hat{n}_2, \hat{n}_3$  下的投影矢量  $\hat{n}' = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ 

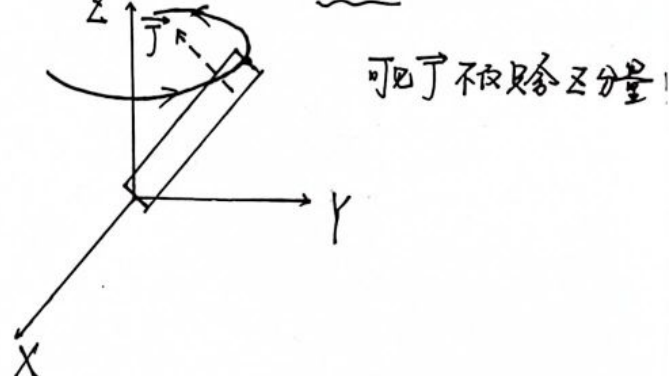
$$\text{则 } I_n' = \alpha^2 I_1 + \beta^2 I_2 + \gamma^2 I_3.$$

// 角动量:

$$\vec{J} = \vec{I}' \cdot \vec{\omega} \xrightarrow{\text{转置}} \begin{pmatrix} \omega_1 I_1 \\ \omega_2 I_2 \\ \omega_3 I_3 \end{pmatrix}$$

$$J_n = \vec{J} \cdot \hat{n} = (\vec{\omega} \cdot \vec{I}) \cdot \hat{n} = (\omega_n \hat{n} + \omega_k \hat{k}) \cdot \vec{I} \\ = \omega_n (\hat{n} \cdot \vec{I} \cdot \hat{n}) + \omega_k \hat{k} \cdot \vec{I} \cdot \hat{n}$$

$$J = \omega_n I_n + \omega_k I_{kn}$$

可见  $\vec{J}$  不取各  $Z$  分量!

// 动能,

$$T = \frac{1}{2} \iiint v^2 dm = \frac{1}{2} \iiint \vec{v} \cdot (\vec{\omega} \times \vec{r}) dm$$

$$= \frac{1}{2} \vec{\omega} \cdot \iiint (\vec{r} \times \vec{v}) dm = \frac{1}{2} \vec{\omega} \cdot (\vec{I} \cdot \vec{\omega})$$

$$\text{详细: } \frac{1}{2} \sum_{i=1}^3 I_i \omega_i^2 = \frac{1}{2} \sum_{i=1}^3 \frac{J_i^2}{I_i}$$

邮编: 100871

www.pku.edu.cn

考虑转动:  $\vec{I}' \Rightarrow OIO^T, \vec{\omega} \Rightarrow O\omega$

$$\vec{\omega}'^T \vec{I}' \vec{\omega}' = (O\omega)^T (OIO^T) (O\omega)$$

$$= \omega^T I \omega \Rightarrow \text{动能是标量}$$

故只需考虑主轴下的动能选取即可。

比较:

质点组

刚体

动能  $\sum_i \frac{1}{2} m_i v_i^2$

$\sum_i \frac{1}{2} I_i \omega_i^2$

(角)动量  $\sum_i m_i \vec{r}_i \times \vec{v}_i$

$\vec{I} \cdot \vec{\omega}$   
 $\vec{I} = \sum_i m_i (\vec{r}_i \otimes \vec{r}_i)$

可见形式上有某种相似性!

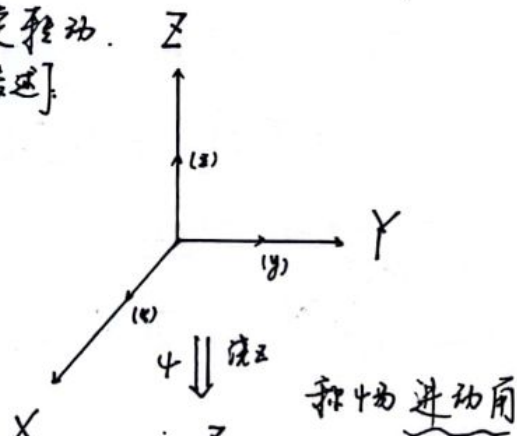
回到上节课内容:

1 欧拉角

11 坐标系  $\Rightarrow$  刚体本体系, 共6个自由度, 3个是位移。

剩下3个是转动。

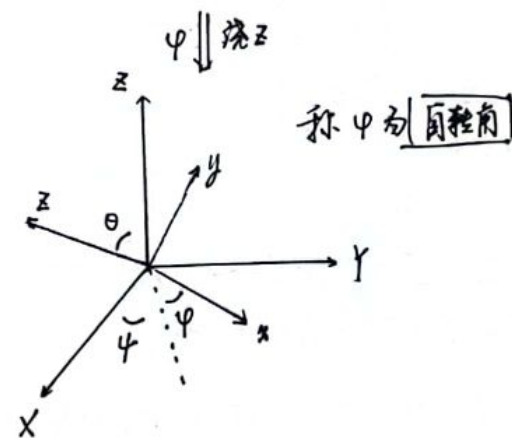
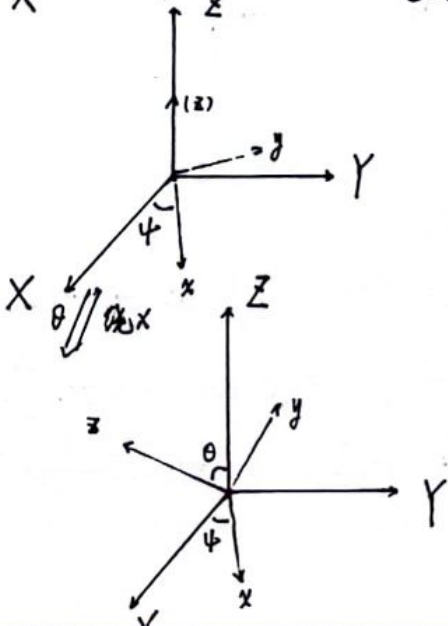
[欧拉表述]



称  $\psi$  为进动角

称  $\theta$  为章动角

章动角



称  $\psi$  为进动角

记其分别对应的矩阵为:

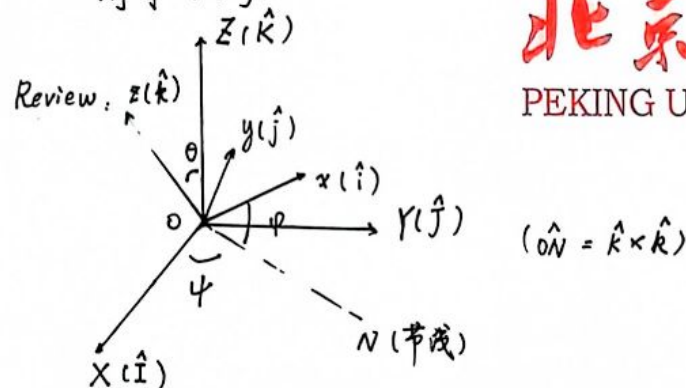
$$\vec{D} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \vec{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

$$\vec{B} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{B} \vec{C} \vec{D} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$



11-20 刚体动力学



通过  $\vec{a} = \vec{B} \vec{C} \vec{D} \cdot \vec{A}$  可得坐标变换至本体系下。  
同理可变换角速度  $\vec{\omega} \rightarrow \vec{\omega}$ , 假定给  $\dot{\theta}, \dot{\phi}, \dot{\psi}$ 。

$$\vec{\omega} = \dot{\theta} \cdot \vec{ON} + \dot{\phi} \vec{k} + \dot{\psi} \vec{k}$$

$$\text{而 } \vec{ON} = \cos\phi \cdot \hat{i} - \sin\phi \cdot \hat{j} \quad (\text{图})$$

对  $\vec{k}$  分解:  $\vec{k} \perp \vec{ON}, \vec{k} \perp \vec{ON}$

$\vec{k}$  在  $\vec{k}$  上投影后余下  $\vec{j}$ ,  $\vec{j} \perp \vec{ON}$

$$\therefore \vec{k} = \cos\theta \cdot \vec{k} + \sin\theta \sin\phi \cdot \hat{i} + \sin\theta \cos\phi \cdot \hat{j}$$

$$\Rightarrow \begin{cases} \omega_1 = \dot{\psi} \sin\theta \sin\phi + \dot{\theta} \cos\phi \\ \omega_2 = \dot{\psi} \sin\theta \cos\phi - \dot{\theta} \sin\phi \\ \omega_3 = \dot{\psi} \cos\theta + \dot{\phi} \end{cases}$$

由上也可看出  $\vec{\omega}$  作为瞬时转动轴。

$$\omega_1 = \vec{\omega} \cdot \hat{i}, \quad \omega_2 = \vec{\omega} \cdot \hat{j}, \quad \omega_3 = \vec{\omega} \cdot \hat{k}$$

$$\text{其中 } \vec{\omega} \cdot \hat{k} = \dot{\psi} + \dot{\theta} \vec{ON} \cdot \hat{k} + \dot{\phi} \hat{k} \cdot \hat{k} \neq \dot{\psi}$$

这是由于选择的并非正交系, 不可点乘投影。

转动能量,

$$T = \frac{1}{2} I_i \omega_i^2 = \frac{1}{2} [(I_1 - I_2) (\dot{\theta} \cos\phi + \dot{\psi} \sin\theta \sin\phi)^2 + I_2 (\dot{\theta}^2 + \dot{\psi}^2 \sin^2\theta) + I_3 (\dot{\psi} \cos\theta + \dot{\phi})^2]$$

$$\mathcal{L} = T - V, \quad \text{可得 } \left\{ \frac{\partial \mathcal{L}}{\partial \dot{\theta}}, \frac{\partial \mathcal{L}}{\partial \dot{\phi}}, \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \right\} = \dots$$

角动量定理,  $\vec{J} = \vec{N}$ , 主轴系非惯性系!

地址: 中国 北京市海淀区颐和园路5号

# 北京大学

PEKING UNIVERSITY

主轴系下  $\vec{r} = x_i \hat{e}_i$

$$\frac{d\vec{r}}{dt} = \left( \frac{d}{dt} x_i \right) \hat{e}_i + x_i \frac{d}{dt} (\hat{e}_i)$$

$$= \frac{d}{dt} x_i \hat{e}_i + x_i \vec{\omega} \times \hat{e}_i$$

$$= \frac{d}{dt} \vec{r} + \vec{\omega} \times \vec{r} \quad \left( \frac{d}{dt} \text{ 随体微分} \right)$$

同理对  $\vec{J} = J_i \hat{e}_i$ :

$$\frac{d\vec{J}}{dt} = \vec{N} \Rightarrow \vec{J} \frac{d}{dt} \vec{\omega} = I_i \omega_i \hat{e}_i$$

整理有:

$$\begin{cases} I_1 \dot{\omega}_1 - (I_2 - I_3) \omega_2 \omega_3 = N_1 \\ I_2 \dot{\omega}_2 - (I_3 - I_1) \omega_3 \omega_1 = N_2 \\ I_3 \dot{\omega}_3 - (I_1 - I_2) \omega_1 \omega_2 = N_3 \end{cases}$$

在以下三种情形下有解:

① Euler 情形:  $\vec{N} = 0$  (自由进动)

② Lagrangian 情形:  $I_1 = I_2$ , 重心在  $x$  轴 ( $z$  轴上)

③  $I_1 = I_2 \neq I_3$ , 重心在  $x_1 x_2$  平面内。

④: 不妨  $\vec{\omega} = \omega \hat{e}_1$  沿  $x$  轴。

$$\text{则 } I_1 \dot{\omega}_1 - (I_2 - I_3) \omega_2 \omega_3 = 0$$

$$\text{有 } \omega_1 = \text{const}; \quad \omega_2 = \omega_3 = 0, \text{ 而 } \vec{\omega} = \omega_1 \hat{e}_1 \therefore (\omega_2 = \omega_3 = 0)$$

当  $\vec{\omega}$  不沿主轴, 讨论特殊情形

$$I_1 = I_2 = I \neq I_3: \begin{cases} I \dot{\omega}_1 - (I - I_3) \omega_2 \omega_3 = 0 \\ I \dot{\omega}_2 = (I_3 - I) \omega_3 \omega_1 \\ I_3 \dot{\omega}_3 = 0 \Rightarrow \omega_3 = \text{const} \end{cases}$$

$$\Rightarrow \ddot{\omega}_1 = - \left( \frac{I - I_3}{I} \right) \omega_3^2 \omega_1$$

$$\Rightarrow \omega_1 = \Omega \cos \left( \frac{I - I_3}{I} \omega_3 t + \beta_1 \right)$$

邮编: 100871

www.pku.edu.cn

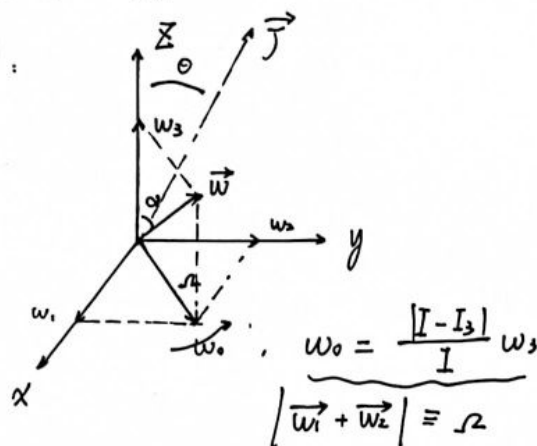


对称性解得:  $\omega_2 = \Omega \cos\left(\frac{|I_3 - I|}{I} \omega_3 t + \beta_2\right)$

$\omega_1, \omega_2$  代入 ① 式有:  $\begin{cases} \Omega_1 = \Omega_2 = \Omega \\ \beta_1 = \beta_2 + \frac{\pi}{2} = \beta \end{cases}$

$$\Rightarrow \begin{cases} \omega_1 = \Omega \cos\left(\frac{|I - I_3|}{I} \omega_3 t + \beta\right) \\ \omega_2 = \Omega \sin\left(\frac{|I - I_3|}{I} \omega_3 t + \beta\right) \\ \omega_3 = \text{const} \end{cases}$$

画图:



$$\tan \alpha = \frac{\Omega}{\omega_3} = \text{const}$$

分析  $\vec{J} = \vec{I} \cdot \vec{\omega} = I_3 \omega_3 \hat{k} + I(\omega_1 \hat{i} + \omega_2 \hat{j})$

拉格朗日  $I_3 \omega_3 \hat{k} + I \Omega \hat{p}$

$$\tan \theta = \frac{I \Omega}{I_3 \omega_3}; \quad \begin{cases} I < I_3, \theta < \alpha \\ I > I_3, \theta > \alpha \end{cases}$$

取空间系  $\vec{J}$  与  $\vec{z}(\hat{k})$  重合.

则  $\vec{\omega}$  在  $\vec{J}$  方向投影为  $\dot{\psi}$ ,  $\omega_3$  方向 ( $\hat{k}$ ) 投影为  $\dot{\psi}$

$I < I_3, \theta < \alpha$   $\begin{cases} \dot{\psi} > 0 \\ \dot{\psi} < 0 \end{cases}$  (顺逆时针)

$I > I_3, \theta > \alpha$   $\begin{cases} \dot{\psi} > 0 \\ \dot{\psi} > 0 \end{cases}$

由  $\begin{cases} \omega_1 = \dot{\psi} \sin \theta \sin \varphi + \dot{\theta} \cos \varphi \\ \omega_2 = \dot{\psi} \sin \theta \cos \varphi - \dot{\theta} \sin \varphi \end{cases} \Rightarrow \text{本题下 } \dot{\theta} = 0, \theta = \text{const}$

$\Rightarrow$  取  $\dot{\psi} > 0$  之时,  $\dot{\psi} = \frac{\Omega}{\sin \theta}$

11.23

刚体动力学

# 回顾：自由进动

(接上节课)  $\tan \theta = \frac{I_2 \omega_2}{I_3 \omega_3} = \frac{I}{I_3} \tan \alpha = \text{const.}$

依此可得  $\dot{\theta} = 0$ ,  $\theta = \text{const}$

同时有:  $\dot{\psi} = \frac{\omega_3}{\sin \theta}$ ;  $\omega_3 = \frac{I_3 - I}{I} \omega_3$

$\dot{\psi} = \omega_3 - \dot{\psi} \cos \theta = \frac{I - I_3}{I} \omega_3$ , 与  $\omega_3$  反号.

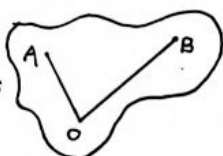
// 辨析概念:  $\dot{\psi}$  为绕  $\hat{k}$  轴转动的角速度.

$\omega_3$  为绕  $\hat{k}$  轴转动的角速度. 两者不相同!

例: 考虑如下刚体.

$$\vec{V}_{O \rightarrow A} = \vec{\omega}_{OA} \times \vec{OA}$$

$$\vec{V}_{A \rightarrow B} = \vec{\omega}_{AB} \times \vec{AB}$$



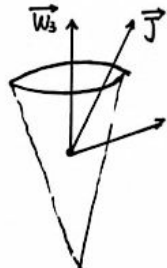
虽然  $\vec{V}_{OA} = -\vec{V}_{AB}$ , 然而  $\vec{OA}$ ,  $\vec{AB}$  无大小/相关联是

任意的  $\Rightarrow$  故  $\vec{\omega}_{OA}$  与  $\vec{\omega}_{AB}$  不是一回事.

解释: 物理中  $B \rightarrow A$  /  $A \rightarrow B$  分析时不在一参考系, 瞬时轴不能直接从 A 变为 B, 一般需要平动/转动变换.

称上述过程为 正则进动

图解:

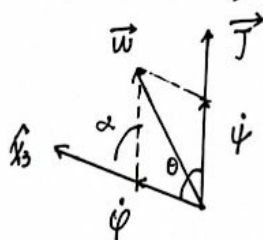


质心选于参考点处.

方便处理 ( $V=0$ ).

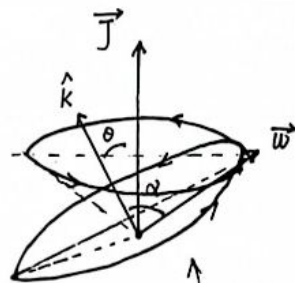
// 当  $I_3 < I$  时:

// 回顾上节课的矢量投影:



北京大学

PEKING UNIVERSITY



考虑在上图所示的圆锥:

在  $I_1 = I_2 = I < I_3$  时:  $\omega_3$  和  $\dot{\psi}$  同号.

$\omega$  绕了: 空间锥面;

$\omega$  绕  $\hat{k}$ : 本体锥面.

$$\frac{\dot{\psi}}{\omega_3} = \frac{\sin \alpha}{\sin(\alpha - \theta)} \Leftrightarrow \dot{\psi} \sin(\alpha - \theta) = \omega_3 \sin \alpha.$$

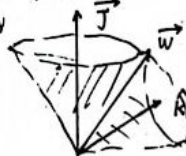
上式两边同乘  $L_{OA}$ , 得: 两者瞬时切线(点)处速度相同, 即不发生相对滑动.

$\Rightarrow$  空间锥面绕本体锥面纯滚.

称为“内卷”

同理, 在  $I_1 = I_2 = I > I_3$  时:

引入“外卷”概念.



(这个对)  
(这个也是内卷).

// 天文领域应用: 地球 极移

即地球并非完美的球体.

即地球经纬度随时间变化.

$$\frac{2\pi}{\omega_3} \approx 300 \text{ days}$$

(精确计算发现  $\approx 14 \text{ months}$  ... 差得远??)

// “椭球”思想引入: 惯量椭球

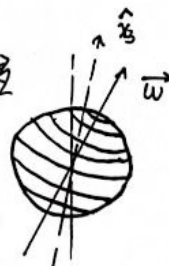
对非对称的三轴刚体,  $\vec{I} = \text{diag}\{I_1, I_2, I_3\}$

定义  $F(\vec{r}) = \vec{r} \cdot \vec{I} \cdot \vec{r}$ , 其中  $\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\text{记 } F(\vec{r}) = I_1 x^2 + I_2 y^2 + I_3 z^2 \equiv F_0.$$

//

此定义决定了大致曲线形状.

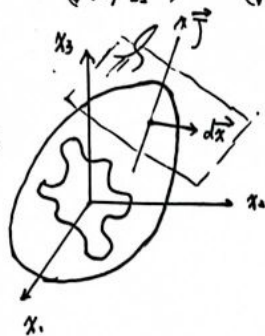


化简参数方程得:

$$\frac{x_1^2}{(\sqrt{F_0/I_1})^2} + \frac{x_2^2}{(\sqrt{F_0/I_2})^2} + \frac{x_3^2}{(\sqrt{F_0/I_3})^2} = 1$$

意义:

可以很快给出  
刚体大致的  
形状。



简化计算:

① 沿  $\hat{n}$  轴的转动惯量:

$$I_n = \hat{n} \cdot \vec{I} \cdot \hat{n} = \frac{F_0}{r^2} \quad (\hat{n} = \frac{\vec{r}}{r})$$

$$I_n \propto \frac{1}{r^2}$$

② 沿  $\vec{\omega}$  的角动量: ( $\vec{\omega} = \omega \cdot \hat{n}$ )

$$J_\omega = (\frac{\vec{\omega}}{\omega}) \cdot \vec{I} \cdot \vec{\omega} = \frac{\omega F_0}{r^2} \propto \frac{1}{r^2}$$

③ 角动量方向 (梯度):

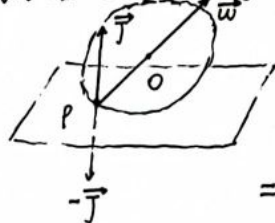
$$\nabla_\omega F(\vec{\omega}) = \begin{pmatrix} \frac{2\omega_1}{(\sqrt{F_0/I_1})^2} \\ \frac{2\omega_2}{(\sqrt{F_0/I_2})^2} \\ \frac{2\omega_3}{(\sqrt{F_0/I_3})^2} \end{pmatrix} \begin{matrix} \propto \omega_1 I_1 \\ \propto \omega_2 I_2 \\ \propto \omega_3 I_3 \end{matrix} \quad (*)$$

$$(*) = \frac{2}{F_0} \vec{J}$$

在上图的切平面中,  $d\vec{r} \cdot \vec{J} = 0$ .

// 三轴刚体的动力学:

1. 自由进动:  $\vec{N} = 0, \vec{J} = \text{const}$



OP 在角动量方向的投影

$$\begin{aligned} \vec{OP} \cdot \frac{\vec{J}}{J} &= \vec{r} \cdot \frac{\vec{J}}{J} \\ &= (r \frac{\vec{\omega}}{\omega}) \cdot \frac{\vec{I} \cdot \vec{\omega}}{J \omega} \omega \end{aligned}$$

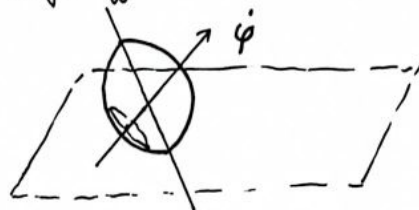
$$= \sqrt{F_0} \cdot \frac{\vec{\omega} \cdot \vec{I} \cdot \vec{\omega}}{J} \cdot \frac{1}{\sqrt{I_\omega \omega^2}}$$

$$= \sqrt{F_0} \cdot \frac{\sqrt{2T}}{J} \equiv \text{const}$$

即 O 到切面距离恒定  
绝迹

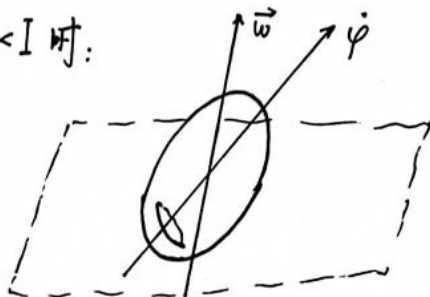
称之为 Poinsot 描迹

$I_3 > I$  时:



( $I_3 > I$ ) 自转与进动方向相反。

$I_3 < I$  时:



本体系角动量的演化:

$$2T = \sum_i I_i \omega_i^2 = \frac{L_1^2}{I_1} + \frac{L_2^2}{I_2} + \frac{L_3^2}{I_3}$$

$$(L_i = I_i \omega_i)$$

$$\Rightarrow \begin{cases} L_1^2 + L_2^2 + L_3^2 = \text{const} \\ \frac{L_1^2}{I_1} + \frac{L_2^2}{I_2} + \frac{L_3^2}{I_3} = 1 \Rightarrow \text{也是椭圆} \end{cases}$$



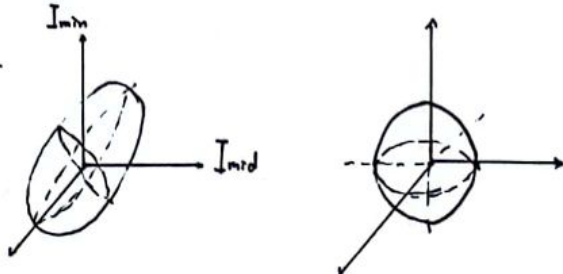
11.27

刚体力学 & 哈密顿力学开课

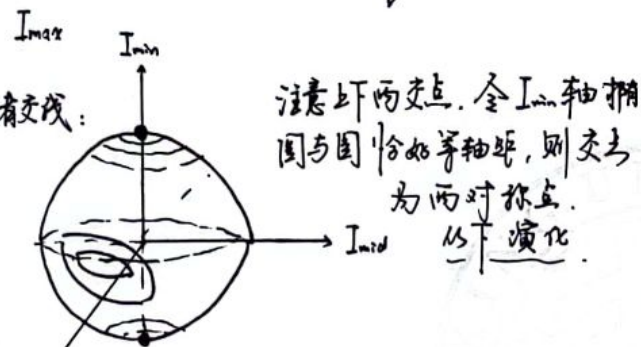
\* 接上节课:

$$\begin{cases} \frac{L_1^2}{2T L_1} + \frac{L_2^2}{2T L_2} + \frac{L_3^2}{2T L_3} = 1 & (A) \\ L_1^2 + L_2^2 + L_3^2 = \text{const} & (B) \end{cases}$$

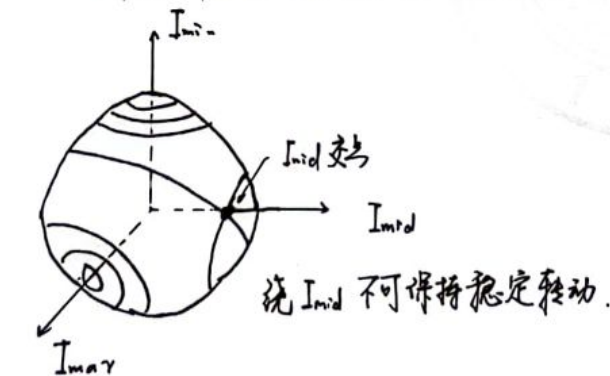
(A) 简图:



两者交线:



\* 特别分析  $I_{mid}$  处相等轴距及演化情况:

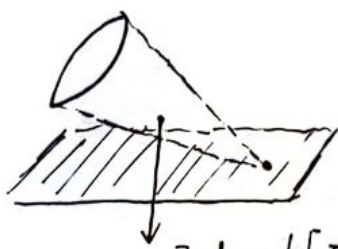


// 对称重刚体. Lagrangian 情形:

$$I_1 = I_2 = I, \quad I \neq I_3$$

由已知量导出

$$L = T - V$$



$$即 L = \frac{1}{2} [I(\dot{\theta}^2 + \dot{\psi}^2 \sin^2 \theta) + I_3(\dot{\psi} \cos \theta + \dot{\varphi})^2] - mgl \cos \theta$$

地址: 中国 北京市海淀区颐和园路5号

北京大学

PEKING UNIVERSITY

可依此定出:

$$E = \frac{1}{2} I(\dot{\theta}^2 + \dot{\psi}^2 \sin^2 \theta) + \frac{1}{2} I_3(\dot{\psi} + \dot{\varphi} \cos \theta)^2 + mgl \cos \theta$$

$$P_{\varphi} = J_3 = \frac{\partial L}{\partial \dot{\varphi}} = (I_3 \sin^2 \theta + I_3 \cos \theta) \dot{\psi} + I_3 \cos \theta \dot{\varphi}$$

$$P_{\theta} = J_3 = \frac{\partial L}{\partial \dot{\theta}} = I_3(\dot{\psi} \cos \theta + \dot{\varphi})$$

$P_{\varphi}$  是  $J_3$  在空间系  $Z$  轴下的投影, (即  $J_z$ )

$P_{\theta}$  是  $J_3$  在本体系  $z(\hat{k}/x_3)$  轴下的投影  $J_3$

$$\begin{cases} \dot{\psi} = \frac{J_3 - I_3 \cos \theta}{I \sin^2 \theta} \\ \dot{\varphi} = \frac{J_3}{I_3} - \frac{J_3 - I_3 \cos \theta}{I \sin^2 \theta} \cos \theta \end{cases}$$

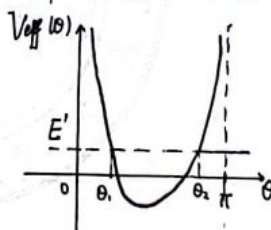
代入方程有:

$$\frac{I}{2} \dot{\theta}^2 + \frac{(J_3 - I_3 \cos \theta)^2}{2I \sin^2 \theta} + \frac{I_3^2}{2I_3} + mgl \cos \theta = E$$

$$令 V_{eff}(\theta) = \frac{(J_3 - I_3 \cos \theta)^2}{2I \sin^2 \theta} - mgl(1 - \cos \theta)$$

$$得 \frac{I}{2} \dot{\theta}^2 + V_{eff}(\theta) = E - \frac{I_3^2}{2I_3} - mgl \triangleq E'$$

$$即 t = \int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{2(E' - V_{eff})/I}}$$



$$\theta_2 \leq \theta \leq \theta_1$$

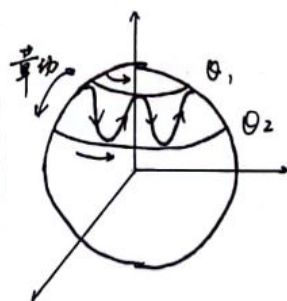
$$E' = V_{eff}(\theta)_{min}$$

否则不满足定角转动条件

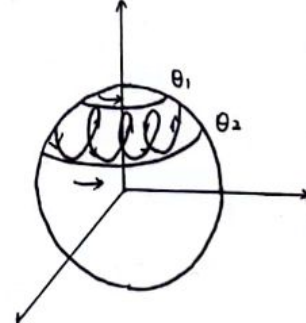
$$E' = V_{eff}(min) \Rightarrow \theta = const, \text{ 正规进动.}$$

$$E' > V_{min} \Rightarrow \psi \text{ 决定进动角速度是否改变符号}$$

$\psi$  变号:



$\psi$  变号:



$\psi$  取值范围:



邮编: 100871

www.pku.edu.cn



/ Appendix:

地球的“岁差”：周期的18.6年，振幅9.2角秒。

/ Extensions & Consequences:

则真正怎么算呢？→再清楚这套体系概念。

如：“惯性系”的具体概念？可以推出作用力么？

以下进入下一章：哈密顿力学

哈密顿量 / 正则方程。

引入：Lagrangian Equations:  $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$

$$\begin{cases} \dot{p} = f_1(\dots) \\ \dot{q} = f_2(t, q, p) \end{cases}$$

Legendre 变换的数学基础，

$$\begin{aligned} dF(x, y) &= \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy \triangleq u dx + v dy \\ &\Downarrow \\ d(F - vy) &= u dx - y dv \\ &\Downarrow \\ dG(x, v) &= u dx - y dv \triangleq \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial v} dv. \end{aligned}$$

将其代入 Lagrangian Equations 得，

$$dL = \frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial \dot{q}} d\dot{q} + \frac{\partial L}{\partial t} dt$$

$$d(\vec{p} \cdot \dot{\vec{q}} - L) = \dot{\vec{q}} d\vec{p} - \frac{\partial L}{\partial q} dq - \frac{\partial L}{\partial t} dt$$

$$\text{记 } \vec{p} \cdot \dot{\vec{q}} - L = H, \quad dH = \frac{\partial H}{\partial p} dp + \frac{\partial H}{\partial t} dt + \frac{\partial H}{\partial q} dq$$

$$\Rightarrow \begin{cases} \dot{\vec{p}} = -\frac{\partial H}{\partial q} \\ \dot{\vec{q}} = \frac{\partial H}{\partial p} \quad (\text{正则方程}) \end{cases}, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

$$\frac{\partial L}{\partial \dot{q}} \Leftrightarrow \text{广义动量}, \quad [H] = \left[ \frac{1}{T} \right] \text{ 为其量纲.}$$

$$\text{特别地, } L = L_0 + L_1 + L_2, \quad H = p_0 q_0 - L$$

$$\Rightarrow H = L_0 - L.$$

$$\text{例: } L = \frac{1}{2} m v^2 - e(\varphi - \vec{v} \cdot \vec{A}),$$

$$H = \frac{1}{2} m v^2 + e\varphi$$

$$\text{① 求 } p: \quad \vec{p} = \frac{\partial L}{\partial \vec{v}} = m\vec{v} + e\vec{A}$$

$$\text{则 } \frac{1}{2} m v^2 = \frac{(\vec{p} - e\vec{A})^2}{2m}$$

$$\therefore H = \frac{(\vec{p} - e\vec{A})^2}{2m} + e\varphi. \quad \text{正则方程 } \dot{q} = \frac{\partial H}{\partial p} = \frac{\vec{p} - e\vec{A}}{m}$$

$$\text{利用 } \nabla(\vec{A} \cdot \vec{A}) = \vec{A} \cdot \nabla \vec{A} + \vec{A} \times \nabla \times \vec{A}$$

$$\text{得, } \dot{\vec{p}} = -e\nabla\varphi + e\vec{v} \cdot \nabla \vec{A} + e\vec{v} \times (\nabla \times \vec{A})$$

$$\text{且 } \dot{\vec{p}} = m\dot{\vec{v}} + e \frac{\partial \vec{A}}{\partial t} + e\vec{v} \cdot \frac{\partial \vec{A}}{\partial x} \quad (A = A(x, t))$$

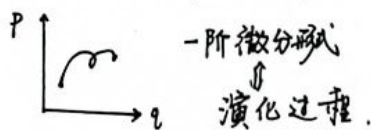
$$(\text{此式表明 } \vec{p} = m\vec{v} + e\vec{A})$$

$$\text{对比得 } m\dot{\vec{v}} = e(-\nabla\varphi - \frac{\partial \vec{A}}{\partial t}) + e\vec{v} \times (\nabla \times \vec{A})$$

$$\text{即为: } \quad \underline{\underline{\vec{E}}} \quad \underline{\underline{\vec{B}}}$$

Hamilton 正则方程:

# 回顾: Hamilton 力学定义了“相空间”:



“q与p的地位开始变得等同”.

正则方程: 
$$\begin{cases} \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha} \\ \dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \end{cases} \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

例: Kepler 轨道

由  $L \sim kL$  形式不变, 取  $\mu=1$

①  $L = \frac{\dot{r}^2}{2} + \frac{r^2 \dot{\theta}^2}{2} - V(r)$

$p_r = \frac{\partial L}{\partial \dot{r}} = \dot{r}, \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = r^2 \dot{\theta}$

②  $H = p_\alpha \dot{q}_\alpha - L = \frac{\dot{r}^2}{2} + \frac{r^2 \dot{\theta}^2}{2} + V(r)$ , 化为:  

$$\begin{cases} p_r = \frac{\partial H}{\partial \dot{r}} = \dot{r} \\ \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{r^2} \end{cases} \quad H = \frac{p_r^2}{2} + \frac{p_\theta^2}{2r^2} + V(r)$$

有:  $\dot{r} = -\frac{\partial H}{\partial p_r} = -\frac{p_r}{r^2} - \frac{\partial V}{\partial r}$

$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0$

# Note: 一定把 Hamilton 量化为只含 p, q, t 的含参多式

几个错误典例:

① 直接套用 (\*) 式:

$\dot{p}_r = -r\dot{\theta}^2 - \frac{\partial V}{\partial r}$ , 而实际上  $\dot{p}_r = r\ddot{\theta}^2 - \frac{\partial V}{\partial r}$

② 代入  $r\dot{\theta} \equiv J$ , 也会有差错 (略去).

(个人补: 会推出  $\dot{\theta} = \frac{\partial H}{\partial p_\theta} = 0$  实际上并非  $\dot{\theta} = 0$ )

// 几个先前讲过的等式:

①  $\frac{\partial L}{\partial q_\alpha} = 0 \Leftrightarrow p_\alpha$  守恒.

在体系下  $H = p_\alpha \dot{q}_\alpha - L$ ,  $\frac{\partial H}{\partial q_\alpha} = -\frac{\partial L}{\partial q_\alpha} = 0$ ,  
 $\dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha} = \frac{\partial L}{\partial q_\alpha} = 0$  ✓

地址: 中国 北京市海淀区颐和园路5号

②:  $\frac{\partial L}{\partial t} = 0 \begin{cases} \text{变换过程不含时间: } T=T_2, \quad h=T+V \\ \text{显含时间: } h=T_2-T_0+V \end{cases}$

北京大学 在 Hamilton 体系下:

PEKING UNIVERSITY

若  $\frac{\partial H}{\partial t} = 0$ ,  $\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial q_\alpha} \dot{q}_\alpha + \frac{\partial H}{\partial p_\alpha} \dot{p}_\alpha$

由  $\begin{cases} \frac{\partial H}{\partial q_\alpha} = -\dot{p}_\alpha \\ \frac{\partial H}{\partial p_\alpha} = \dot{q}_\alpha \end{cases} \rightarrow \frac{\partial H}{\partial t} = \frac{dH}{dt}$

$\frac{\partial H}{\partial t} = 0 \Leftrightarrow H \equiv \text{const.}$

例: 旋转参考系下的运动

$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \vec{\omega} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

①  $\frac{d\vec{r}}{dt} = \frac{d'\vec{r}}{dt} + \vec{\omega} \times \vec{r}$

$T = \frac{1}{2} m \left( \frac{d\vec{r}}{dt} \right)^2 \Rightarrow$

$T = \frac{m}{2} (\vec{v} + (\vec{\omega} \times \vec{r}))^2 = \frac{m}{2} v^2 + \frac{m}{2} (\vec{\omega} \times \vec{r})^2 + m \vec{v} \cdot (\vec{\omega} \times \vec{r})$

②  $L = T - V$ ,

$\vec{p} = \frac{\partial L}{\partial \dot{\vec{r}}} = \frac{\partial L}{\partial \vec{v}} = m\vec{v} + m(\vec{\omega} \times \vec{r})$

③:  $H = \vec{p} \cdot \vec{v} - L = \frac{m}{2} v^2 - \frac{m}{2} (\vec{\omega} \times \vec{r})^2 + V$

(便捷法:  $H = T_2 - T_0 + V$ )  $[(\vec{\omega} \times \vec{r}) \perp \vec{v}]$

化为  $H(p, q, t) = \frac{p^2}{2m} - \vec{p} \cdot (\vec{\omega} \times \vec{r}) + V$

④:  $\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{r}} = \vec{p} \times \vec{\omega} - \frac{\partial V}{\partial \vec{r}}$

化作分解:  $\dot{\vec{p}} = m\ddot{\vec{r}} + m\vec{\omega} \times \vec{r} + m\vec{\omega} \times \dot{\vec{r}}$

则有:  $\ddot{\vec{r}} = -\frac{\partial V}{m\partial \vec{r}} - \vec{\omega} \times \vec{r} - 2(\vec{\omega} \times \dot{\vec{r}}) - \vec{\omega} \times (\vec{\omega} \times \vec{r})$

$\frac{d'}{dt} \left( \frac{m\vec{v}^2}{2} \right) = \vec{F} \cdot \frac{d\vec{r}}{dt} = (m\ddot{\vec{r}}) \cdot \frac{d\vec{r}}{dt}$

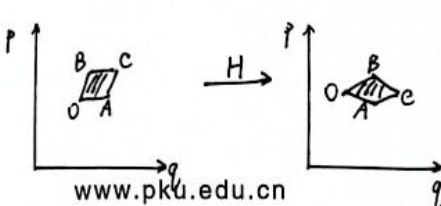
$= \left( -\frac{\partial V}{\partial \vec{r}} + m\vec{\omega} \times (\vec{\omega} \times \vec{r}) \right) \cdot \frac{d\vec{r}}{dt}$

(利用  $(\vec{\omega} \times \vec{r}) \cdot \vec{v} = 0$ )

$\therefore d' \left( \frac{m\vec{v}^2}{2} \right) = \left( -\frac{\partial V}{\partial \vec{r}} + m\vec{\omega} \times (\vec{\omega} \times \vec{r}) \right) \cdot d'\vec{r}$

$= -d \left[ V - \frac{m}{2} (\vec{\omega} \times \vec{r})^2 \right]$

// 刘维尔定理:



邮编: 100871

www.pku.edu.cn





刘维尔定理:

力学系统置于  $q, p$  中;

$$\dot{q}^i = \omega^{ik} \partial_k H, \quad \partial_j \dot{q}^i = 0.$$

$$dS' = \left| \frac{\partial(u,v)}{\partial(x,y)} \right| dS \quad (= \text{元体积}).$$

推广有:  $d^n \vec{S}_{at} = \det \frac{\partial \vec{S}(at)}{\partial \vec{S}(0)} d^n \vec{S}_0$

记  $M_{ij} = \frac{\partial \dot{q}^i(at)}{\partial \dot{q}^j(0)} = \frac{\partial \dot{q}^i(at)}{\partial \dot{q}^j(0)}$

不妨写出  $M_{ij}^{-1} = \frac{1}{M_{ij}} = \frac{\partial \dot{q}^j(0)}{\partial \dot{q}^i(at)}$

$$M^{-1} M = M_{ij}^{-1} M_{jk} = \frac{\partial \dot{q}^j(0)}{\partial \dot{q}^i(at)} \cdot \frac{\partial \dot{q}^k(at)}{\partial \dot{q}^j(0)}$$

$$= \frac{\partial \dot{q}^k(0)}{\partial \dot{q}^i(0)} = \delta_{ik};$$

以下结论:  $\ln \det M = \text{Tr} \ln M$

证明: 将  $M$  对角化,  $M = P^{-1} A P$

则  $\det M = \det A = \prod \lambda_\alpha$  (时相体积)

$$\ln \det M = \ln \prod \lambda_\alpha = \sum \ln \lambda_\alpha$$

而  $\ln M = \ln (P^{-1} A P) = P^{-1} \ln A P$

$$\text{Tr} \ln M = \text{Tr} \ln A = \sum \ln \lambda_\alpha$$

$\therefore$  证毕! (by qs)

(稍微不严谨地写作)  $\vec{M} = e^{\vec{N}}$ , 写作  $M = 1 + N + \frac{N^2}{2} \dots$

$$\frac{d}{dt} \vec{M} = \dot{N} + \frac{N}{1} \cdot \dot{N} + \frac{N^2}{2} \cdot \dot{N} + \dots = \dot{N} M$$

(实际上  $\frac{d}{dt} (\frac{N^2}{2}) = \frac{1}{2} (\dot{N} N + N \dot{N})$ )

即  $\frac{d}{dt} \ln M = \frac{d}{dt} N = \frac{\dot{N}}{M}$

$$\text{令 } J = \det \frac{\partial \vec{S}(at)}{\partial \vec{S}(0)}$$

$$\frac{dJ}{dt} = \frac{d}{dt} e^{\ln \det M} = \frac{d}{dt} e^{\text{Tr} \ln M}$$

$$= e^{\text{Tr} \ln M} \cdot \frac{d}{dt} (\text{Tr} \ln M) = J \cdot \text{Tr} \left( \frac{\dot{N}}{M} \right)$$

$$= J M_{ij}^{-1} \dot{M}_{ji} = J \frac{\partial \dot{q}^i(0)}{\partial \dot{q}^j(at)} \cdot \frac{\partial \dot{q}^j(at)}{\partial \dot{q}^i(0)}$$

地址: 中国 北京市海淀区颐和园路5号

$$[M_{ij} M_{ji} = \text{Tr}(M)]$$

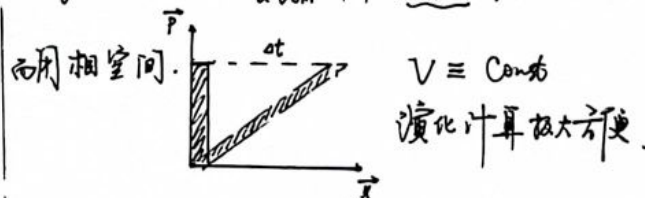
$$\begin{aligned} (\text{接上}) &= J \frac{\partial \dot{q}^i(0)}{\partial \dot{q}^j(at)} \cdot \frac{\partial \dot{q}^j(at)}{\partial \dot{q}^i(at)} \cdot \frac{\partial \dot{q}^i(at)}{\partial \dot{q}^i(0)} \\ &= J \delta_{ij} \frac{\partial \dot{q}^j(at)}{\partial \dot{q}^i(at)} = J \frac{\partial \dot{q}^i(at)}{\partial \dot{q}^i(at)} \\ &= J \cdot \frac{\partial \dot{q}^i}{\partial \dot{q}^i} \equiv 0 \end{aligned}$$

由  $\frac{dJ}{dt} = 0$  得  $J = C$ , 取  $\lim_{t \rightarrow 0} J(at) = 1$

$\therefore J = \det M = 1$ , 刘维尔定理得证;

// Examples & Extensions: “体积不变”的应用便利.

定义空间密度  $\rho = \frac{dN}{dV}$ . 分布发散  $\rightarrow (1 \sim 10^{10})$   
又有  $\rho$  涨成量子空间. 考虑分布聚集  $\rightarrow (1 \sim 10^0)$   
得理由“经典密度”  $\frac{dN}{dV_{\text{cell}}}$  计算, 误差.



4. Poisson Bracket  $[f, g]$ , 泊松括号

$$[f, g] = \frac{\partial f}{\partial q_\alpha} \frac{\partial g}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha} \frac{\partial g}{\partial q_\alpha}$$

一些重要结论:

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q} \dot{q} + \frac{\partial f}{\partial p} \dot{p} \\ &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q} \cdot \frac{\partial H}{\partial p} + \frac{\partial f}{\partial p} \cdot \left( -\frac{\partial H}{\partial q} \right) \\ &= \frac{\partial f}{\partial t} + [f, H] \end{aligned}$$

$$[H, H] = 0 \Leftrightarrow \frac{dH}{dt} = \frac{\partial H}{\partial t}$$

例: 一维谐振子.

$$L = \frac{m}{2} \dot{x}^2 + \left( \frac{k}{2} \right) x^2 = \frac{m}{2} \dot{x}^2 - \frac{k}{2} x^2$$

$$H = \frac{p^2}{2m} + \frac{k}{2} x^2$$

$$\dot{E}_k = \frac{d}{dt} \left( \frac{m}{2} \dot{x}^2 \right) = \left[ \frac{p^2}{2m}, H \right]$$

$$= -\frac{p}{m} \cdot kx = -kx \cdot \dot{x} = -\frac{d}{dt} \left( \frac{k}{2} x^2 \right)$$

$$\dot{q}_\alpha = [q_\alpha, H], \quad \dot{p}_\alpha = [p_\alpha, H]$$

举例:  $\dot{p}_\alpha = -\frac{\partial p_\alpha}{\partial p_\beta} \frac{\partial H}{\partial q_\beta} = -\delta_{\alpha\beta} \frac{\partial H}{\partial q_\beta} = -\frac{\partial H}{\partial q_\alpha}$  ✓

邮编: 100871

www.pku.edu.cn



$$[p_\alpha, q_\alpha] = 0, [p_\alpha, p_\beta] = 0,$$

$$[q_\alpha, p_\beta] = \delta_{\alpha\beta}$$

$$[f, g] = -[g, f]; [f, g+h] = [f, g] + [f, h]$$

$$[f, Cg] = C[f, g]; [f, gh] = g[f, h] + [f, g]h$$

$$\frac{\partial}{\partial t}[f, g] = [\frac{\partial f}{\partial t}, g] + [f, \frac{\partial g}{\partial t}]$$

$$\text{Jacobi 恒等式: } [f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0$$

$$\text{证明: } [f, [g, h]] \triangleq [f, k] = \frac{\partial f}{\partial q_\mu} \frac{\partial k}{\partial p_\mu} - \frac{\partial f}{\partial p_\mu} \frac{\partial k}{\partial q_\mu}$$

$$= \frac{\partial f}{\partial q_\mu} \cdot \frac{\partial}{\partial p_\mu} \left[ \frac{\partial g}{\partial q_\mu} \frac{\partial h}{\partial p_\mu} - \frac{\partial g}{\partial p_\mu} \frac{\partial h}{\partial q_\mu} \right] - \frac{\partial f}{\partial p_\mu} \cdot \frac{\partial}{\partial q_\mu} \left[ \frac{\partial g}{\partial q_\mu} \frac{\partial h}{\partial p_\mu} - \frac{\partial g}{\partial p_\mu} \frac{\partial h}{\partial q_\mu} \right]$$

$$\text{同理 } [g, [h, f]] = \frac{\partial g}{\partial q_\mu} \cdot \frac{\partial}{\partial p_\mu} \left[ \frac{\partial h}{\partial q_\mu} \frac{\partial f}{\partial p_\mu} - \frac{\partial h}{\partial p_\mu} \frac{\partial f}{\partial q_\mu} \right] - \frac{\partial g}{\partial p_\mu} \cdot \frac{\partial}{\partial q_\mu} \left[ \frac{\partial h}{\partial q_\mu} \frac{\partial f}{\partial p_\mu} - \frac{\partial h}{\partial p_\mu} \frac{\partial f}{\partial q_\mu} \right]$$

$$\text{同理 } [h, [f, g]] = \frac{\partial h}{\partial q_\mu} \cdot \frac{\partial}{\partial p_\mu} \left[ \frac{\partial f}{\partial q_\mu} \frac{\partial g}{\partial p_\mu} - \frac{\partial f}{\partial p_\mu} \frac{\partial g}{\partial q_\mu} \right] - \frac{\partial h}{\partial p_\mu} \cdot \frac{\partial}{\partial q_\mu} \left[ \frac{\partial f}{\partial q_\mu} \frac{\partial g}{\partial p_\mu} - \frac{\partial f}{\partial p_\mu} \frac{\partial g}{\partial q_\mu} \right]$$

$$\left( \frac{\partial f}{\partial q_\mu} \cdot \frac{\partial}{\partial p_\mu} \left[ \frac{\partial g}{\partial q_\mu} \frac{\partial h}{\partial p_\mu} \right] \right) \text{ 记为 } f_{\mu} \cdot g_{\mu} \cdot h_{\mu}$$

$$(1) = f_{\mu} \cdot g_{\mu} \cdot h_{\mu} - f_{\mu} \cdot g_{\mu} \cdot h_{\mu}$$

$$(2) = f_{\mu} \cdot g_{\mu} \cdot h_{\mu} - f_{\mu} \cdot g_{\mu} \cdot h_{\mu}$$

$$(3) = f_{\mu} \cdot g_{\mu} \cdot h_{\mu} - f_{\mu} \cdot g_{\mu} \cdot h_{\mu}$$

$$\text{①与⑥消, ②与⑦消, ③与⑩消}$$

$$\text{⑤与⑭消, ⑧与⑮消, ⑨与⑰消} \quad (\text{by 12})$$

例: 刚体角动量

$$\vec{r} = (x, y, z), \quad \vec{p} = (p_x, p_y, p_z)$$

$$J_x = y p_z - z p_y, \quad J_y = z p_x - x p_z, \quad J_z = x p_y - y p_x$$

$$[J_x, J_y] = \frac{\partial J_x}{\partial x} \frac{\partial J_y}{\partial p_y} - \frac{\partial J_x}{\partial p_x} \frac{\partial J_y}{\partial x} = x p_z - y p_z = J_z$$

$$\text{同理有 } [J_y, J_z] = J_x, [J_z, J_x] = J_y$$

$$\text{而本体系下取 } H = \sum_{i=1}^3 \frac{J_i^2}{2 I_i}, \text{ 此时会有 } [J_i, J_j] = -J_k$$

$$\text{由角动量守恒, 由 } \vec{r} = \{0, \psi, \varphi\}; \quad p = \{p_\theta, p_\varphi, p_\psi\}$$

$$\text{先证明以下公式: } J_i = \vec{J} \cdot \hat{e}_i$$

$$J_i = \omega \cdot I_i = \omega \cdot (\hat{e}_i \cdot \vec{I} \cdot \hat{e}_i)$$

$$= (\omega \hat{e}_i) \cdot \vec{I} \cdot \hat{e}_i = (\omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3) \cdot \vec{I} \cdot \hat{e}_i$$

$$= \vec{\omega} \cdot \vec{I} \cdot \hat{e}_i = \vec{J} \cdot \hat{e}_i \quad (\text{利用 } I_{ij} = \delta_{ij} \cdot I')$$

$$\text{由刚体: } \mathcal{L} = \sum_{i=1}^3 \frac{I_i \omega_i^2}{2} - V(\theta, \varphi, \psi)$$

$$\begin{cases} \omega_1 = \dot{\psi} \sin \theta \sin \varphi + \dot{\theta} \cos \varphi \\ \omega_2 = \dot{\psi} \sin \theta \cos \varphi - \dot{\theta} \sin \varphi \\ \omega_3 = \dot{\psi} \cos \theta + \dot{\varphi} \end{cases}$$

$$\Rightarrow p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = I_1 \omega_1 \cos \varphi - I_2 \omega_2 \sin \varphi = \frac{J_{1 \cos \varphi} - J_{2 \sin \varphi}}{2}$$

$$p_\varphi = J_1 \sin \theta \sin \varphi + J_2 \sin \theta \cos \varphi + J_3 \cos \theta$$

$$p_\psi = J_3$$

$$\Rightarrow \begin{cases} J_1 = p_\theta \cos \varphi + \frac{p_\varphi \sin \varphi}{\sin \theta} - p_\psi \cot \theta \sin \varphi \\ J_2 = -p_\theta \sin \varphi + \frac{p_\varphi \cos \varphi}{\sin \theta} - p_\psi \cot \theta \cos \varphi \\ J_3 = p_\psi \end{cases}$$

$$\Rightarrow \text{由: } [J_1, J_2] = \frac{\partial J_1}{\partial p_\theta} \frac{\partial J_2}{\partial p_\varphi} - \frac{\partial J_1}{\partial p_\varphi} \frac{\partial J_2}{\partial p_\theta}$$

$$\Rightarrow p_\varphi = -J_3, \text{ 可得角动量守恒!}$$

$$H = [J_i, H] = [J_i, \sum_{i=1}^3 \frac{I_i \omega_i^2}{2} + V]$$

$$= \frac{J_1}{I_1} [J_1, J_1] + \frac{J_2}{I_2} [J_1, J_2] + [J_1, V]$$

$$= -\frac{J_2 J_1}{I_2} + \frac{J_2 J_1}{I_2} + M_1$$

$$\Leftrightarrow I_1 \dot{\omega}_1 = -(I_3 - I_2) \omega_2 \omega_3 + M_1$$

转换为 Euler 方程!

对于函数  $y=f(x)$ : 如果  $\frac{\partial f}{\partial x}\bigg|_{x_0}=0$ , 称为“稳定点”

12-7.

≡ Review:  $\vec{J}_1 = -(\frac{1}{2} - \frac{1}{2}) \vec{J}_2 + [\vec{J}_1, \vec{V}]$

$[\vec{J}_1, \vec{V}] = \frac{\partial \vec{J}_1}{\partial \vec{p}} \cdot \frac{\partial \vec{V}}{\partial \vec{p}} - \frac{\partial \vec{J}_1}{\partial \vec{p}} \cdot \frac{\partial \vec{V}}{\partial \vec{p}}$

由于  $\frac{\partial \vec{V}}{\partial \vec{p}} = 0$ , 且有:  $(\vec{J}_1 = p_\theta \cos \varphi + p_\varphi \frac{\sin \varphi}{\sin \theta} - p_\varphi \cot \theta \sin \varphi)$

$\therefore [\vec{J}_1, \vec{V}] = -\frac{\partial \vec{J}_1}{\partial \vec{p}_\theta} \cdot \frac{\partial \vec{V}}{\partial \theta} - \frac{\partial \vec{J}_1}{\partial \vec{p}_\varphi} \cdot \frac{\partial \vec{V}}{\partial \varphi} - \frac{\partial \vec{J}_1}{\partial \vec{p}_\varphi} \cdot \frac{\partial \vec{V}}{\partial \varphi}$

※ 注意:  $-\frac{\partial \vec{V}}{\partial \theta} = N_\theta$ ,  $-\frac{\partial \vec{V}}{\partial \varphi} = N_\varphi$ ,  $-\frac{\partial \vec{V}}{\partial \varphi} = N_\varphi$

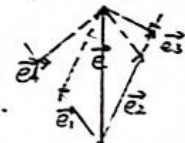
即:  $[\vec{J}_1, \vec{V}] = N_1$ ,  $N_1$  与各轴分量之合成方式与  $P_1/P_{\theta, \varphi}$  相同

※ 考虑角速度联系  $\begin{cases} \omega_1 = \dot{\varphi} \sin \theta \sin \varphi + \dot{\theta} \cos \varphi \\ \omega_2 = \dot{\varphi} \sin \theta \cos \varphi - \dot{\theta} \sin \varphi \quad (\text{参见 11-20 讲义}) \\ \omega_3 = \dot{\varphi} \cos \theta + \dot{\varphi} \end{cases}$

为何  $\varphi, \dot{\varphi}, \dot{\theta}$  前系数, 即“合成方式”不同?

从矢量的“分解”与“投影”:

$\vec{e}_1 + \vec{e}_2 = \vec{e}$ ; 投影为  $\vec{e}_3, \vec{e}_4$



考虑以上方程,  $\vec{\omega}$  是分解式:

$\vec{\omega} = \dot{\theta} \cdot \vec{e}_\theta + \dot{\varphi} \cdot \vec{e}_\varphi + \dot{\varphi} \cdot \vec{e}_\varphi = \omega_1 \vec{e}_1 + \omega_2 \vec{e}_2 + \omega_3 \vec{e}_3$

而:  $\begin{cases} p_\theta = I_1 \omega_1 \cos \varphi - I_2 \omega_2 \sin \varphi \\ p_\varphi = I_1 \omega_1 \sin \varphi + I_2 \omega_2 \cos \varphi + I_3 \omega_3 \cos \theta \\ p_\varphi = I_3 \omega_3 \end{cases}$

是角动量的投影. ( $p_\theta$  在  $\omega_1$  上已有全向投影, 而  $I_2 \omega_2$  仍在  $p_\varphi$  中. 若为分解则不会出现.)

也就是说,  $\vec{N} \neq N_\theta \vec{e}_\theta + N_\varphi \vec{e}_\varphi + N_\varphi \vec{e}_\varphi$ .

物理意义: 由角动量定理, 投影力矩仍可产生绕轴旋转

数学论证:  $N_\theta = -\frac{\partial \vec{V}}{\partial \theta} = -\frac{\partial \vec{V}}{\partial \vec{p}} \cdot \frac{\partial \vec{p}}{\partial \theta} = \frac{\partial \vec{V}}{\partial \vec{p}} \cdot (\vec{e}_\theta \times \vec{r})$

$\Rightarrow N_\theta = \vec{e}_\theta \cdot (\vec{r} \times (-\frac{\partial \vec{V}}{\partial \vec{p}}))$  为投影!

※ 更深层的数学框架: (谢鑫论述)

$p_\theta$  是  $\vec{J}$  在  $\vec{e}_\theta$  上的投影,  $\vec{J}$ : “总角动量”,  $\vec{e}_\theta$ : 绕轴

① 首先, 转动轴有其特殊性. 取  $(\theta + \varphi, \varphi, \varphi) // (\varphi, \varphi, \varphi)$  为轴“或”广义坐标“均会使“坐标轴”失去物理含义.

② 其次, 各向的  $\vec{p}$  投影量纲对应, 提供了运算便利.

$[p_\theta] = [\vec{J}]$ : 若取  $(R\theta, \varphi, \varphi)$  为轴“投影”

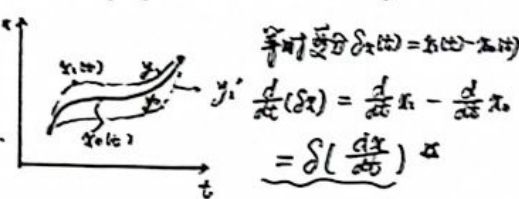
则  $p_{\theta\theta} \neq p_\theta$ : 量纲不会对应, 亦可无意义.

故上述都是在随体系坐标轴选为基础这一根上进行的有其完备性.

北京大学

PEKING UNIVERSITY

称  $y = F(f(x))$  为“泛函”, 如  $\int x(t) dt$

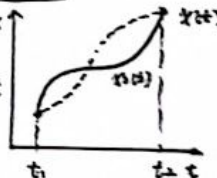


// Euler - Lagrangian Equation:

设  $f(t, x, \dot{x}) = F(t) = \int_{t_1}^{t_2} f dt$

$F(x+\delta x) - F(x) =$

$\int_{t_1}^{t_2} f(t, x+\delta x, \dot{x}+\delta \dot{x}) dt - \int_{t_1}^{t_2} f(t, x, \dot{x}) dt$   
 $= \int_{t_1}^{t_2} (\frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial \dot{x}} \delta \dot{x}) dt$



※  $\delta \dot{x} \cdot \frac{\partial f}{\partial \dot{x}} = \frac{\partial f}{\partial \dot{x}} \cdot \frac{d}{dt}(\delta x) = \frac{d}{dt}(\frac{\partial f}{\partial \dot{x}} \delta x) - \delta x \frac{d}{dt}(\frac{\partial f}{\partial \dot{x}})$

代入上式  $= \frac{\partial f}{\partial x} \delta x \bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} (\frac{\partial f}{\partial x} - \frac{d}{dt}(\frac{\partial f}{\partial \dot{x}})) \delta x dt$   
 $(+ 0(\delta x^2) + 0(\delta \dot{x}^2))$

端点处  $\delta x = 0$

故  $\frac{\partial F}{\partial \delta x} = 0 \Leftrightarrow \frac{\partial f}{\partial x} = \frac{d}{dt}(\frac{\partial f}{\partial \dot{x}})$ ,  $f \sim L(t, x, \dot{x})$

即:  $S = \int L dt$  取极值

作用量  $\rightarrow$  “最小作用量原理” ✓

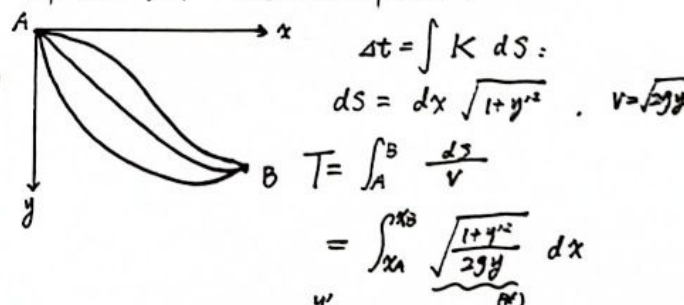
//  $S$  取稳定值(最小值): Hamilton 原理.

小性质,  $L \sim L + \frac{d u(t, x)}{dt}$ :

$S' = \int_{t_1}^{t_2} (L + \frac{du}{dt}) dt = S + u \bigg|_{t_1}^{t_2}$

$\frac{\partial S'}{\partial x} = \frac{\partial S}{\partial x}$  ✓

例, 最速下降曲线求解思路:

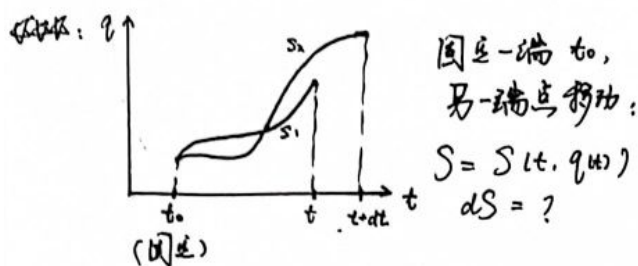


记  $u = L(x, y, \frac{y'}{\sqrt{1+y'^2}})$ ,  $x \sim t$

用  $\frac{\partial L}{\partial y} = \frac{d}{dx} \frac{\partial L}{\partial y'}$  求解.

(亦可用初积分来简化计算).





$$S_1 = \int_{S_1} L(q, \dot{q}, t) dt, \quad S_2 = \int_{S_2} L(q, \dot{q}, t) dt$$

$$\Rightarrow S_2 = \int_{t_0}^t L dt + \int_t^{t+dt} L dt$$

$$\therefore dS = \int_{t_0}^t \delta L dt + L(t, q_{s_2}, \dot{q}_{s_2}) dt$$

$$= \int_{t_0}^t \left( \frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i dt + \left( \frac{\partial L}{\partial \dot{q}_i} \right)_{s_1} \delta q_i \Big|_t + L(t, q_{s_2}, \dot{q}_{s_2}) dt$$

$$L_{s_1} dt + O(\delta q dt) \rightarrow \text{沿 } S_1 \text{ 展开}$$

$$\text{写成 } dU(t, q) = \frac{\partial U}{\partial t} dt + \frac{\partial U}{\partial q} dq \text{ 形式:}$$

$$dq = q_{s_2}(t+dt) - q_{s_1}(t) = q_{s_2}(t+dt) - q_{s_2}(t) + q_{s_2}(t) - q_{s_1}(t) \\ = \left( \frac{dq}{dt} \right)_{s_2} dt + \delta q \Leftarrow \delta q = dq - \left( \frac{dq}{dt} \right)_{s_2} dt \\ = dq - \left( \frac{dq}{dt} \right)_{s_1} dt = dq - \dot{q} dt$$

$$\therefore dS = p_i dq_i - p_i \dot{q}_i dt + L dt$$

$$= p_i dq_i - (p_i \dot{q}_i - L) dt = p_i dq_i - H dt$$

$$\begin{cases} \frac{\partial S}{\partial t} = -H \\ \frac{\partial S}{\partial q_i} = p_i \end{cases}, \quad S = \int (p_i \dot{q}_i - H) dt$$

$$\delta S = \int (\dot{q}_i \delta p_i + p_i \delta \dot{q}_i - \delta H) dt$$

$$\text{代入 } \begin{cases} p_i \delta \dot{q}_i = \frac{d}{dt} (p_i \delta q_i) - \dot{p}_i \delta q_i \\ \delta H = \frac{\partial H}{\partial p_i} \delta p_i + \frac{\partial H}{\partial q_i} \delta q_i \end{cases}$$

$$\Rightarrow p_i \delta \dot{q}_i \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left( \underbrace{\left( \frac{\partial H}{\partial p_i} p_i - \frac{\partial H}{\partial q_i} \right)}_{\text{推出正则方程}} \right) dt = 0$$

$$\text{若 } S = \int (p_i dq_i - H) dt$$

$$\frac{\partial H}{\partial t} = 0 \Leftrightarrow \int H dt = \text{const.}$$

$$\text{Maupertuis 原理, } S_0 = \int p_i dq_i$$

特例: = 准静态.

$$H = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + V(x, y) \equiv E$$

$$\Rightarrow dt = \sqrt{\frac{m(dx^2 + dy^2)}{2(E-V)}} = \sqrt{\frac{m}{2(E-V)}} ds$$

$$S_0 = \int \vec{p} \cdot d\vec{x} = \int \vec{p} \cdot \vec{v} dt$$


$$= \int 2(E - V_{\text{eff}}) \sqrt{\frac{m}{2(E-V)}} ds$$

$$\delta S_0 = 0 \Rightarrow \text{导出 } y = y(x),$$

而无时间演化


$\Downarrow$   
 $H dt$  项略去

#Review: 最小作用量原理.

$$S = \int L dt.$$


$$\delta S = 0 \Leftrightarrow \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q}.$$

相空间中



$$\delta S = 0 \Rightarrow \dot{q}_i = \frac{\partial H}{\partial p_i}, \quad p_i = - \frac{\partial H}{\partial q_i}$$

变上限定端点的  $\delta S$  表达式:

$$dS = p_i dq_i - H dt$$

$$\frac{\partial S}{\partial q_i} = p_i, \quad \frac{\partial S}{\partial t} = -H.$$

$$S = S(q, t).$$

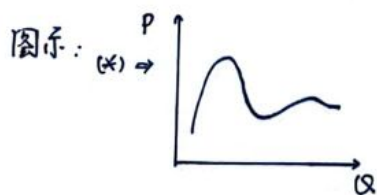
正则变换:

已知变量记作  $p, q, h$ , 满足  $h = p q_i - L$ 

$$S = \int p_i dq_i - h dt$$

若给定  $P(t, q, p)$ ,  $Q(t, q, p)$  及  $H(t, q, p)$ 

$$\begin{cases} \dot{p}_i = -\frac{\partial H}{\partial q_i} \\ \dot{q}_i = \frac{\partial H}{\partial p_i} \end{cases}, \quad H = p_i \dot{q}_i - L, \quad S = \int p_i dq_i - H dt$$

若给定  $q_i \rightarrow Q_i$ ,  $p_i$  如何变化?

$$\text{关系式: } dS = p_i dq_i - h dt \quad ①$$

$$dS = p_i dQ_i - H dt + dF \quad ②$$

$$① = ② \Leftrightarrow dF = p_i dq_i - p_i dQ_i + (H - h) dt$$

称  $F(t, q, Q)$  为 生成函数

$$\frac{\partial F}{\partial q_i} = p_i, \quad \frac{\partial F}{\partial Q_i} = -p_i, \quad \frac{\partial F}{\partial t} = H - h.$$

地址: 中国 北京市海淀区颐和园路5号

例如:  $Q_i = p_i$  时,

北京大学

PEKING UNIVERSITY

已知  $\frac{\partial F}{\partial q_i} = p_i = Q_i$ 猜:  $F = q_i Q_i + Q_i Q_i$ 

$$p_i = -\frac{\partial F}{\partial Q_i} = -q_i - Q_i$$

$$H = \frac{\partial F}{\partial t} + h = h$$

F: I型生成函数. 常用  $F + p_i Q_i = \Phi$ .

$$d\Phi = dF + p_i dQ_i + Q_i d p_i$$

$$\therefore d\Phi = p_i dq_i + Q_i dp_i + (H - h) dt$$

 $\Phi(t, q, p)$ : II型生成函数.

$$\frac{\partial \Phi}{\partial q_i} = p_i, \quad \frac{\partial \Phi}{\partial p_i} = Q_i, \quad \frac{\partial \Phi}{\partial t} = H - h.$$

例:  $P_i = 3p_i$ . 问  $Q_i = ?$ 

用 I 型:  $\frac{\partial F(t, q, Q)}{\partial q_i} = p_i = \frac{1}{3} P_i$

$$\frac{\partial F}{\partial Q_i} = -p_i;$$

用 II 型:  $\frac{\partial \Phi(t, q, P)}{\partial q_i} = p_i = \frac{P_i}{3};$

$$\Phi = \frac{1}{3} q_i P_i, \quad \frac{\partial \Phi}{\partial P_i} = Q_i = \frac{q_i}{3}, \quad H = h$$

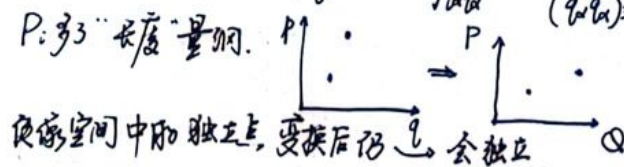
$$\therefore p_i dq_i = P_i dQ_i$$

生成函数:

I 型:  $\Phi(t, q, Q)$ ; II 型:  $\Phi(t, q, P)$ III 型:  $\Phi(t, p, Q)$ ; IV 型:  $\Phi(t, p, P)$ 思考:  $Q_i = \frac{q_i}{\sqrt{q_i q_i}} \sim \frac{1}{|\vec{r}|}, \quad p_i = ?$ 

$$\frac{\partial \Phi(t, q, P)}{\partial P_i} = Q_i = \frac{q_i}{\sqrt{q_i q_i}}$$

$$\Phi = \frac{q_i P_i}{\sqrt{q_i q_i}}, \quad \frac{\partial \Phi}{\partial q_i} = p_i = \frac{P_i}{\sqrt{q_i q_i}} - \frac{q_i P_i q_i}{(q_i q_i)^{3/2}}$$

 $P_i$  为“长度”量纲.

正则变换的应用:

$$\frac{\partial L}{\partial q_i} = 0, \quad p_i = 0; \quad \vec{P} = (J_1, J_2, J_3), \quad J_i = \frac{\partial L}{\partial \dot{q}_i}$$

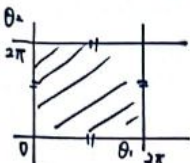
$$\dot{q} = \dot{Q} = \frac{\partial H}{\partial J} = \text{const}, \quad \frac{\partial L}{\partial \dot{q}} = -\vec{P} = 0$$

邮编: 100871

www.pku.edu.cn



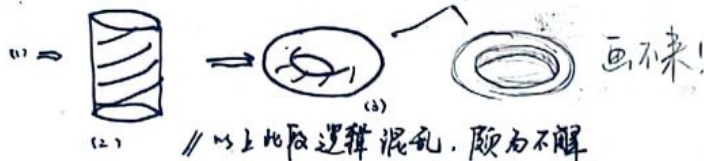
讨论运动轨迹:  $\dot{\theta}_1 = \text{const}$ ,  $\dot{\theta}_2 = \text{const}$

$$\frac{\Delta \theta_1}{\Delta \theta_2} = \text{const}$$


$\Rightarrow$  (1)

如若  $\frac{\dot{\theta}_1}{\dot{\theta}_2} = \frac{m}{n}$ , 则在“填满”空间.

由对称性, 作对称与“卷”: 反是个“甜甜圈”!



$$\text{算 } \frac{df}{dt} = \frac{\partial f}{\partial t} + [f, H] \sim \left\{ \begin{matrix} (p, q) \\ (p, q) \end{matrix} \right\} F?$$

$$\text{下证 } [h, g]_{p, q} = [h, g]_{p, q}$$

记  $w = q_1 \sim q_s, p_1 \sim p_s$ ;  $W = Q_1 \sim Q_s, P_1 \sim P_s$ .

$$J_{\alpha\beta} = \begin{pmatrix} \vec{0} & \vec{I} \\ -\vec{I} & \vec{0} \end{pmatrix}$$

分以下两步证明.

$$[A, B]_{p, q} = J_{\alpha\beta} \frac{\partial}{\partial w_\alpha} A \frac{\partial}{\partial w_\beta} B$$

11

$$\frac{\partial A}{\partial q_\alpha} \frac{\partial B}{\partial p_\alpha} - \frac{\partial A}{\partial p_\alpha} \frac{\partial B}{\partial q_\alpha}$$

$$[w_\alpha, w_\beta]_w = J_{ij} \frac{\partial}{\partial w_i} w_\alpha \frac{\partial}{\partial w_j} w_\beta = J_{ij} \delta_{i\alpha} \delta_{j\beta} = J_{\alpha\beta}$$

$$\text{下证 } [w_\alpha, w_\beta]_{p, q} = [w_\alpha, w_\beta]_{p, q}$$

$$\text{一方面: } [w_\alpha, H]_{p, q} = J_{ij} \frac{\partial}{\partial w_i} w_\alpha \left[ \frac{\partial H}{\partial w_j} \right]$$

$$\text{另一方面: } [w_\alpha, H]_{p, q} = \dot{w}_\alpha = J_{\alpha\beta} \frac{\partial H}{\partial w_\beta} = J_{\alpha\beta} \left[ \frac{\partial H}{\partial w_j} \right] \frac{\partial w_j}{\partial w_\beta}$$

$$\Rightarrow J_{ij} \frac{\partial w_\alpha}{\partial w_i} = J_{\alpha\beta} \frac{\partial w_j}{\partial w_\beta} \quad \text{两边} \times \frac{\partial w_\alpha}{\partial w_j}$$

$$\Leftarrow [w_\alpha, w_\beta]_{p, q} = J_{\alpha\beta} = [w_\alpha, w_\beta]_{p, q} \quad \checkmark$$

// 空间变换,  $d^{2n} \vec{w} = \begin{vmatrix} \frac{\partial w_1}{\partial w_1} & \dots & \frac{\partial w_n}{\partial w_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial w_1}{\partial w_n} & \dots & \frac{\partial w_n}{\partial w_n} \end{vmatrix} d^{2n} \vec{w}$

$$= |\vec{J}| d^{2n} \vec{w}$$

$$\text{下证 } |\vec{J}| = 1$$

12.18 正则变换与 H-J 方程.

1 正则变换的性质:

① 不改变 Poisson 括号

② 不改变相空间中对应体积 (4).

$$(4): d^n \vec{W} = |q_{ab}| d^n \vec{w}$$

$$[W_a, W_b]_{PQ} = J_{ab} \frac{\partial W_b}{\partial w_a} \frac{\partial W_a}{\partial w_b}$$

$$= T_{\alpha\beta} J_{ab} (T_{b\beta})^T \stackrel{(4)}{=} [W_a, W_b]_{pq} = J_{\alpha\beta}$$

取行列式值:  $|J| |J| |J| = |J|$ ,

$$|J| = \pm 1, \text{ 舍去负号项 (体积不可为负).}$$

正则变换的意义:

$$\frac{\partial L}{\partial q_\alpha} = 0, \alpha=1 \sim s \text{ 时, } p_1 \sim p_s \text{ 守恒.}$$

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} = \text{const}, q_\alpha = \Omega_\alpha t + q_{\alpha 0}$$

\* 哈密顿-雅可比方法:

由正则变换,  $H = h + \frac{\partial F}{\partial t}$  需要

$$\text{找 } (P, Q) \text{ 使得 } \begin{cases} \dot{Q} = \frac{\partial H}{\partial P} = 0 \\ \dot{P} = -\frac{\partial H}{\partial Q} = 0 \end{cases}$$

先给出结论: (4) 写作  $h + \frac{\partial S}{\partial t} = 0$ . (由  $dS = pdq - h dt$  得)\*  $F = F(t, q, Q)$ ,  $S = S(t, q)$ ,  $Q$  在何处?Ans:  $Q \equiv \text{const}$ . 将  $S$  中常数项看作 0 即得.

\* 补充: 这里老师跳步飞快 (为了赶进度?), 详细推导如下:

考虑 II 型生成函数形式:  $S = S(t, q, P)$ .

$$h + \frac{\partial S}{\partial t} = 0, \text{ 首先由已知, } p_1 \sim p_s \text{ 均为常数.}$$

$$\text{且 } \frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial q_\alpha} \dot{q}_\alpha, \text{ 由变换式 } \frac{\partial S}{\partial q_\alpha} = p_\alpha$$

$$\text{得 } \frac{dS}{dt} = p_\alpha \dot{q}_\alpha - h = L, S = \int L dt \text{ 恰为作用量.}$$

以下讨论求解过程:  $[S(t, q, P) = W(q, P) + f(t)]^{(4)}$  分离为微分

$$\text{例: } L = \frac{m}{2} \dot{x}^2 - \frac{k}{2} x^2, h = \frac{p^2}{2m} + \frac{k}{2} x^2$$

$$\Rightarrow \text{由 (4): } h + \frac{\partial S}{\partial t} = 0 \text{ 给出 } \begin{cases} h = E \\ f'(t) = -E \end{cases}$$

$$h = h(q_1, \dots, q_s, \frac{\partial W}{\partial q_1}, \dots, \frac{\partial W}{\partial q_s}).$$

北京大学  
PEKING UNIVERSITY

代入例子,

$$\text{得 } \frac{1}{2m} \left( \frac{\partial W}{\partial x} \right)^2 + \frac{k}{2} x^2 = E,$$

$$W(x, E) = \sqrt{mk} \int \sqrt{\frac{2E}{k} - x^2} dx$$

变换后  $H = E$ ,  $E$  为仅含初条件的,  $E = P$ 

$$\Rightarrow \dot{Q} = \frac{\partial H}{\partial P} = \frac{\partial H}{\partial E} = 1$$

$$\Rightarrow Q = t - t_0$$

$$\text{同理 } \dot{Q} = \frac{\partial W}{\partial P} = \frac{\partial W}{\partial E}$$

$$= \sqrt{\frac{m}{k}} x c \sin(x \sqrt{\frac{k}{2E}}).$$

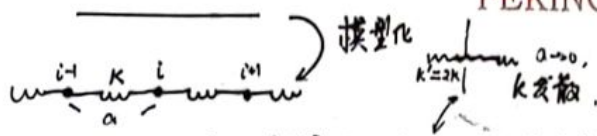
$$\Rightarrow x = \sqrt{\frac{2E}{k}} \sin(\sqrt{\frac{k}{m}} t - \sqrt{\frac{k}{m}} t_0).$$

\* 仍有一例 (抛物体), 参见课程讲义.

过于繁杂, 不再抄写于下.

其他见作业.

## 12.21 连续介质力学

例：线密度  $\mu = \frac{M}{L}$  的弹性杆。 $m = \mu a$ , 弹性模量  $\gamma = \lim_{a \rightarrow 0} a k$ , 记偏导为  $\frac{\partial}{\partial x}$ 。

$$T = \sum \frac{1}{2} m \dot{y}_i^2, \quad V = \sum \frac{1}{2} k (y_{i+1} - y_i)^2$$

$$L = T - V = \frac{1}{2} \sum [m \dot{y}_i^2 + (-k) (y_{i+1} - y_i)^2]$$

$$\triangleq a \sum \left[ \frac{1}{2} \frac{m}{a} \dot{y}_i^2 + \frac{1}{2} \frac{k}{a} (y_{i+1} - y_i)^2 \right]$$

$$\text{显然有: } \lim_{a \rightarrow 0} \frac{y_{i+1} - y_i}{a} = \frac{dy}{dx} \frac{a}{a} = \frac{dy}{dx}$$

$$\text{用 } x \text{ 换表: } L = \frac{1}{2} \int dx \cdot [\mu \dot{y}^2 - \gamma \left( \frac{dy}{dx} \right)^2]$$

记  $L = \int \mathcal{L} dx$ ,  $\mathcal{L}$  为拉氏密度。名词及概念阐释:  $y = y(t, x) \rightarrow$  “标记”

(空间的坐标与物理量)

先分析受力平衡条件:

$$m \ddot{y}_i = k(y_{i+1} - y_i) - k(y_i - y_{i-1})$$

$$\Rightarrow a \mu \ddot{y}_i = \gamma \left[ \frac{y_{i+1} - y_i}{a} - \frac{y_i - y_{i-1}}{a} \right] = \gamma \left[ \frac{dy}{dx} \Big|_{x_i} - \frac{dy}{dx} \Big|_{x_{i-1}} \right]$$

$$\Rightarrow \mu \ddot{y}_i = \gamma \left[ \frac{dy}{dx} \Big|_{x_i} - \frac{dy}{dx} \Big|_{x_{i-1}} \right] \Big|_{a \rightarrow 0}$$

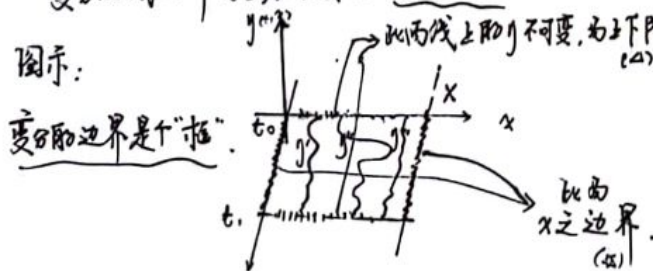
$$\Rightarrow \mu \ddot{y} = \gamma \frac{d^2 y}{dx^2} \quad (*)$$

下由变分法及最小作用量原理导出 (\*) 方程。

$$\text{作用量 } S = \int L dt = \iint \mathcal{L} dx dt, \quad \mathcal{L} = \mathcal{L}(t, x, y, \dot{y}, \frac{dy}{dx})$$

变分法: 最小作用量原理要求  $\delta S = 0$ 

图示:



地址: 中国 北京市海淀区颐和园路5号

证明: (1)  $t_0, t_1$  时刻,  $x$  上取每点任取一确定。(2) 时间轴上划定  $x$  之范围, 亦为  $y$  之边界。

北京大学

PEKING UNIVERSITY

$$\text{由: } \delta S = \iint dx dt \cdot \left[ \frac{\partial \mathcal{L}}{\partial y} \delta y + \frac{\partial \mathcal{L}}{\partial \dot{y}} \delta \left( \frac{dy}{dt} \right) + \frac{\partial \mathcal{L}}{\partial \left( \frac{dy}{dx} \right)} \delta \left( \frac{dy}{dx} \right) \right]$$

分部积分: ( $\delta$  与  $\frac{d}{dx}$  可交换)

$$\frac{\partial \mathcal{L}}{\partial \left( \frac{dy}{dx} \right)} \frac{d}{dx} (\delta y) = \frac{d}{dx} \left( \frac{\partial \mathcal{L}}{\partial \left( \frac{dy}{dx} \right)} \delta y \right) - \left( \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial \left( \frac{dy}{dx} \right)} \right) \delta y$$

(2) 项, 对  $dx dt$  积分后, 边界处  $\delta y = 0$ , 故此项为 0

$$\therefore \delta S = \iint dx dt \left( \frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{y}} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial \left( \frac{dy}{dx} \right)} \right) \delta y = 0$$

$$\Rightarrow \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{y}} + \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial \left( \frac{dy}{dx} \right)} = \frac{\partial \mathcal{L}}{\partial y}$$

$$\text{由 } L = \frac{\mu}{2} \dot{y}^2 - \frac{\gamma}{2} \left( \frac{dy}{dx} \right)^2$$

$$\rightarrow \mu \ddot{y} - \gamma \frac{d^2 y}{dx^2} = 0 \quad \text{证明完毕!}$$

如下, 开始一般化场论论述。

记号: 记  $y_\alpha(t, \vec{x})$ ,  $\alpha = 1, 2, \dots, S$ 笛卡尔/三维坐标下记作  $y_\alpha(x_\mu) \sim \begin{cases} x_1 = x \\ x_2 = y \\ x_3 = z \\ x_4 = t \end{cases}$ 

$$\rightarrow L = L(x_\mu, y_\alpha, p_{\alpha\mu}), \quad p_{\alpha\mu} = \frac{\partial y_\alpha}{\partial x_\mu}$$

$$\text{相应地 } S = \iiint L dt dx dy dz,$$

$$\delta S = 0 \Rightarrow \frac{\partial}{\partial x_\mu} \frac{\partial L}{\partial p_{\alpha\mu}} - \frac{\partial L}{\partial y_\alpha} = 0$$

Review:  $L \rightarrow L + \frac{dU(t, \vec{x})}{dt}$ , 结果不变。

$$\text{记 } L' = L + \frac{\partial F_\mu(x_\mu, y_\alpha)}{\partial x_\mu}$$

$$\text{证: } S' - S = \int d^4 x \frac{\partial F_\mu}{\partial x_\mu} = \int F_\mu dS_\mu \quad (*)$$

$$\delta(S' - S) = \delta \int d^4 x \frac{\partial F_\mu}{\partial x_\mu} \quad (\text{看流版})$$

(2)  $S_\mu$  边界上  $F_\mu$  不变,  $\delta(S' - S) = 0$ 

$$\int d^4 x \frac{\partial F_\mu}{\partial x_\mu} = \iiint dt dx dy dz \left( \frac{\partial F_t}{\partial t} + \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right)$$

[不看流版证明]

$$\text{第项} = \iiint (F_t(t_1) - F_t(t_0)) dx dy dz$$

$$\delta(F_t(t_1) - F_t(t_0)) = 0$$

$$\text{时后 } n \text{ 项同理} \Rightarrow \int d^4 x \frac{\partial F_\mu}{\partial x_\mu} = 0$$

邮编: 100871

www.pku.edu.cn



例：气体声场，均匀，给出初压  $P_0$  和初始密度  $\mu_0$ 。  
用  $\vec{r}(t, \vec{x})$  表示场， $\gamma, V$  表示功/势能密度。  
 $\mathcal{L} = \gamma - V$ 。由于考虑微扰条件， $\mu \approx \mu_0$

$$\therefore \gamma = \frac{\mu}{2} \dot{\vec{r}}^2 \approx \frac{\mu_0}{2} \dot{\vec{r}}^2$$

$$V = \frac{U_0 + \Delta U}{V_0 + \Delta V} \approx \frac{U_0}{V_0} \left(1 + \frac{\Delta U}{U_0}\right) \left(1 - \frac{\Delta V}{V_0}\right)$$

$$V = \frac{U_0}{V_0} + \frac{\Delta U}{V_0} - \frac{U_0}{V_0} \frac{\Delta V}{V_0} + O(\Delta U \Delta V)$$

而  $\Delta U = - \int_{V_0}^{V_0 + \Delta V} P dV$ ，绝热过程  $P V^\gamma = C$

$$\frac{\partial P}{\partial V} = -C \gamma V^{-\gamma-1} = -\gamma \frac{P}{V}$$

$$\therefore \Delta U = - \int_{V_0}^{V_0 + \Delta V} \left(P_0 + \frac{\partial P}{\partial V} \Delta V\right) dV$$

$$= -P_0 \Delta V + \frac{\partial P_0}{2 V_0} (\Delta V)^2$$

(\*) 求解  $\left( \frac{\Delta V}{V_0} = \nabla \cdot \vec{r} \right)$

$$(\Rightarrow \iiint \nabla \cdot \vec{r} d^3 \vec{x} = \oint \vec{r} \cdot d\vec{S})$$

$$\therefore V = -\frac{\gamma}{\gamma-1} P_0 \nabla \cdot \vec{r} + \frac{\partial P_0}{2} (\nabla \cdot \vec{r})^2$$

$$\therefore \mathcal{L} = \frac{1}{2} \left[ \mu_0 \dot{\vec{r}}^2 + \frac{2\gamma}{\gamma-1} P_0 \nabla \cdot \vec{r} - \partial P_0 (\nabla \cdot \vec{r})^2 \right]$$

$$\frac{\partial \mathcal{L}}{\partial \eta_{i,j}} = \frac{1}{2} \left[ \mu_0 \eta_{i,j} + \frac{2\gamma}{\gamma-1} P_0 \eta_{i,i} - \partial P_0 \eta_{i,i} \eta_{j,j} \right]$$

$\forall (i=1,2,3)$  此时:  $\frac{\partial \mathcal{L}}{\partial \eta_{i,i}} = \frac{\gamma}{\gamma-1} P_0 - \partial P_0 \eta_{i,i}$

$$\frac{\partial \mathcal{L}}{\partial \eta_{1,2}} = \frac{\partial \mathcal{L}}{\partial \eta_{2,1}} = 0, \quad \frac{\partial \mathcal{L}}{\partial \eta_{1,4}} = \mu_0 \eta_{1,4}, \quad \frac{\partial \mathcal{L}}{\partial \eta_4} = 0$$

取 1 时，代入  $\frac{\partial}{\partial x_\mu} \frac{\partial \mathcal{L}}{\partial \eta_{\mu,\mu}} - \frac{\partial \mathcal{L}}{\partial \eta_\mu} = 0$

$$\text{得: } \frac{\partial}{\partial x_\mu} \frac{\partial \mathcal{L}}{\partial \eta_{\mu,\mu}} = 0$$

$$\text{即: } -\partial P_0 \eta_{i,i,i} + \mu_0 \eta_{i,i,i} = 0$$

(此时  $i$  再对 1 特)

$$\Rightarrow \mu_0 \ddot{\eta}_x = \partial P_0 \frac{\partial}{\partial x} (\nabla \cdot \vec{r})$$

同理整理得:  $\mu_0 \ddot{\vec{r}} = \partial P_0 \nabla (\nabla \cdot \vec{r})$

等式两边取散度，记  $\frac{\Delta V}{V} = \sigma$  代入。

$$\mu_0 \ddot{\sigma} = \gamma P_0 \nabla^2 \sigma$$

$$\Rightarrow \ddot{\sigma} = \frac{\partial P_0}{\mu_0} \nabla^2 \sigma \text{ 为波动方程}$$

(如  $\ddot{\mu} = c^2 \frac{\partial^2 \mu}{\partial x^2}$  解为  $K_1 e^{k(c t - x)}$ )

$$\text{本例中, } c = \sqrt{\frac{\partial P_0}{\mu_0}}$$

取  $\gamma = \frac{7}{5}$ ,  $P_0 \sim 10^5 \text{ Pa}$ ,  $\mu_0 \sim 1.3 \text{ kg/m}^3$

得  $V_f \approx 331.5 \text{ m/s}$  恰为  $0^\circ \text{C}$  时声速!

// 补充论述,

$$\textcircled{1} \text{ 为可是 } \left( \frac{\partial}{\partial x_\mu} \right) \frac{\partial \mathcal{L}}{\partial \eta_{\mu,\mu}} - \frac{\partial \mathcal{L}}{\partial \eta_\mu} = 0$$

(不是  $\frac{d}{dx_\mu}$  ?) (参考  $\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q}$ )

由于  $x_\mu$  彼此间有联系，故可直接用  $\frac{d}{dx_\mu}$ 。

之前用  $\eta$  因为除  $\eta$  外只含  $t$  这“标记”。



12.25 Christmas!

相对论性分析力学.

Remark<sup>(1)</sup>: 上节课声场 " $\frac{1}{r}$ " 系数由来:

$$U_0 = \int_{V_0} P dV, \text{ 代入 } PV^\gamma = C.$$

$$U_0 = \int_{V_0}^{\infty} C V^{-\gamma} dV = \frac{1}{-\gamma+1} (C V^{-\gamma+1}) \Big|_{V_0}^{\infty}$$

$$= \frac{1}{\gamma-1} P_0 V_0, \text{ 即 } \frac{U_0}{V_0} = \frac{1}{\gamma-1} P_0.$$

$$\mathcal{V} = \frac{U_0}{V_0} + \frac{\Delta U}{V_0} - \frac{U_0}{V_0} \cdot \frac{\Delta V}{V_0}$$

$$= \frac{1}{\gamma-1} P_0 + (-P_0 \cdot \frac{\Delta V}{V_0} + \frac{\partial P_0}{\partial V_0} (\frac{\Delta V}{V_0})^2) - \frac{1}{\gamma-1} P_0 \cdot \frac{\Delta V}{V_0}$$

$$= \frac{1}{\gamma-1} P_0 - \frac{\gamma}{\gamma-1} P_0 \frac{\Delta V}{V_0} + \frac{\gamma P_0}{2} (\frac{\Delta V}{V_0})^2$$

以下同.

$$\text{Remark}^{(2)}: \mathcal{V} = \frac{U_0 + \Delta U}{V_0 + \Delta V} = \frac{U_0}{V_0} (1 + \frac{\Delta U}{U_0}) (1 - \frac{\Delta V}{V_0})$$

陈老师给出了一阶小量近似, 实则保留到  $\frac{\partial P_0}{\partial V_0} (\frac{\Delta V}{V_0})$  项. 下标  $\alpha = \overline{1, \dots, n}$ .

作完整的二阶近似如下.

$$\mathcal{V} = \frac{U_0}{V_0} (1 + \frac{\Delta U}{U_0}) (1 - \frac{\Delta V}{V_0} + (\frac{\Delta V}{V_0})^2)$$

$$= \frac{U_0}{V_0} + \frac{\Delta U}{V_0} - \frac{U_0 \Delta V}{V_0^2} - \frac{\Delta U \Delta V}{V_0^2} + \frac{U_0}{V_0} (\frac{\Delta V}{V_0})^2$$

$$\text{代入 } \frac{\Delta U}{V_0} = -P_0 \cdot \frac{\Delta V}{V_0} + \frac{\partial P_0}{\partial V_0} (\frac{\Delta V}{V_0})^2, \text{ 得 } \mathcal{V} = \text{所求量}$$

$$\text{取 } \mathcal{V} = -P_0 \frac{\Delta V}{V_0} + \frac{\gamma P_0}{2} (\frac{\Delta V}{V_0})^2 - \frac{1}{\gamma-1} P_0 \cdot \frac{\Delta V}{V_0} - (-P_0 \frac{\Delta V}{V_0} + \frac{\partial P_0}{\partial V_0} (\frac{\Delta V}{V_0})^2) \cdot \frac{\Delta V}{V_0} + \frac{1}{\gamma-1} P_0 \cdot (\frac{\Delta V}{V_0})^2$$

$$= -\frac{\gamma}{\gamma-1} P_0 \frac{\Delta V}{V_0} + (\frac{\partial P_0}{\partial V_0} + \frac{\gamma}{\gamma-1} P_0) \cdot (\frac{\Delta V}{V_0})^2$$

$$= -\frac{\gamma}{\gamma-1} P_0 \frac{\Delta V}{V_0} + \frac{\partial(\gamma P_0)}{2(\gamma-1)} P_0 (\frac{\Delta V}{V_0})^2$$

$$\therefore \mathcal{L} = \frac{1}{2} \mu \dot{\vec{r}}^2 + \frac{\gamma}{\gamma-1} P_0 (\vec{r} \cdot \vec{r}) - \frac{\partial(\gamma P_0)}{2(\gamma-1)} P_0 (\vec{r} \cdot \vec{r})^2$$

$$= \frac{1}{2} [\mu \dot{r}_i \dot{r}_i + \frac{2\gamma}{\gamma-1} P_0 r_{ii} - \frac{\partial(\gamma P_0)}{\partial r_i} P_0 r_{ii} r_{ii}]$$

$$\text{最终有 } \mu_0 \ddot{\sigma} = \frac{\partial(\gamma P_0)}{\partial r_i} P_0 \sigma^i \sigma^i \quad (\approx 90 \text{ m/s})$$

$$C = \sqrt{\frac{\partial(\gamma P_0)}{\mu_0(\gamma-1)}} P_0 \gg 343 \text{ m/s}$$

地址: 中国 北京市海淀区颐和园路5号 ???

// 以下为课程内容:

$$\text{考虑 } \vec{F} = m\vec{a} \Rightarrow S = \int \mathcal{L} dt d\vec{r} dy dz$$

北京大学

相对论情形下,

例: 自由粒子 ( $dt, d\vec{x}$ )

PEKING UNIVERSITY 与固有时 (proper time)

$$ds^2 = (cd\tau)^2 = (cdt)^2 - dx^2 - dy^2 - dz^2.$$

$$\Leftrightarrow ds^2 = (cdt, d\vec{x}) \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} cdt \\ d\vec{x} \end{pmatrix}$$

规定闵氏度规  $\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$  (约定)

$$\Rightarrow ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \equiv \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} d\tau^2 \quad (\text{记 } \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{d\tau} \text{ 降指标})$$

$$\text{记 } (\eta^{-1}) = \eta^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad \eta^{\mu\nu} \eta_{\nu\mu} = \delta_\mu^\mu$$

$$\text{同理“升指标” } \eta^{\mu\nu} dx_\mu = dx^\nu \Rightarrow \begin{pmatrix} \vec{S} \rightarrow \vec{v} \\ S \end{pmatrix} \text{ fig. 1}$$

$$\text{洛伦兹变换: } x^\mu = \left( \frac{\partial x^\mu}{\partial x^\nu} \right) x^\nu \Rightarrow \begin{pmatrix} \frac{\partial x^\mu}{\partial x^\nu} \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \triangleq \Lambda_\nu^\mu$$

$$x_\mu = \eta_{\mu\nu} x^\nu = \eta_{\mu\nu} \Lambda_\rho^\nu x^\rho = \eta_{\mu\nu} \Lambda_\rho^\nu \eta^{\rho\sigma} x_\sigma$$

$$\text{记 } \Lambda_\mu^\mu = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \eta_{\mu\nu} \Lambda_\rho^\nu \eta^{\rho\sigma}$$

称之为“协变张量”

选取计算式与路径, 使其不依赖于参考系.

$$\text{对 } \eta_{\mu\nu} \text{ 仍有: } ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = (cdt')^2 - (d\vec{x}')^2$$

关于度规之理解

$$\mathcal{V} = \frac{U_0 + \Delta U}{V} \text{ 应写“拉氏量与固有时之比”}$$

否则: 取  $U_\infty = \frac{1}{\gamma-1} P_0 V_0$  相差常数却影响了结果计算!!! (即影响了  $\frac{\Delta V}{V_0}$  项)

$$\therefore \text{只按 } \mathcal{V} = \frac{U_0}{V_0} + \frac{\Delta U}{V_0} \text{ 即可!!!}$$

即  $\mathcal{V} \Leftarrow \mathcal{V} + \Delta \mathcal{V}$  陈老师混淆概念

啦~

邮编: 100871

www.pku.edu.cn

回到自由粒子的作用量

$$\text{记作 } S = \int ds, \quad \delta S = \int \delta \sqrt{dx^\mu dx_\mu} = \int \delta \sqrt{dx^\mu dx_\mu}$$

$$\Rightarrow \delta S = \int \frac{(8 dx^\mu) dx_\mu + dx^\mu \delta(dx_\mu)}{2 \sqrt{dx^\mu dx_\mu}}$$

$\delta(dx^\mu dx_\mu) = \delta(dx_\mu) dx^\mu$ , 理由如下:

$$\delta(\eta_{\mu\nu} dx^\mu dx^\nu) = \eta_{\mu\nu} \delta dx^\mu dx^\nu + \eta_{\mu\nu} dx^\mu \delta dx^\nu \quad (\text{两者同!})$$

$$\therefore \text{上式} = \int \frac{dx^\mu}{ds} d(\delta x_\mu)$$

$$= \int d\left(\frac{dx^\mu}{ds} \delta x_\mu\right) - \frac{d}{ds}\left(\frac{dx^\mu}{ds}\right) ds \delta x_\mu = 0$$

$$\Rightarrow \frac{d}{ds}\left(\frac{dx^\mu}{ds}\right) = 0 \quad (\text{代入 } cd\tau = ds)$$

$$\Rightarrow \frac{d}{dt} U^\mu = 0$$

$$\text{代入: } S = \int \sqrt{(cdt)^2 - (d\vec{x})^2}$$

$$= \int c dt \sqrt{1 - \beta^2} \xrightarrow{\beta \ll 1} \int c dt \left(1 - \frac{\beta^2}{2}\right)$$

$$= -\int C \cdot \frac{v^2}{2c}$$

$$\Rightarrow \underline{C = -mc} \quad \text{即 } S = -mc \int ds = \int dt \left(-\frac{mc^2}{\gamma}\right)$$

$$\Rightarrow \underline{L = -\frac{mc^2}{\gamma}} = -mc \sqrt{c^2 - v^2}$$

$$\underline{P = \frac{\partial L}{\partial \vec{v}} = \frac{mc \vec{v}}{\sqrt{c^2 - v^2}} = \gamma m \vec{v}}$$

将: 放入电磁场

$$A^\mu = (\phi, \vec{A}). \quad \text{可知 } L \propto A^\mu \text{ 而非 } \underline{A^\mu A_\mu}$$

$$\therefore \text{猜 } S = \int -mc ds + c_1 \underline{A^\mu dx_\mu}$$

$$= \int -mc ds + c_1 (\phi c dt - \vec{A} \cdot d\vec{x})$$

$$= \int dt [-mc \sqrt{c^2 - v^2} + c_1 (\phi \cdot c - \vec{A} \cdot \vec{v})]$$

$$\left[ \text{对比 } L = \frac{mv^2}{2} - e(\phi - \vec{A} \cdot \vec{v}) \text{ 得} \right]$$

$$\Rightarrow \underline{c_1 = -\frac{e}{c}}$$

$$S = \int -mc ds - \frac{e}{c} A^\mu dx_\mu$$

$$0 = \delta S = \int mc \frac{d^2 x^\mu}{ds^2} \delta x^\mu ds - \frac{e}{c} \left[ \frac{\partial A_\mu}{\partial x^\nu} \delta x^\nu \frac{dx^\mu}{ds} ds + A^\mu \delta(dx_\mu) \right]$$

$$(\times) = -\frac{e}{c} \int \left( \frac{\partial A_\mu}{\partial x^\nu} \frac{dx^\nu}{ds} - \frac{\partial A_\nu}{\partial x^\mu} \frac{dx^\mu}{ds} \right) \delta x^\nu ds$$

如何得来?  $\Rightarrow$  见右半面

$$(\times) = -\frac{e}{c} \int \frac{\partial A_\mu}{\partial x^\nu} \frac{dx^\nu}{ds} \delta x^\mu ds + d(A_\mu \delta x^\mu) - \frac{\partial A_\mu}{\partial x^\nu} \frac{dx^\nu}{ds} \delta x^\mu ds$$

合并即得

$$\text{问: 为何 } F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu (= -F_{\nu\mu})$$

$$\Rightarrow mc \frac{d^2 x_\mu}{c^2 dt^2} - \frac{e}{c} F_{\mu\nu} \frac{dx^\nu}{cdt} = 0$$

$$\Rightarrow m \frac{dU_\mu}{dt} + \frac{e}{c} F_{\mu\nu} U^\nu = 0$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \quad \begin{cases} \text{[4][1]} -\frac{\partial A_\mu}{\partial x^\nu} + \frac{\partial A_\nu}{\partial x^\mu} = B_{\mu\nu} \\ \vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \\ \vec{B} = \nabla \times \vec{A} \end{cases}$$

$$\text{由 } U^\mu = \begin{pmatrix} \frac{cdt}{ds} \\ \frac{dx}{ds} \\ \frac{dy}{ds} \\ \frac{dz}{ds} \end{pmatrix} = \begin{pmatrix} \gamma c \\ -\gamma \vec{v} \end{pmatrix}$$

得到4个分量式:

$$0. \quad m \frac{d}{dt}(\gamma \vec{v}) = -\frac{e}{c} (\vec{E}) \cdot (\gamma \vec{v})$$

$$\text{即 } \underline{\frac{d}{dt}(\gamma mc^2) = e \vec{E} \cdot \vec{v}}$$

$$1 \sim 3. \quad \underline{\frac{d}{dt}(\gamma m \vec{v}) = e \vec{E} + \frac{e}{c} (\vec{v} \times \vec{B})}$$

默认  $c=1$  或直接将  $c$  代入  $A^\mu$  分量中

$A^\mu$  分量便  $\propto$  (量纲  $V \cdot m/s$ )



陈强

12.28

——“看看山顶是什么样的”

地电初学。

//Klein,  $S = -mc \int ds - \frac{e}{c} \int A_\mu dx^\mu$

而  $S = \int dt dx dy dz: \mathcal{L}$ . 如何转换成协变的?

$$\begin{pmatrix} \bar{t} \\ \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \quad dt dx dy dz = d\bar{t} d\bar{x} d\bar{y} d\bar{z} \quad [2]$$

$$\Rightarrow [?] = \frac{\partial x^\mu}{\partial \bar{x}^\mu} = \gamma(1-\beta^2) = 1$$

$\therefore dt dx dy dz$  随坐标系变换, 是标量!

$\therefore \mathcal{L}$  应是标量.

1 电磁场:  $\eta_\mu(x^\mu) \rightsquigarrow A_\mu(x^\mu)$

$$\mathcal{L}(x^\mu, \eta_\mu, \eta_{\mu,\nu}) = \mathcal{L}(x^\mu, A_\mu, F_{\mu\nu})$$

其中  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Question:  $A_\mu x^\mu, F_{\mu\nu} x^\mu x^\nu$  会进入  $\mathcal{L}$  中吗?

会! 但  $x^\mu SA_\mu$  而  $\mathcal{L}$  的变化与时空位置无关.

最终给出  $\mathcal{L} = \mathcal{L}_f + \mathcal{L}_{int}(\dots, J^\mu)$

[f, free, int, interaction]  $J^\mu = \frac{e}{c} U^\mu, x^\mu = A_\mu x^\mu$

结论:  $\mathcal{L} = -\frac{E_{\mu\nu} F_{\mu\nu}}{16\pi} - A_\mu J^\mu$

//补充知识 — EM 单位制.

Coulomb's Law  $\frac{1}{4\pi\epsilon_0} \frac{Q_1 Q_2}{r^2} = F$ .

/ESU 单位制:  $\frac{Q_1 Q_2}{r^2} = F$ . 取  $c=1$ . ( $Q, q$  单位是 C.)

此时 Maxwell 方程化为,

$$\vec{E} = 4\pi\vec{p}; \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}; \quad \nabla \cdot \vec{B} = \frac{\partial \vec{E}}{\partial t} + 4\pi(\vec{p}\vec{v});$$

$$\nabla \cdot \vec{B} = 0$$

$c=1$  给出  $dx^\mu = \begin{pmatrix} dt \\ d\vec{x} \end{pmatrix}$

$$\delta S = \int d^4x \left( -\frac{F_{\mu\nu} \delta F_{\mu\nu}}{8\pi} - J^\mu \delta A_\mu \right)$$

[Trick:  $\delta x^\mu(x_\mu) = x^\mu(\delta x_\mu)$ ]

$$-\frac{1}{8\pi} F^{\mu\nu} \delta F_{\mu\nu} = -\frac{F^{\mu\nu}}{8\pi} (\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu)$$

[Trick:  $\partial$  与  $\delta$  可交换]

地址: 中国 北京市海淀区颐和园路5号

$$\delta S = -\frac{F^{\mu\nu}}{8\pi} \partial_\mu \delta A_\nu \cdot \frac{F^{\mu\nu}}{8\pi} \partial_\nu \delta A_\mu$$

$$= \frac{F^{\mu\nu}}{4\pi} \partial_\mu \delta A_\nu$$

北京大学

PEKING UNIVERSITY

[Trick: 反对称矩阵.  $\partial_\mu \delta A_\nu = -\partial_\nu \delta A_\mu$ ]

$$= \partial_\mu \left[ \frac{F^{\mu\nu}}{4\pi} \delta A_\nu \right] - \delta A_\nu \cdot \partial_\mu \left( \frac{F^{\mu\nu}}{4\pi} \right)$$

规范边界, 值为零

$$\Rightarrow \partial_\mu F^{\mu\nu} = -4\pi J^\nu \Rightarrow \partial_\nu F^{\mu\nu} = 4\pi J^\mu$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \quad F^{\mu\nu} = \eta^{\mu\alpha} \eta^{\nu\beta} F_{\alpha\beta}$$

$$\mu=0: \partial_1 F^{01} + \partial_2 F^{02} + \partial_3 F^{03} = 4\pi \cdot \frac{\rho}{c} \cdot c$$

$$\text{即 } \nabla \cdot \vec{E} = 4\pi\rho$$

$$\mu=1: \frac{\partial}{\partial t}(-E_x) + \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) = 4\pi\rho v_x$$

$$\mu=2: \frac{\partial}{\partial t}(-E_y) + \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) = 4\pi\rho v_y$$

$$\mu=3: \frac{\partial}{\partial t}(-E_z) + \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) = 4\pi\rho v_z$$

$$\Rightarrow \nabla \times \vec{B} = \frac{\partial \vec{E}}{\partial t} + 4\pi\rho\vec{v}$$

此两项符合  $J^\mu$ ;

此外, 均应满足  $\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0$

$$\Rightarrow \begin{cases} \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \nabla \cdot \vec{B} = 0 \end{cases}$$

此两项直接针对场, 故无  $J^\mu$ .

//如何从作用量“看出”守恒量?

$$\mathcal{L} = \mathcal{L}(x_\mu, \eta_\mu, \eta_{\mu,\nu})$$

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = 0 \text{ 可推出 } [?] \text{ (中括号内为 } \frac{\partial \mathcal{L}}{\partial \eta_\mu})$$

$$\frac{d\mathcal{L}}{dx^\mu} = \frac{\partial \mathcal{L}}{\partial x^\mu} + \frac{\partial \mathcal{L}}{\partial \eta_\mu} \eta_{\mu,\mu} + \frac{\partial \mathcal{L}}{\partial \eta_{\mu,\nu}} \eta_{\mu,\nu,\mu}$$

$$(\text{由 } \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial \eta_{\mu,\nu}} = \frac{\partial \mathcal{L}}{\partial \eta_\mu})$$

$$= \frac{\partial \mathcal{L}}{\partial x^\mu} + \frac{d}{dx^\nu} \left( \frac{\partial \mathcal{L}}{\partial \eta_{\mu,\nu}} \eta_{\mu,\nu} \right)$$

$$\text{而 } \frac{d\mathcal{L}}{dx^\mu} = \delta_\mu^\nu \frac{d\mathcal{L}}{dx^\nu}$$

$$\rightarrow -\frac{\partial \mathcal{L}}{\partial x^\mu} = \frac{d}{dx^\nu} \left( \frac{\partial \mathcal{L}}{\partial \eta_{\mu,\nu}} \eta_{\mu,\nu} - \mathcal{L} \delta_\mu^\nu \right)$$

$$RHS \triangleq \frac{d}{dx^\nu} \gamma_\mu^\nu$$

邮编: 100871

www.pku.edu.cn



$$T^{\mu\nu} = g^{\mu\nu} \frac{\partial \mathcal{L}}{\partial \eta_{\mu\nu}} - \mathcal{L} \delta^{\mu\nu}$$

$$\mu=0: \Rightarrow \frac{d}{dt} T^{00} + \nabla \cdot T^0 = 0$$

$$(\text{如: } \frac{d}{dt} \rho + \nabla \cdot (\rho \vec{v}) = 0)$$

能量密度.

$T_{00}$ :  $\mu=0$ ,  $T_{00}$ : 能量密度;  $T_{0i}$ : 能流密度.

$$\mathcal{L} = \frac{1}{2} \mu \dot{\eta}^2 - \frac{1}{2} Y \left( \frac{d\eta}{dx} \right)^2 \rightarrow \text{(单向杆)}$$

$$T_{00} = \frac{1}{2} \mu \dot{\eta}^2 + \frac{Y}{2} \left( \frac{d\eta}{dx} \right)^2 \text{ 为能量密度;}$$

$$T_{01} = \eta \dot{\eta} \left( -Y \frac{d\eta}{dx} \right)$$

$$\frac{d T_{00}}{dt} + \frac{\partial}{\partial x} T_{01} = 0 \Rightarrow \frac{d}{dt} \mathcal{L} = 0$$

局域能量守恒

$$\mu \text{ 取 } 1, T_{10} = \frac{d\eta}{dx} \mu \eta$$

$$\frac{\mu \eta dx}{dx + d\eta} = \mu \eta - \mu \eta \frac{d\eta}{dx}$$

$$\Rightarrow \frac{d}{dt} T_{10} + \frac{d}{dx} T_{11} = 0$$

$$\Rightarrow \text{动量密度.}$$

$T_{\mu\nu}$  能量张量

$$\mu \begin{cases} 0: \text{能量} \\ 1-3: \text{动量} \end{cases} \left\{ \begin{array}{l} 0: \text{密度 (沿时间轴传播)} \\ 1-3: \text{流量 (空间轴传播)}. \end{array} \right.$$

## 1. Basic rules &amp; formula:

$$\vec{r} = \frac{d\vec{r}}{dt} = \vec{v}; \quad \vec{a} = \ddot{\vec{r}} = \dot{\vec{v}} = \frac{d\vec{v}}{dt} = \frac{d}{dt} \left( \frac{d\vec{r}}{dt} \right).$$

极坐标中, 记住  $\vec{r} = r\vec{e}_r$  以及  $\begin{cases} \dot{\vec{e}}_r = \dot{\theta}\vec{e}_\theta \\ \dot{\vec{e}}_\theta = -\dot{\theta}\vec{e}_r \end{cases}$  即可.

相对论变换:  $\begin{pmatrix} cdt' \\ dx' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} cdt \\ dx \end{pmatrix} \Rightarrow$  (其中  $\beta = \frac{v}{c}$ ,  $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ )

爱因斯坦求和: ( $A_i \lambda_i$  代表  $\sum_i A_i \lambda_i$ ).

## 2. 虚位移: “变分, 等时性”

$$\begin{cases} d\vec{r}_i = \frac{\partial \vec{r}_i}{\partial q_\alpha} dq_\alpha + \frac{\partial \vec{r}_i}{\partial t} dt \\ \delta \vec{r}_i = \frac{\partial \vec{r}_i}{\partial q_\alpha} \delta q_\alpha \end{cases} \quad \left. \vphantom{\begin{cases} d\vec{r}_i \\ \delta \vec{r}_i \end{cases}} \right\} \text{ 稳定的约束条件下, } \delta \vec{r}_i \propto d\vec{r}_i. \quad \delta \vec{r}_i \text{ 可有多个.}$$

\* 目前题目中大多出现  $\delta \vec{r}_i$  仅与一个广义坐标有关. 群无一双相关择例:

曲线  $z = x^2 + y^2$ , 记  $\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$

$$\text{则有: } \begin{cases} \delta z = \delta(r^2) = 2r \delta r \\ \delta x = \delta(r \cos \theta) = \cos \theta \delta r - r \sin \theta \delta \theta \\ \delta y = \delta(r \sin \theta) = \sin \theta \delta r + r \cos \theta \delta \theta \end{cases}$$

3. 虚功原理.  $\delta W = \vec{F} \cdot \delta \vec{r} = 0$ . ( $\vec{F}_i \cdot \delta \vec{r} = 0$ ).

$$\# \text{ 广义力: } \delta W = \vec{F}_i \cdot \delta \vec{r}_i = \vec{F}_i \cdot \left( \frac{\partial \vec{r}_i}{\partial q_\alpha} \delta q_\alpha \right) = \left( \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha} \right) \delta q_\alpha = Q_\alpha \delta q_\alpha$$

记  $Q_\alpha = \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha}$  为广义力.

备注: Lagrangian Equations 这一套体系中, 求解  $Q_\alpha$  时利用矢量点乘, 消去了矢量合成之烦.

故, 算  $Q_\alpha$  时“直接无脑加”, 不用考虑力的方向.

$$\text{例: HW(T8). } \begin{cases} y = y(x) \\ z = z(x) \end{cases} \begin{cases} F_x = \frac{mgx}{2c} \\ F_z = -mg \end{cases} \text{ 在给出 } \begin{cases} F_y = 0 \\ y'(x) = 0 \\ z(x) = -x \end{cases} \text{ 时, 仅不可表出位移,}$$

$$Q_\alpha = \vec{F}_x \cdot \frac{\partial x}{\partial \alpha} + \vec{F}_z \cdot \frac{\partial z}{\partial \alpha} = mg \left( 1 + \frac{x}{2c} \right). \quad (\text{而不是笔者误以为的 } mg \sqrt{1 + \frac{x^2}{4c^2}})$$

$$\# \vec{F}_i = -\nabla V, \text{ 故 } \vec{F}_i \cdot \delta \vec{r}_i = -\nabla V \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha} \delta q_\alpha = -\frac{\partial V}{\partial q_\alpha} \delta q_\alpha$$

即对保守势  $V$ , 广义力  $Q_\alpha$  有  $Q_\alpha = -\frac{\partial V}{\partial q_\alpha}$ .

平衡条件下  $\frac{\partial V}{\partial q_\alpha} = 0$ . 在(某种)条件下, 处于极值

} \* 在所有平衡体系中.

$$Q_\alpha = -\frac{\partial V}{\partial q_\alpha} = 0 \quad (\text{而非})$$

\* 虚功原理解题: ①  $\left\{ \begin{array}{l} 1. \text{用广义坐标表出需求解点位} \\ 2. \vec{F}_i \cdot \delta \vec{r}_i = 0 \text{ 化解, 化简 } \delta \vec{r}_i \\ 3. \text{得出关系式} \end{array} \right\}$  归为“可用广义坐标表出型”

[具体见 HW1. T1.]  $\Rightarrow$  传统式

②  $\left\{ \begin{array}{l} 1. \text{利用 } \delta W = \vec{F}_i \cdot \delta \vec{r}_i = -\frac{\partial V}{\partial q_\alpha} \delta q_\alpha = 0 \\ 2. \text{写出 } V \text{ 的表达式} \\ 3. \text{求解 } \frac{\partial V}{\partial q_\alpha} \delta q_\alpha = 0 \end{array} \right\}$  归为“可用广义坐标变分法型”

(1) [具体见 HW1. T2]  $\Rightarrow (V = V(\theta), q_\alpha = \theta) \Rightarrow (\frac{\partial V}{\partial \theta} = 0)$

(2). 课例: 复合摆

(\*)  $\left\{ \begin{array}{l} \vec{F}_1 = m_1 \vec{g} \\ \vec{F}_2 = m_2 \vec{g} \\ \vec{F}_3 = F \end{array} \right\} \left\{ \begin{array}{l} \vec{r}_1 = (l_1 \cos \theta_1 + l_2 \cos \theta_2) \hat{x} + (l_1 \sin \theta_1 + l_2 \sin \theta_2) \hat{y} \\ \vec{r}_2 = (l_2 \cos \theta_2) \hat{x} + (l_2 \sin \theta_2) \hat{y} \end{array} \right.$

$\left\{ \begin{array}{l} Q_{\theta_1} = \vec{F}_1 \cdot \frac{\partial \vec{r}_1}{\partial \theta_1} + \vec{F}_2 \cdot \frac{\partial \vec{r}_2}{\partial \theta_1} + \vec{F}_3 \cdot \frac{\partial \vec{r}_2}{\partial \theta_1} = 0 \\ Q_{\theta_2} = \vec{F}_1 \cdot \frac{\partial \vec{r}_1}{\partial \theta_2} + \vec{F}_2 \cdot \frac{\partial \vec{r}_2}{\partial \theta_2} + \vec{F}_3 \cdot \frac{\partial \vec{r}_2}{\partial \theta_2} = 0 \end{array} \right. (Q_{\theta_1} \neq 0 \neq Q_{\theta_2})$

(\*)  $V = -m_1 g l_1 \cos \theta_1 - m_2 g (l_1 \cos \theta_1 + l_2 \cos \theta_2) - F (l_2 \sin \theta_2)$   
 $\frac{\partial V}{\partial \theta_1} = 0; \quad \frac{\partial V}{\partial \theta_2} = 0$

[思考]: 若 F 并非沿水平面, 则须给定其前

可分解于  $\vec{F}$  分别用虚功原理代入求解 (若  $\theta$ , 必须已知)

4. d'Alembert 原理 // Lagrangian Equations

由  $\vec{F}_i = m_i \ddot{\vec{r}}_i$  推广得  $(\vec{F}_i - m_i \ddot{\vec{r}}_i) \cdot \delta \vec{r}_i = 0$

# 由 d'Alembert 原理推广至 Lagrangian Equations:

$$m_i \ddot{\vec{r}}_i \cdot \delta \vec{r}_i = m_i \cdot \left( \frac{d}{dt} \dot{\vec{r}}_i \right) \cdot \left( \frac{\partial \vec{r}_i}{\partial q_\alpha} \delta q_\alpha \right) = m_i \left( \frac{d}{dt} \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha} \right) \delta q_\alpha$$

$$\text{由 } \frac{\partial \vec{r}_i}{\partial q_\alpha} = \frac{\partial \vec{r}_i}{\partial \dot{q}_\alpha} / \text{先有 } \frac{d}{dt} \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha} = \frac{d}{dt} \left( \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha} \right) - \dot{\vec{r}}_i \cdot \left( \frac{d}{dt} \frac{\partial \vec{r}_i}{\partial q_\alpha} \right) \quad (*)$$

$$\text{再由 } \frac{d}{dt} \frac{\partial \vec{r}_i}{\partial \dot{q}_\alpha} = \frac{\partial \vec{r}_i}{\partial q_\alpha} / (*) \leftarrow \frac{d}{dt} \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha} = \frac{d}{dt} \left( \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha} \right) - \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_\alpha}$$

$$\therefore m_i \left( \frac{d}{dt} \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha} \right) \delta q_\alpha = \frac{d}{dt} \left[ \frac{\partial (m_i \frac{\vec{r}_i \cdot \dot{\vec{r}}_i}{2})}{\partial \dot{q}_\alpha} \right] - \frac{\partial (m_i \frac{\vec{r}_i \cdot \dot{\vec{r}}_i}{2})}{\partial q_\alpha} = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial T}{\partial q_\alpha}$$

再由  $\vec{F}_i \cdot \delta \vec{r}_i = Q_\alpha \delta q_\alpha$  得:

$$\left( \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial T}{\partial q_\alpha} - Q_\alpha \right) \delta q_\alpha = 0 \Leftrightarrow \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial T}{\partial q_\alpha} = Q_\alpha \quad (**)$$

(\*\*) 式即为一般意义 T 的 Lagrangian Equations.

# 保守系下的方程变形:

$$\text{由于 } \left\{ \begin{array}{l} Q_\alpha = -\frac{\partial V}{\partial q_\alpha} \\ \frac{\partial V}{\partial \dot{q}_\alpha} = 0 \end{array} \right. \text{ 代入: 记 } L = T - V, \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\alpha} - \frac{\partial L}{\partial q_\alpha} = 0$$



#. d'Alembert 原理及 Lagrangian Equations 应用到解题中:

1. 一般情况下不用推导、变形. 牢记  $\begin{cases} L = T - V \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \end{cases}$
2. 一定区分 V 是保守势! ( $\frac{\partial V}{\partial q_i} = 0$  不能忘!)

※. 特殊情形: 广义有势体系,  $U = U(t, q, \dot{q})$  且可解.

切记:  $Q_i = -\frac{\partial U}{\partial q_i} + \frac{d}{dt} \frac{\partial U}{\partial \dot{q}_i}$  (凑形式) 这样令  $L = T - U$  仍可成立 Lagrangian Equations.

[例1] HW(T6).  $U = V(\vec{r}) + \vec{r} \cdot \vec{J}$ . 首先确认  $V(\vec{r})$  为保守势.

其次,  $\vec{J} = \vec{r} \times \vec{p} = m \vec{r} \times \vec{v}$  (得  $U = U(q, \dot{q})$ ).  $m \vec{r} \cdot (\vec{r} \times \vec{v}) = \begin{cases} m \vec{r} \cdot (\vec{v} \times \vec{r}) \\ m \vec{v} \cdot (\vec{r} \times \vec{r}) \end{cases}$

故广义力  $Q_i = -\frac{\partial U}{\partial r} + \frac{d}{dt} \frac{\partial U}{\partial v} = -\nabla V(\vec{r}) - m(\vec{v} \times \vec{r}) + \frac{d}{dt} [m(\vec{r} \times \vec{r})] = \underline{-\nabla V(\vec{r}) + 2m(\vec{r} \times \vec{v})}$

我们不难发现  $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$  ( $L = T - U$ ) 与  $\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_i$  在此题中恒等!

[例2]. 稳定电磁场中.  $\vec{F} = e\vec{E} + e\vec{v} \times \vec{B}$ .

显然地, 记  $\vec{B} = \nabla \times \vec{A}$ , (由于  $\nabla \cdot \vec{B} = 0 \sim$  Maxwell Equations), 记  $\vec{E} = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t}$

则  $\vec{F} = -e\nabla\varphi - e\frac{\partial \vec{A}}{\partial t} + e\vec{v} \times (\nabla \times \vec{A}) = -e\nabla\varphi - e\frac{\partial \vec{A}}{\partial t} + e\nabla(\vec{v} \cdot \vec{A}) - e\vec{v} \cdot (\nabla \vec{A})$

$[\vec{A} \times (\nabla \times \vec{B}) = \nabla(\vec{A} \cdot \vec{B}) - \vec{A} \cdot \nabla \vec{B}] = -e\nabla(\varphi - \vec{v} \cdot \vec{A}) - e(\frac{\partial \vec{A}}{\partial t} + \vec{v} \cdot (\nabla \vec{A}))$

$[\vec{v} \cdot (\nabla \vec{A}) = \frac{\partial \vec{A}}{\partial t} \cdot \frac{\partial \vec{A}}{\partial \vec{A}} + \frac{\partial \vec{A}}{\partial t} \cdot \frac{\partial \vec{A}}{\partial \vec{A}} + \frac{\partial \vec{A}}{\partial t} \cdot \frac{\partial \vec{A}}{\partial \vec{A}}] = -e\nabla(\varphi - \vec{v} \cdot \vec{A}) - e\frac{d\vec{A}}{dt}$

$[\frac{\partial(e\varphi - e\vec{v} \cdot \vec{A})}{\partial v} = \underline{-e\vec{A}}] = \underline{-\nabla[e\varphi - e\vec{v} \cdot \vec{A}] - \frac{d}{dt}[-e\vec{A}]}$   
 $\rightarrow$  取广义势  $U = e\varphi - e\vec{v} \cdot \vec{A}$

故代入有:  $L = T - U = \frac{m}{2}v^2 - e(\varphi - \vec{v} \cdot \vec{A})$ .

## 5. 初积分 (进阶!!!)

#. 广义能量积分.

引理. 定义  $f(q_1, q_2, \dots, q_n) = \lambda^k f(q_1, q_2, \dots, q_n)$  此类函数为 k 次齐次函数

k 次齐次函数  $f$  满足:  $q_i \frac{\partial f}{\partial q_i} = k \cdot f$

引入:  $\frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i$

地址: 中国 北京市海淀区颐和园路5号

邮编: 100871

www.pku.edu.cn

在主动全保守力、完整的系统中，

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \quad \text{而} \quad \ddot{q}_i = \frac{d}{dt} \dot{q}_i$$

$$\therefore \frac{dL}{dt} = \frac{\partial L}{\partial t} + \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \dot{q}_i + \frac{\partial L}{\partial q_i} \left( \frac{d}{dt} q_i \right) = \frac{\partial L}{\partial t} + \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right)$$

$$\text{记 } p_i = \frac{\partial L}{\partial \dot{q}_i} \text{ 为 "广义动量"} \Rightarrow \frac{d}{dt} (p_i \dot{q}_i - L) = - \frac{\partial L}{\partial t}$$

$$\# \text{ 记广义能量函数 } H = p_i \dot{q}_i - L = \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L, \quad \frac{\partial H}{\partial t} = - \frac{\partial L}{\partial t}$$

若  $L$  不是  $t$  的显函数,  $L = L(q, \dot{q})$  则  $\frac{\partial H}{\partial t} = 0$  称  $H$  为 Jacobi 积分 / 广义能量积分.

在广义有势体系中,  $L = T - V$ . 记  $T = \frac{m_i}{2} \dot{q}_i^2 = T_2(\dot{q}_i) + T_1(\dot{q}_i) + T_0$

变换  
到  
显含  $t$   $\left\{ \begin{array}{l} \text{其中 } T_2 \text{ 代表 } \dot{q}_i \text{ 平方项} \\ \text{则由 } \left\{ \begin{array}{l} \frac{\partial T_2}{\partial \dot{q}_i} = 2T_2 \\ \frac{\partial T_1}{\partial \dot{q}_i} = T_1 \end{array} \right\} \Rightarrow H = \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L = T_2 - T_0 + V \end{array} \right.$

有时称  $H = T_2 - T_0 + V$  为 Hamilton 量.

特别地,  $\frac{\partial L}{\partial t} = 0$  时,  $E = T + V$  守恒

# 广义动量积分:

若  $L$  不含某个广义坐标  $q_i$ , 即  $\frac{\partial L}{\partial q_i} = 0$ , 称其为循环坐标 / 可遗坐标.

则  $\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = 0 \Leftrightarrow$  广义动量  $p_i = \frac{\partial L}{\partial \dot{q}_i}$  守恒.

# 初积分应用至解题:

# 一般条件下直接无脑  $H = T + V$  守恒.

# 不建议坐标变换推出惯性力

(例. HW1 (T3) 直接  $T'' = \frac{1}{2} m \omega^2 r^2$ , 如果  $\omega$  给定且恒定).


# 出现多项项的, 注意归类分离

如电磁场中  $L = \frac{m}{2} v^2 - e\varphi + e\vec{v} \cdot \vec{A}$   $\left\{ \begin{array}{l} T_0 = 0 \\ T_1' = e\vec{v} \cdot \vec{A} \text{ (和含动能中 } \vec{v} \text{ 之项)} \\ T_2 = \frac{m}{2} v^2 \\ V = -e\varphi \end{array} \right. \Rightarrow H = \frac{m}{2} v^2 + e\varphi \equiv \text{const}$

或直接用  $-V = T_0'(x)$  即  $\left\{ \begin{array}{l} T_1' = -e\varphi \\ T_1' = e\vec{v} \cdot \vec{A} \\ T_2' = T_2 = \frac{m}{2} v^2 \end{array} \right\} \Rightarrow H = \frac{m}{2} v^2 + e\varphi \equiv \text{const}$

6. 线性非完整约束 // Lagrange 乘子法.

引理. Lagrange 乘子法.

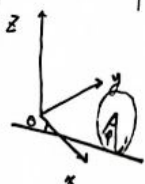
给定  $\begin{cases} z = f(x, y) \\ g(x, y) = 0 \end{cases}$   构造  $F = f + \lambda g$  求极值  $\begin{cases} \frac{\partial F}{\partial x} = 0 \\ \frac{\partial F}{\partial y} = 0 \\ \frac{\partial F}{\partial \lambda} = 0 \quad (g=0) \end{cases}$

\* 线性非完整约束:

$$A_{r\alpha} \dot{q}_\alpha + B_r = 0. \quad \text{取微分, } A_{r\alpha} dq_\alpha + B_r dt = 0 \Leftrightarrow A_{r\alpha} \delta q_\alpha = 0$$

应用,  $\begin{cases} \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial T}{\partial q_\alpha} - Q_\alpha = \lambda_r A_{r\alpha} & (1) \\ A_{r\alpha} \dot{q}_\alpha + B_r = 0 & (2) \end{cases}$  分别求解.

Lagrange 乘子法在解题中的应用.

\* 提前标好  $A_{r\alpha}$ , 表明是对第  $\alpha$  个广义坐标的第  $r$  个约束系数.例: (课堂例) [纯滚圆盘] 四个广义坐标  $x, y, \theta, \varphi$ 

$$\begin{cases} \dot{x} \sin \theta - \dot{y} \cos \theta = 0 \\ \dot{x} \cos \theta + \dot{y} \sin \theta = r \dot{\varphi} \end{cases} \rightarrow \text{取 } \delta \text{ 得 } \begin{cases} A_{11} \delta x - A_{12} \delta y = 0 \\ A_{21} \delta x + A_{22} \delta y - A_{2\varphi} \delta \varphi = 0 \end{cases}$$

$$\text{写出 } T = \frac{m}{2} (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I_2 \dot{\varphi}^2 + \frac{1}{2} I_1 \dot{\theta}^2 \quad \begin{cases} I_1 = \frac{mr^2}{4} \\ I_2 = \frac{mr^2}{2} \end{cases}$$

记  $V = 0$

(1) 分别给出:  $\begin{cases} m\ddot{x} = \lambda_1 A_{11} + \lambda_2 A_{21} = \lambda_1 \sin \theta + \lambda_2 \cos \theta & (1) \\ m\ddot{y} = \lambda_1 A_{12} + \lambda_2 A_{22} = -\lambda_1 \cos \theta + \lambda_2 \sin \theta & (2) \\ I_1 \ddot{\theta} = 0 & (3) \\ I_2 \ddot{\varphi} = -\lambda_2 r & (4) \end{cases}$  (2) 给出:  $\begin{cases} \dot{x} \sin \theta - \dot{y} \cos \theta = 0 & (5) \\ \dot{x} \cos \theta + \dot{y} \sin \theta = r \dot{\varphi} & (6) \end{cases}$

求解: 首先有  $\dot{\theta} = \omega$  定常,  $\theta = \omega t + \theta_0$ ;  $(1) \cdot \sin \theta - (2) \cdot \cos \theta \Rightarrow m\ddot{x} \sin \theta - m\ddot{y} \cos \theta = \lambda_1$ 

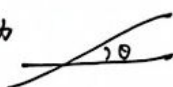
$$\Rightarrow \text{化为 } m \frac{d}{dt} (\dot{x} \sin \theta - \dot{y} \cos \theta) - m (\dot{x} \cos \theta + \dot{y} \sin \theta) \dot{\theta} \Rightarrow \lambda_1 = -m r \dot{\varphi} \dot{\theta}, \quad \text{同理 } \lambda_2 = m r \ddot{\varphi}$$

$$\Rightarrow (I_2 - m r^2) \ddot{\varphi} = 0, \Rightarrow \varphi = C t + \varphi_0$$

$$\therefore \dot{x} = r \dot{\varphi} \cos \theta = r C \cos(\omega t + \theta_0) \quad \text{同理得 } y.$$

$$\text{得 } (x - x_0)^2 + (y - y_0)^2 = \frac{r^2 C^2}{\omega^2}$$

\* 常用约束条件: 沿某方向运动



$$\Rightarrow \dot{x} \sin \theta - \dot{y} \cos \theta = 0 \quad (*)$$

\* 常用 (\*) 技巧: 令  $\begin{cases} \dot{x} = k \cos \theta \\ \dot{y} = k \sin \theta \end{cases} \Rightarrow \ddot{x} = K' (\cos \theta)' + \frac{dk}{dt} \cos \theta, \quad \text{若 } \theta = \omega t + \theta_0 \Rightarrow \ddot{x} = -\omega k \sin \theta + \frac{dk}{dt} \cos \theta$

地址: 中国 北京市海淀区颐和园路5号

邮编: 100871

www.pku.edu.cn



## # Background Knowledge:

1 两体运动, 约化质量  $\mu = \frac{Mm}{M+m}$ ;  $M \gg m$  时  $\mu \approx m$

1 极坐标下  $T = \frac{1}{2} m' v'^2 = \frac{1}{2} \mu [\dot{r}^2 + r^2 \dot{\varphi}^2]$

1 第一宇宙速度  $v_1 = \sqrt{\frac{GM}{R}}$ ; 第二宇宙速度  $\frac{1}{2} m v'^2 - \frac{GM_0 m}{R} = 0$ ,  $v = \sqrt{\frac{2GM_0}{R}}$ ,  $M_0$  太阳质量.

借助地球公转可使  $v_3 = (\sqrt{2}-1) \sqrt{\frac{GM_0}{R}}$

1 定义  $V(r) = -\int_0^{\infty} F(r) dr$ ,  $V|_{r \rightarrow \infty} = 0$ , 平方反比下  $V = -\frac{GMm}{r}$

## # Variations on Lagrangian Equations:

# Define  $L = \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\varphi}^2) - V(r)$

(1)  $\frac{\partial L}{\partial \dot{\varphi}} = \mu r^2 \dot{\varphi} \equiv \text{const}$ , 取  $r^2 \dot{\varphi} = h$

(2).  $\frac{\partial L}{\partial t} = 0$ ,  $H = \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\varphi}^2) + V(r) = E$  (\*)

(\*)  $\Leftrightarrow \frac{1}{2} \mu \dot{r}^2 + \frac{1}{2} \mu \frac{h^2}{r^2} + V(r) = E$

# Define  $\frac{\mu h^2}{2r^2} + V(r) = V_{\text{eff}}(r)$

## # Differential Equations given:

$$\textcircled{1} \frac{dr}{\sqrt{\frac{2}{\mu}(E - V_{\text{eff}}(r))}} = dt \quad \overset{\dot{\varphi} = \frac{h}{r^2}}{\parallel} \quad \frac{h dr}{r^2 \sqrt{\frac{2}{\mu}(E - V_{\text{eff}}(r))}} = d\varphi$$

$$\# \frac{dr}{dt} = \frac{h}{r^2} \frac{dr}{d\varphi} = \frac{-h du}{d\varphi} \quad (\text{let } u = \frac{1}{r}) \Leftrightarrow \dot{r} = -h \frac{du}{d\varphi}$$

## \textcircled{2} Binet's Equations

$$1 \quad \frac{1}{2} \mu h^2 \left[ \left( \frac{du}{d\varphi} \right)^2 + u^2 \right] + V\left(\frac{1}{u}\right) = E$$


$$1 \quad -\mu h^2 u^2 \left[ \frac{d^2 u}{d\varphi^2} + u \right] = F\left(\frac{1}{u}\right).$$

# # Kepler's Inverse square law & Solutions,

$$V(p) = -\frac{GMm}{R}$$

$$p = \frac{\frac{h^2 \mu}{GMm}}{1 + \left( \sqrt{1 + \frac{2Eh^2 \mu}{G^2 M^2 m^2}} \right) \cos(\psi + C)} \xrightarrow[\mu \approx m]{M \gg m} p = \frac{\frac{h^2}{GM}}{1 + \sqrt{1 + \frac{2Eh^2}{G^2 M^2 m}} \cos(\psi + C)} \triangleq \frac{p}{1 + e \cos(\psi + C)} \Leftrightarrow \begin{cases} p = \frac{h^2}{GM} \\ e = \sqrt{1 + \frac{2Eh^2}{G^2 M^2 m}} \\ p \sim \begin{cases} \psi_0 \\ p_0 \end{cases} \end{cases}$$

其中  $h = r^2 \dot{\psi} = p v_{\psi}$ . 若给定初条件可求得.

例: (HW # T.)  $v_0 = \sqrt{\frac{GM}{4R}}$ .   $\Rightarrow h = R \cdot \sqrt{\frac{GM}{8R}} = \sqrt{\frac{GM R}{8}} \Rightarrow p = \frac{R}{8 + 5\sqrt{2} \cos(\psi + C)}$

$$E = -\frac{GMm}{R} + \frac{1}{2}m \cdot \frac{GM}{4R} = -\frac{7GMm}{8R}$$

C 由初条件  $\begin{cases} \psi = 0 \\ p = R \end{cases}$  给出  $\beta = \arccos -\frac{\sqrt{2}}{10}$ .

## # Parameters in Elliptical Orbit.

半长轴  $a = \frac{1}{2} \left[ \frac{p}{1-e} + \frac{p}{1+e} \right] = \frac{GMm}{2|E|} \quad (\Delta)$

$b = h \sqrt{\frac{\mu}{2|E|}} \approx h \sqrt{\frac{m}{2|E|}} \quad (\Leftrightarrow E \sim |E|)$

$$\tau = \frac{\pi ab}{p^2 \dot{\psi}/2} = \frac{2\pi a^2 \sqrt{1-e^2}}{h} = \pi GM \sqrt{\frac{a^3}{2E^3}}$$

显然地 (Trivial).  $\frac{\tau^2}{a^3} \equiv \frac{4\pi^2}{GM}$ , (The Reader is invited to do it as an exercise).

## # (From Textbook, not mentioned) Kepler's Equations

(Define / Simulate  $\mu \approx m$ )

(1)  $E < 0, \Leftrightarrow p - a = -ae \cos \psi \Rightarrow \frac{\sqrt{GM}}{a^{3/2}} t = \psi - e \sin \psi$

(2)  $E = 0 \Leftrightarrow p = \frac{h^2}{2GM} (1 + \psi^2) \Rightarrow t = \frac{h^3}{2k^3} (\psi + \frac{1}{3} \psi^3) + \text{const}$

(3)  $E > 0 \Leftrightarrow p + a = ae \cosh \psi \Rightarrow \frac{\sqrt{GM}}{a^{3/2}} t = e \sinh \psi - \psi$

## # Energy in Elliptical Orbit:

$|E| = -\frac{GMm}{2a} \quad (\text{由 } \Delta)$

## # Stability of Elliptical Orbits;

essence of the Elliptical Orbit:

$$\left. \frac{\partial V_{\text{eff}}}{\partial p} \right|_{p=r_0} = 0$$

1# Stability:  $\left. \frac{\partial^2 V_{\text{eff}}}{\partial \rho^2} \right|_{\rho=r_0} > 0$ . 由  $\frac{\partial V_{\text{eff}}}{\partial \rho} = \frac{\partial V}{\partial \rho} - \frac{\mu h^2}{r_0^3} = 0$

$$\frac{\partial^2 V_{\text{eff}}}{\partial \rho^2} = \frac{\partial^2 V}{\partial \rho^2} + \frac{3\mu h^2}{r_0^4} = -\left. \frac{\partial F(\rho)}{\partial \rho} \right|_{\rho=r_0} - \frac{3}{r_0} \cdot F(\rho)|_{\rho=r_0} = -F'(r_0) - 3F(r_0)$$

$$\Leftrightarrow \frac{F'(r_0)}{F(r_0)} + \frac{3}{r_0} > 0 \quad (\text{由于 } F < 0).$$

$$1\# \omega = 2\pi \sqrt{\frac{m}{\left. \frac{\partial^2 V_{\text{eff}}}{\partial \rho^2} \right|_{\rho=r_0}}}$$

例1. 库泊,  $V = -\frac{km}{\rho^n}$

$$\Rightarrow V_{\text{eff}} = -\frac{km}{\rho^n} + \frac{mh^2}{2\rho^2}$$

$$\begin{cases} \frac{dV_{\text{eff}}}{d\rho} \Big|_{\rho=r_0} = 0 \Rightarrow \frac{1}{r_0^{n+2}} = \frac{h^2}{kn} \\ \frac{d^2V_{\text{eff}}}{d\rho^2} \Big|_{\rho=r_0} > 0 \Rightarrow \frac{-knn(n+1)}{r_0^{n+3}} + \frac{3mh^2}{r_0^4} > 0 \end{cases} \Rightarrow 3 > n+1, \quad n < 2$$

例2. HWTo.  $V = -k \frac{e^{-a\rho}}{\rho} \Rightarrow V_{\text{eff}} = -k \frac{e^{-a\rho}}{\rho} + \frac{J^2}{2m\rho^2}$  (此  $J = mh$  角动量守恒).

$$\left\{ \frac{dV_{\text{eff}}}{d\rho} \Big|_{\rho=r_0} = 0 \Rightarrow J_0^2 = km e^{-a\rho_0} (a\rho_0^2 + \rho_0) \Rightarrow V_0 = \frac{ke^{-a\rho_0}}{2\rho_0} (a\rho_0 + 1) < 0 \Rightarrow a\rho_0 + 1 < 0 \right.$$

1# Judge:  $\frac{3}{\rho_0} F(\rho_0) + F'(\rho_0) = -ke^{-a\rho_0} \frac{1}{\rho_0^3} [1 + a\rho_0(1 - a\rho_0)] < 0$ , Stable!

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{1}{m} \left[ -\frac{3}{\rho_0} F(\rho_0) - F'(\rho_0) \right]} = 2\pi \sqrt{m\rho_0^3 / k e^{-a\rho_0} (1 + a\rho_0 - a^2\rho_0^2)}$$

# Extension: Bertrand's Theorem (Not required):

Binet's Equations (1) States that:  $\frac{d^2 u}{d\varphi^2} + u = -\frac{m}{\rho^2 u^2} f\left(\frac{1}{u}\right) = J(u)$

let  $u' = u_0 + \xi$ ,  $\xi(\varphi) \ll u_0$ .

$$J(u_0 + \xi) = J(u_0) + J_0' \xi + \frac{1}{2} J_0'' \xi^2 + \frac{1}{6} J_0''' \xi^3 + o(\xi^4)$$

Circuit of circular Orbit given:  $\frac{d^2 u_0}{d\varphi^2} + u_0 = J(u_0)$

$$\Rightarrow \frac{d^2 \xi}{d\varphi^2} + (1 - J_0') \xi = \frac{J_0''}{2} \xi^2 + \frac{J_0'''}{6} \xi^3$$



$$\text{let } \rho^2 = 1 - J_0' \Rightarrow \frac{d^2 \xi}{d\varphi^2} + \rho^2 \xi = 0, \quad \rho^2 > 0 \text{ 特解 } e^{\pm i\rho\varphi};$$

$$1 - J_0' > 0 \Rightarrow J_0' < 1$$

# 闭合轨道:  $\xi \sim \cos(\rho\varphi)$ , let  $\rho = \frac{m}{n} \in \mathbb{Q}$

$$J'(u_0) = -\frac{2J}{u} + \frac{J}{f} \frac{df}{du} = -2 + \frac{J}{f} \frac{df}{du}$$

$$\rho^2 = 1 - J_0' = 3 - \frac{u_0}{f(u_0)} \frac{df}{du} \Big|_{u=u_0} \Rightarrow \frac{df}{dr} = (\rho^2 - 3) \frac{f}{r},$$

$$\text{假设 } f = -kr^\gamma = -kr^{\rho^2-3} \Rightarrow \gamma > -3$$

再将  $J_0''$ ;  $J_0'''$  代入有:  $\rho^2(1-\rho^2)(4-\rho^2)=0 \Rightarrow \rho=1 \text{ 或 } 2$

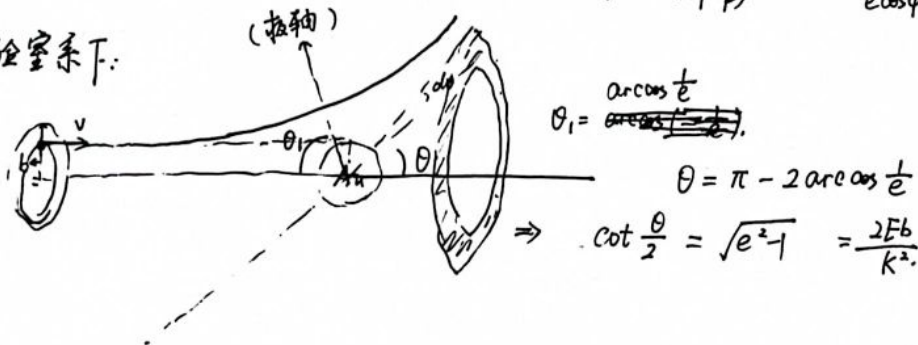
$f \propto r^{-2}$  /  $f \propto r$  轨道闭合.

# 散射问题:

$$\# \text{ 记 } k = \frac{qQ}{4\pi\epsilon_0}, \quad V = \frac{k}{\rho} > 0$$

$$\text{记 } p = \frac{mh^2}{k}, \quad e = \sqrt{1 + \frac{2Eh^2 m}{k^2}}, \quad \rho = \frac{-p}{1 - e \cos(\varphi - \beta)} = \frac{p}{e \cos \varphi' - 1}$$

# 考虑实验室系:



$$\text{若是一串粒子: } 2\pi b db \cdot I = \sigma I d\Omega = \sigma I \int_0^\pi \sin\theta d\theta d\varphi = 2\pi \sigma I \sin\theta d\theta$$

$$\Rightarrow \sigma = \frac{b db}{\sin\theta d\theta}; \quad \sigma = \frac{dS}{d\Omega}$$

$$\frac{(\sum l n_{\pm}) dS}{\sum} = \frac{dN}{N}$$

$$dS = \frac{dN}{N} \cdot \frac{1}{l n_{\pm}} \text{ 可由测量得到!}$$

$$\therefore \text{估计: } \cot \frac{\theta}{2} = \frac{1}{k} \sqrt{2E\mu} h; \quad E = \frac{1}{2} v_0^2, \quad h = bv_0, \quad \cot \frac{\theta}{2} = \frac{2E}{k} db$$

$$\text{代入} \Rightarrow \sigma(\theta) = \left| \frac{k}{4E} \right|^2 \frac{1}{\sin^4 \frac{\theta}{2}}$$

1/ 推论: 总散射截面  $\sigma_T = \int_0^\pi \left(\frac{k}{4E}\right)^2 \frac{1}{\sin^4 \frac{\theta}{2}} 2\pi \sin\theta d\theta = \infty$

$$(\sigma_T = \int_{4\pi} \sigma d\Omega = 2\pi \int_0^\pi \sigma(\theta) \sin\theta d\theta)$$

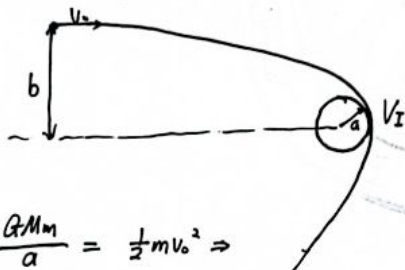
\* 实验室系:  $\tan \Theta_{\text{rel}} = \frac{\sin\theta}{\cos\theta + \frac{m_1}{m_2}} \quad (*)$ , ( $m_1 = m_\alpha$ ,  $m_2 = m_{Au}$ )

$m_1 \ll m_2$  时  $\tan \Theta_{\text{rel}} = \tan\theta$

(\*) 的推导: (书 P119)  $\begin{cases} V_0 = \frac{m_1}{m_1+m_2} V_i \\ V_{i0}' = \frac{m_1}{m_1} V_0 \\ V_{20}' = -V_0 \end{cases} \Rightarrow \text{散射后 } V_0 \text{ 不变}, V_i' = V_{i0}'$

$$\Rightarrow \tan \Theta_r = \frac{V_i' \sin\theta_1}{V_0 + V_i' \cos\theta_1} = \frac{\sin\theta_1}{\frac{m_1}{m_1} + \cos\theta_1}$$

例: (Hw. 7.5) 拦截面



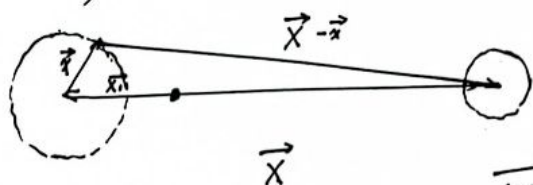
$$\frac{1}{2} m V_1'^2 - \frac{GMm}{a} = \frac{1}{2} m V_0^2 \Rightarrow$$

$$V_1'^2 = V_0^2 + \frac{2GM}{a} \triangleq V_0^2 + u^2;$$

$$h = V_0 b = V_1 a. \therefore (V_0 = \frac{h}{b}) \quad b = \frac{h}{V_0}$$

$$\sigma_T = \int \sigma d\Omega = \int_0^b 2\pi b' db' = \pi b^2 = \pi \cdot \frac{h^2}{V_0^2} = \pi a^2 \left(1 + \frac{u^2}{V_0^2}\right).$$

\* 潮汐现象:



$$1/4 (?) \Omega^2 = \frac{G(m_1+m_2)}{R^3}, \quad |\vec{r}| = r$$

$$\Phi(r) = -\frac{Gm_1}{r} - \frac{GM_2}{|\vec{r}-\vec{R}|} - \frac{1}{2} \Omega^2 (\vec{r} + \vec{r}')^2$$

$$\text{记 } R = |\vec{R}| \Rightarrow$$

$$\frac{1}{|\vec{r}-\vec{R}|} \approx \frac{1}{R} \left[ 1 + \frac{\vec{R} \cdot \vec{r}}{R} - \frac{r^2}{2R^2} + \frac{3}{2} \left( \frac{\vec{R} \cdot \vec{r}}{R} \right)^2 + \dots \right] \quad (*)$$

(\*) 式得:  $\frac{1}{|R-\vec{r}|} = \frac{1}{R} \left[ \left( 1 - \frac{\vec{r} \cdot \vec{R}}{R^2} \right)^{-\frac{1}{2}} \right] = \frac{1}{R} \left[ 1 - \frac{2\vec{r} \cdot \vec{R}}{R^2} + \frac{r^2}{R^2} \right]^{-\frac{1}{2}}$

对其进行级数展开:  $(1-x)^{-\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \dots$  (容许因不可抗力导致的小小误差)

保留对 R 阶项以得 (\*) 式

代入 (1) 得:  $\Phi(x) = -\frac{Gm_1}{r} - \frac{Gm_2}{R} - \frac{1}{2}R^2(\vec{r}^2 + \vec{r}^2) + \frac{Gm_2 \vec{r}^2}{2R^3} - \frac{3}{2} \frac{Gm_2}{R} \left( \frac{\vec{r} \cdot \vec{R}}{R} \right)^2$

[疑问]:  $-\frac{1}{2}R^2(\vec{r} + \vec{r})^2$  为何变化? 比得:

$\frac{R^2 \cdot \vec{r} \cdot \vec{r}}{R^3} = \frac{G(m_1+m_2)}{R^3} (\vec{r} \cdot \vec{r})$ , 考虑  $\vec{r} = -\frac{\vec{R} m_2}{m_1+m_2}$

得: 上式 =  $-\frac{Gm_2}{R^3} (\vec{r} \cdot \vec{r}) = -\frac{Gm_2}{R^3} (\vec{r} \cdot \vec{R})$ ; 取负得  $\frac{Gm_2}{R^3} (\vec{r} \cdot \vec{R})$

而这恰与 (1) 处式:  $-\frac{Gm_2}{R} \cdot \frac{\vec{r} \cdot \vec{R}}{R} = -\frac{Gm_2}{R^2} (\vec{r} \cdot \vec{R})$  相消!

式中  $-\frac{3}{2} \frac{Gm_2}{R} \left( \frac{\vec{r} \cdot \vec{R}}{R} \right)^2$  项, 称为引潮力。

[Extensions]: 上课时提到, 由于地球表面液体的黏滞阻力,

地球表面水应呈 (b) 状

(a) 为理想态。



// Summary:

$\frac{h dp}{p^2 \sqrt{\frac{2}{\mu}(E - V_{eff})}} = d\psi$ ;  $p = \frac{P}{1 + e \cos(\psi - \beta)}$

$\begin{cases} \frac{\partial V_{eff}}{\partial p} = 0 \\ \frac{\partial^2 V_{eff}}{\partial p^2} > 0 \end{cases} \Leftrightarrow \frac{F'(p_0)}{F(p_0)} + \frac{3}{r_0} > 0$

$\tau = 2\pi \sqrt{\frac{m}{\left| \frac{\partial^2 V_{eff}}{\partial p^2} \right|_{p=p_0}}}$

$p' = \frac{p'}{e' \cos \psi'' - 1}$   $\begin{cases} p' = \frac{mh^2}{k} \\ e' = \sqrt{1 + \frac{2E'm}{k^2}} \end{cases}$

$\cot \frac{\theta}{2} = \frac{2Eb}{k^2}$

$\sigma = \frac{dS}{d\Omega} = \int_0^\pi 2\pi \sigma(\theta) \sin \theta d\theta$

$\sigma(\theta) = \left( \frac{k}{4E} \right)^2 \frac{1}{\sin^4 \frac{\theta}{2}} \Leftrightarrow db = \frac{b}{\sin \theta} \cdot \left( \frac{db}{d\theta} \right)$





## 小振动.

## # Preview &amp; Background Knowledge:

/ 小振动问题一般取质点偏移平衡位置的量为广义坐标

/  $V \propto x^n$  ( $n \geq 0$ ). 使用 Taylor 展开技巧.

/ 弹性势能不考虑正负, 直接无脑加

展开  $V(x)$ :  $V(x) = V(x_0) + \frac{\partial V}{\partial x}|_{x_0} (x-x_0) + \frac{1}{2} \frac{\partial^2 V}{\partial x^2}|_{x_0} (x-x_0)^2 + O((x-x_0)^3)$

由  $\frac{\partial^2 V}{\partial x^2}$  项入拉氏方程: ( $-\frac{\partial V}{\partial x}|_{x_0} = 0$  代表“不受力”)

$$L = \frac{m}{2} \dot{x}^2 - \frac{1}{2} \frac{\partial^2 V}{\partial x^2}|_{x_0} x^2 \triangleq \frac{m}{2} \dot{x}^2 - \frac{k}{2} x^2$$

由  $\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x}$ ,  $m\ddot{x} = -kx$  记  $\omega_0^2 = \frac{k}{m} \Rightarrow \ddot{x} + \omega_0^2 x = 0$

/ Simple Harmonic Oscillation.  $x = K_1 \cos \omega_0 t + K_2 \sin \omega_0 t \stackrel{\Delta}{=} A e^{i\omega_0 t}$

/ 受迫振动: 对复力作 Fourier 展开, 取不同频次的单频力分析:

$$\ddot{x} + \omega_0^2 x = f e^{i\omega t}, \text{ 给出 } \omega \neq \omega_0 \text{ 时 } x = A e^{i\omega_0 t} + \frac{f}{\omega_0^2 - \omega^2} e^{i\omega t};$$

$\omega = \omega_0$  时称为共振.  $x = A e^{i\omega_0 t} + \frac{f t}{2i\omega_0} e^{i\omega_0 t}$  (第2项  $\propto t$ )

/ 阻尼振动:  $\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = 0$

$$\begin{cases} \gamma^2 < \omega_0^2, & x = A_1 e^{-\gamma t} \cdot e^{i\sqrt{\omega_0^2 - \gamma^2} t} + A_2 e^{-\gamma t} e^{-i\sqrt{\omega_0^2 - \gamma^2} t} \\ \gamma^2 > \omega_0^2, & x = A_1 e^{-\gamma t} \cdot e^{\sqrt{\gamma^2 - \omega_0^2} t} + A_2 e^{-\gamma t} e^{-\sqrt{\gamma^2 - \omega_0^2} t} \\ \gamma^2 = \omega_0^2, & x = A e^{-\gamma t} + B t e^{-\gamma t} \end{cases}$$

// 受迫阻尼振动,  $\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = f e^{i\omega t}$  给出特解  $x' = \frac{f}{(\omega_0^2 - \omega^2) + 2\gamma i \omega}$

$\gamma^2 < \omega_0^2$  得  $x = A e^{-\gamma t} e^{i\sqrt{\omega_0^2 - \gamma^2} t} + \frac{f}{(\omega_0^2 - \omega^2) + 2\gamma i \omega} e^{i\omega t}$

$$|A_{\text{稳}}|^2 = |A_{\text{特}}|^2 = \frac{f^2}{(\omega_0^2 - \omega^2)^2 + 4\gamma^2 \omega^2}, \quad |A_m|^2 = \frac{f^2}{4\gamma^2 (\omega_0^2 - \gamma^2)}$$

$$Q = \frac{\omega_0}{2\gamma}$$

## \* Theory of Oscillation:

假设有  $N$  个粒子, 广义坐标取为  $q_1 \sim q_f$ :

$$L = T - V \triangleq \frac{1}{2} T_{\alpha\beta} \dot{q}_\alpha \dot{q}_\beta - \frac{1}{2} V_{\alpha\beta} q_\alpha q_\beta$$

Lagrangian Equations:  $T_{\alpha\beta} \ddot{q}_\beta + V_{\alpha\beta} q_\beta = 0$  取  $\vec{q} = \vec{b} e^{i\omega t}$

$$\Leftrightarrow (\vec{V} - \omega^2 \vec{T}) \vec{b} = 0$$

对任一非平凡解  $\vec{b}$ , 给出  $|\vec{V} - \omega^2 \vec{T}| = 0$ ,  $\vec{q} = \sum_{\vec{k}} \vec{b}_{\vec{k}} e^{i\omega_{\vec{k}} t}$

/ # 小振动, 考虑稳态有  $\vec{b}_r^T \vec{V} \vec{b}_r = \omega_r^2 (\vec{b}_r^T \vec{T} \vec{b}_r) > 0$  ( $\vec{T}$  为动能系数矩阵是  $\frac{1}{2}$  正定)

解的形式写作  $\vec{q} = \vec{B} \begin{pmatrix} f_1 \cos(\omega_1 t + \varphi_1) \\ \vdots \\ f_f \cos(\omega_f t + \varphi_f) \end{pmatrix}$

/ # 记  $\vec{b}_r = \hat{b}_r \cos(\omega_r t + \varphi_r)$ , 称  $\hat{b}_r$  为本征矢量.

//  $\vec{B}$  的特征,  $\vec{B}$  中每列矢量关于  $\vec{T}, \vec{V}$  正交 (A)

/// Extension: proof of (A): [This is proved by C++ x]

$$\text{取 } \begin{cases} \vec{V} \hat{b}_1 = \omega_1^2 \vec{T} \hat{b}_1 \\ \vec{V} \hat{b}_2 = \omega_2^2 \vec{T} \hat{b}_2 \end{cases} \Leftrightarrow \begin{cases} \hat{b}_1^T \vec{V} = \omega_1^2 (\hat{b}_1^T \vec{T}) \\ \hat{b}_2^T \vec{V} = \omega_2^2 (\hat{b}_2^T \vec{T}) \end{cases} \quad \left( \text{若有 } \vec{V}^T = \vec{V}, \vec{T}^T = \vec{T} \right)$$

$$\left( \text{考虑 } (\hat{b}_1^T \vec{V}) \hat{b}_2 = \omega_1^2 (\hat{b}_1^T \vec{T}) \hat{b}_2 \right) \quad \left( \text{或 } \hat{b}_1^T (\vec{V} \hat{b}_2) = \omega_2^2 \hat{b}_1^T (\vec{T} \hat{b}_2) \right) \quad \left. \vphantom{\begin{matrix} \text{考虑} \\ \text{或} \end{matrix}} \right\} \omega_2^2 \hat{b}_1^T \vec{V} \hat{b}_2 = \omega_1^2 \hat{b}_1^T \vec{T} \hat{b}_2$$

$$\Leftrightarrow \omega_2 \neq \omega_1 \text{ 时, } \hat{b}_1^T \vec{V} \hat{b}_2 = 0, \text{ 正交得证! (对 } \vec{T} \text{ 同理).}$$

## / Different Expressions & Normal Coordinates:

对  $q = \sum_{\vec{r}} b_{\vec{r}} \cos(\omega_{\vec{r}} t + \varphi_{\vec{r}})$ ,  $b_{\vec{r}} = \begin{pmatrix} b_{1\vec{r}} \\ \vdots \\ b_{f\vec{r}} \end{pmatrix}$

$$(A_1) \quad \vec{q} = \begin{pmatrix} \vdots \\ \vdots \\ \vdots \end{pmatrix} \cdot \begin{pmatrix} b_{11} \cos(\omega_1 t + \varphi_1) + b_{12} \cos(\omega_2 t + \varphi_2) \cdots + b_{1f} \cos(\omega_f t + \varphi_f) \\ \vdots \\ b_{f1} \cos(\omega_1 t + \varphi_1) + b_{f2} \cos(\omega_2 t + \varphi_2) \cdots + b_{ff} \cos(\omega_f t + \varphi_f) \end{pmatrix}$$

$$(A_2 - \text{Normal Coordinates}) \quad \vec{q} = (b_1, b_2, \dots, b_f) \begin{pmatrix} \cos(\omega_1 t + \varphi_1) \\ \vdots \\ \cos(\omega_f t + \varphi_f) \end{pmatrix} \triangleq \vec{B} \begin{pmatrix} \cos(\omega_1 t + \varphi_1) \\ \vdots \\ \cos(\omega_f t + \varphi_f) \end{pmatrix}$$

简正坐标表示为  $\vec{B}^T \vec{q}$



北京大学  
PEKING UNIVERSITY

produced by Xun-cheng

小振动解题的一般化步骤:

1. 分析系统, 写出拉氏量 ( $T$  与  $V$ ), 取好广义坐标.

2. 求出  $\vec{T}$ ,  $\vec{V}$  (在这步可选为更便广义坐标)

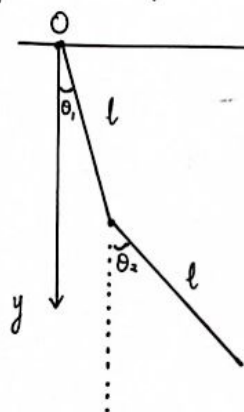
3. 解出  $|\vec{V} - \omega^2 \vec{T}| = 0$  的全部  $\omega$

4. 分别代入  $(\vec{V} - \omega^2 \vec{T}) \cdot \vec{b}_i = 0$  解出  $b_i$

5. 合并写出  $q = B(\vec{z})$

~/ Extensions: 用  $B^*q$  写出真正坐标.

例: (Hw T2 又邵立晶老师往年题)



解: ① 取  $\theta_1, \theta_2$  为广义坐标, 两棒质心表出为:

$$\begin{cases} x_1 = \frac{l}{2} \sin \theta_1 \\ y_1 = \frac{l}{2} \cos \theta_1 \end{cases} \quad \& \quad \begin{cases} x_2 = l \sin \theta_1 + \frac{l}{2} \sin \theta_2 \\ y_2 = l \cos \theta_1 + \frac{l}{2} \cos \theta_2 \end{cases}$$

$$T = \frac{1}{2} m (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m (\dot{x}_2^2 + \dot{y}_2^2) + \frac{m l^2}{24} (\dot{\theta}_1^2 + \dot{\theta}_2^2)$$

$P_i \cdot (\dot{q}_i)$   
勿漏刚体转动!  
(柯尼希定理)

$$V = -mgy_1 - mgy_2 = -mg \cdot \frac{3l}{2} \cos \theta_1 - mg \frac{l}{2} \cos \theta_2 \xrightarrow{\text{Taylor}} \frac{3mgl}{2} \theta_1^2 + \frac{1}{2} mgl \theta_2^2 \quad (\text{保留至 } 2^{\text{nd}} \text{ order})$$

$$\textcircled{2} \quad \vec{T} = \begin{pmatrix} \frac{4ml^2}{3} & -\frac{ml^2}{2} \\ -\frac{ml^2}{2} & \frac{ml^2}{3} \end{pmatrix}, \quad \vec{V} = \begin{pmatrix} \frac{3mgl}{2} & 0 \\ 0 & \frac{1}{2} mgl \end{pmatrix}$$

$$\textcircled{3} \quad \begin{vmatrix} \frac{3mgl}{2} - \frac{4ml^2}{3} \omega^2 & -\frac{ml^2}{2} \omega^2 \\ -\frac{ml^2}{2} \omega^2 & \frac{1}{2} mgl - \frac{ml^2}{3} \omega^2 \end{vmatrix} = 0 \Rightarrow \omega_1 = \sqrt{\frac{3g}{l} (1 + \frac{2\sqrt{5}}{3})}; \omega_2 = \sqrt{\frac{3g}{l} (1 - \frac{2\sqrt{5}}{3})}$$

$$\textcircled{4} \quad \text{代入 } \omega_1 \text{ 有 } b_1 = \begin{pmatrix} 1 \\ -\frac{1}{3} (1 + 2\sqrt{5}) \end{pmatrix}, \quad \text{代入 } \omega_2 \text{ 有 } b_2 = \begin{pmatrix} 1 \\ \frac{1}{3} (2\sqrt{5} - 1) \end{pmatrix}$$

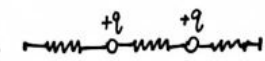
$$\textcircled{5} \quad q = \begin{pmatrix} 1 & 1 \\ -\frac{1}{3} (1 + 2\sqrt{5}) & \frac{1}{3} (2\sqrt{5} - 1) \end{pmatrix} \begin{pmatrix} f_1 \cos(\sqrt{\frac{3g}{l} (1 + \frac{2\sqrt{5}}{3})} t + \varphi_1) \\ f_2 \cos(\sqrt{\frac{3g}{l} (1 - \frac{2\sqrt{5}}{3})} t + \varphi_2) \end{pmatrix}$$

地址: 中国 北京市海淀区颐和园路5号

邮编: 100871 www.pku.edu.cn



# 1# 展开型势函数:

例: (HW T1.14) 

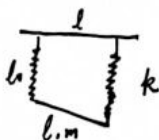
存在库仑斥力  $\Rightarrow V_Q = \frac{Kq^2}{r}$  在引入偏移量  $\begin{matrix} x_1+x_2 \\ \downarrow \\ \varepsilon_1 \\ \downarrow \\ \varepsilon_2 \\ \downarrow \\ x_1-x_2 \end{matrix}$  后

$$V_Q = \frac{Kq^2}{l-\varepsilon_2} = \frac{Kq^2}{l} \left(1 + \frac{\varepsilon_2}{l} + \frac{\varepsilon_2^2}{l^2}\right) \text{ 保留至二阶:}$$

这样求得  $\vec{V}_0 = \begin{pmatrix} 0 \\ 0 \\ \frac{2Kq^2}{l^3} \end{pmatrix}$  加入  $\vec{V}$  矩阵即可.

## // 选取合适的广义坐标

例: (HW T3)



/① 以平衡位置为参考:  (取此处  $V=0$ )

引入  $x_1, x_2$

$$T = \frac{1}{2}m\left(\frac{\dot{x}_1+\dot{x}_2}{2}\right)^2 + \frac{m l^2}{24}\left(\dot{x}_1-\dot{x}_2\right)^2, \quad V = \frac{1}{2}k(x_1^2+x_2^2) + mg\frac{x_1+x_2}{2}$$

$$\Rightarrow \vec{T} = \begin{pmatrix} \frac{m}{4} + \frac{ml^2}{12} & \frac{m}{4} - \frac{ml^2}{12} \\ \frac{m}{4} - \frac{ml^2}{12} & \frac{m}{4} + \frac{ml^2}{12} \end{pmatrix}, \quad \vec{V} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

$$|\omega^2 \vec{T} - \vec{V}| = 0 \text{ 不如计算!}$$

$$T = \frac{1}{2}m\dot{x}^2 + \frac{ml^2}{24}\dot{\theta}^2$$

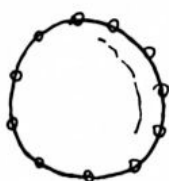
$$V = mgx + \frac{1}{2}k\left[\left(x+\frac{l}{2}\theta\right)^2 + \left(x-\frac{l}{2}\theta\right)^2\right] = mgx + kx^2 + \frac{k l^2}{4}\theta^2$$

$$\vec{T} = \begin{pmatrix} m & 0 \\ 0 & \frac{ml^2}{12} \end{pmatrix}; \quad \vec{V} = \begin{pmatrix} 2k & 0 \\ 0 & \frac{1}{2}kl^2 \end{pmatrix} \Rightarrow |\omega^2 \vec{T} - \vec{V}| = 0 \text{ 如算多了!}$$

/ '补救' 措施: ①中看到  $T$  与  $\frac{x_1+x_2}{2} // \frac{x_1-x_2}{2}$  相关时, 取  $\begin{cases} \varepsilon_1 = \frac{x_1+x_2}{2} \\ \varepsilon_2 = \frac{x_1-x_2}{2} \end{cases}$  即可

$$\text{这样 } \begin{cases} x_1^2+x_2^2 = 2(\varepsilon_1^2+\varepsilon_2^2) \\ x_1^2-x_2^2 = 4\varepsilon_1\varepsilon_2 \end{cases} \checkmark$$

## // Extensions: "环 n 球问题"



$$L = \sum \frac{m}{2} \dot{x}_n^2 + \left(-\frac{k}{2} \sum (x_n - x_{n+1})^2\right)$$

$$\Rightarrow \ddot{x}_n = \frac{k}{m} (-2x_n + x_{n-1} + x_{n+1}) \text{ (记 } \frac{k}{m} = \omega_0^2)$$

$$\text{求精解 } x_n = A e^{i\omega t + ipn} \Rightarrow$$

$$-\omega^2 = \omega_0^2 (e^{ip} + e^{-ip} - 2)$$

$$\text{而 } x_{i+N} = x_i \Leftrightarrow e^{ipN} = 1, \quad ipN = 2k\pi i, \quad k=0 \sim N-1$$

$$\therefore p = \frac{2k\pi}{N} \Leftrightarrow \omega^2 = \omega_0^2 (1 - \cos p)$$

/ 对于  $N=3$ , 可以利用矩阵计算.

$$\vec{T} = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix}, \quad \vec{V} = \begin{pmatrix} 2k & -k & -k \\ -k & 2k & -k \\ -k & -k & 2k \end{pmatrix} \text{ 消元亦可得!}$$

(至此为期中前内容)

略去了 拉格朗日-龙格-楞次  
变量 // 受迫阻尼振动能  
量分析等 Extension Part, 具体  
见完整版课堂笔记)



(注: 所有题都由

陈诚手写 &amp; 做过, 若有错

改系中途思路不当 || 做错导致)

## 第一章. 拉格朗日方程

/ 考点:

1. 达朗贝尔原理 / 拉氏方程的推导
2. 虚功原理解题 (3种)
3. 简单系统的拉氏方程. 初积分. 求解
4. 耗散力. 拉格朗日乘子法
5. 特殊场 / 势表达式下的拉氏方程 & 初积分

$$1. (\vec{F}_i^A - m_i \ddot{\vec{r}}_i) \cdot \delta \vec{r}_i = 0$$

考虑  $m_i (\frac{d}{dt} \dot{\vec{r}}_i) \cdot (\frac{\partial \vec{r}_i}{\partial q_\alpha} \delta q_\alpha) = 0$

$$m_i (\frac{d}{dt} \dot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_\alpha}) \delta q_\alpha = 0$$

$$\frac{d}{dt} \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_\alpha} = \frac{d}{dt} (\dot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_\alpha}) - \dot{\vec{r}}_i \cdot \frac{d}{dt} (\frac{\partial \vec{r}_i}{\partial q_\alpha})$$

利用  $\left\langle \frac{\partial \vec{r}_i}{\partial q_\alpha} = \frac{\partial \vec{r}_i}{\partial q_\alpha} \right\rangle$  得

$$m \left[ \frac{d}{dt} (\dot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_\alpha}) - \dot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_\alpha} \right] \delta q_\alpha = (m_i \ddot{\vec{r}}_i) \delta \vec{r}_i$$

$$\Leftrightarrow \left( \frac{d}{dt} (\frac{\partial T}{\partial \dot{q}_\alpha}) - \frac{\partial T}{\partial q_\alpha} \right) \delta q_\alpha = (m_i \ddot{\vec{r}}_i) \delta \vec{r}_i$$

又有:  $\vec{F}_i^A \delta \vec{r}_i = (\frac{\vec{F}_i^A \partial \vec{r}_i}{\partial q_\alpha}) \delta q_\alpha = Q_\alpha \delta q_\alpha$

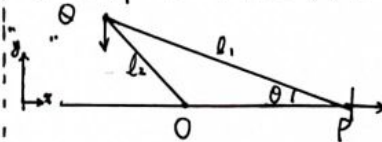
代入  $\Leftrightarrow \frac{d}{dt} (\frac{\partial T}{\partial \dot{q}_\alpha}) - \frac{\partial T}{\partial q_\alpha} = Q_\alpha$

保守势下  $Q_\alpha = -\frac{\partial V}{\partial q_\alpha}$ , ( $\frac{\partial V}{\partial q_\alpha} = 0$ )

记  $L = T - V \Leftrightarrow \frac{d}{dt} (\frac{\partial L}{\partial \dot{q}_\alpha}) = \frac{\partial L}{\partial q_\alpha}$

\* 如果考3就是全列题, 分数勿丢! / 2分题 (易)

2. ① 作主下, 坐标全可表示取(笛卡尔坐标式)系统



以 O 为原点:  $y_a = l \sin \theta$ ,  $\delta y_a = l \cos \theta \delta \theta$

$$x_p = l \cos \theta - \sqrt{l_2^2 - l_1^2 \sin^2 \theta}$$

$$\delta x_p = \left( -l \sin \theta + \frac{l_1^2 \sin \theta \cos \theta}{\sqrt{l_2^2 - l_1^2 \sin^2 \theta}} \right) \delta \theta$$

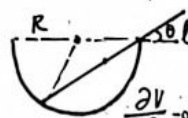
$$\delta W = Q_x \delta x + Q_y \delta y$$

$$= \left( P(-l \sin \theta + \frac{l_1^2 \sin \theta \cos \theta}{\sqrt{l_2^2 - l_1^2 \sin^2 \theta}}) - Q l \cos \theta \right) \delta \theta$$

(注: 此处记得留心 P, Q 与坐标系间方向!!!)

$$\delta W = 0 \Leftrightarrow \frac{Q}{P} = \frac{l \sin \theta}{\sqrt{l_2^2 - l_1^2 \sin^2 \theta}} - \tan \theta$$

② 往年题. 列出势能  $V = V(\theta)$ ,  $\frac{\partial V}{\partial \theta} = 0$  型:



$$V = -mg \cdot 2R \cos \theta \sin \theta + mg \cdot \frac{1}{2} \sin \theta$$

$$\frac{\partial V}{\partial \theta} = 0 \Leftrightarrow -2mgR \cos 2\theta + \frac{mg}{2} \cos \theta = 0$$

$$\Leftrightarrow 4R \cos \theta = l \cos \theta \Leftrightarrow$$

$$\text{记 } \cos \theta = x \in [0, 1]:$$

$$4R(2x^2 - 1) - lx = 0$$

$$8Rx^2 - lx - 4R = 0, \quad x = \frac{l \pm \sqrt{l^2 + 128R^2}}{16R}$$

取 > 0 之解:  $\theta = \arccos \left( \frac{l + \sqrt{l^2 + 128R^2}}{16R} \right)$

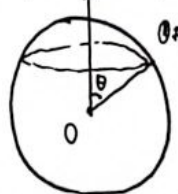
范围限定:  $l + \sqrt{l^2 + 128R^2} < 16R$

$$\text{即 } l^2 + 128R^2 < l^2 + 256R^2 - 32Rl$$

$$\Rightarrow \underline{l < 4R}$$



3. (作业题 2) 虚功可用沿虚位移及参量表示



① 用  $V$  表示:

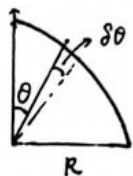
$$V = \sigma \log R \cos \theta + \frac{1}{2} K (2\pi R \sin \theta - l_0)^2$$

$$\frac{\partial V}{\partial \theta} = 0 \Rightarrow$$

$$-\sigma \log R \sin \theta + K (2\pi R \sin \theta - l_0) \cdot 2\pi R \cos \theta = 0$$

$$\Leftrightarrow K (2\pi R \sin \theta - l_0) \cdot 2\pi R \cos \theta = \sigma \log R \sin \theta \quad \checkmark$$

② 用  $\sum Q_i \delta q_i = 0$  表示:



$$\delta W = mg \sin \theta \delta \theta - K(l - l_0) \delta l$$

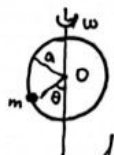
$$\text{代入 } m = \sigma l_0; l = 2\pi R \sin \theta$$

方程与上(1)处等价!

// 总结: 写出  $V$ ,  $\frac{\partial V}{\partial q}$  按为阿便: / 难度: (易)  
注意力的方向, 留心正负号.

3. 这里随意举两例.

例作 13.



$$T = \frac{1}{2} m v^2 = \frac{1}{2} m a^2 \dot{\theta}^2$$

$$V = -mg a \cos \theta - \frac{1}{2} m \omega^2 a^2 \sin^2 \theta$$

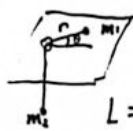
$$L = T - V = \frac{1}{2} m a^2 \dot{\theta}^2 + mg a \cos \theta + \frac{1}{2} m \omega^2 a^2 \sin^2 \theta$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) = \frac{\partial L}{\partial \theta} \Rightarrow m a^2 \ddot{\theta} = -mg a \sin \theta + m \omega^2 a^2 \sin \theta \cos \theta$$

当小球处于平衡位置时,  $\ddot{\theta} = 0$

$$\Leftrightarrow g = \omega^2 a \cos \theta \quad (\Rightarrow g > \omega^2 a \text{ 时无解})$$

例作 14.



$$T = \frac{1}{2} (m_1 + m_2) \dot{r}^2 + \frac{1}{2} m_2 r^2 \dot{\theta}^2$$

$$L = T - V = T - m_2 g r$$

依此分别求初积分:

$$\frac{\partial L}{\partial \dot{\theta}} = m_2 r^2 \dot{\theta} \equiv \text{const.}$$

$$h = \sum p_i \delta q_i - L = \frac{1}{2} (m_1 + m_2) \dot{r}^2 + \frac{1}{2} m_2 r^2 \dot{\theta}^2 + m_2 g r = \text{const.}$$

// 总结: 选如广义坐标; 考虑约束条件

如果涉及刚体考虑  $T = \frac{1}{2} m v_c^2 + \frac{1}{2} I \omega^2$

$$\text{牢记 } \left\{ \begin{array}{l} \frac{\partial L}{\partial \dot{\theta}} = \text{const.}; h = \sum p_i \delta q_i - L (= T + V) = \text{const.} \\ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q} \end{array} \right.$$

/ 难度: (中等)

4. 这里举两例:

(1) 作业 T9. 拉力与摩擦力.



取广义坐标  $x, y, \varphi$

$$\text{给出约束: 沿杆向运动. } A_n \quad A_{12}$$

$$\dot{x} \sin \varphi = \dot{y} \cos \varphi \Rightarrow \sin \varphi \cdot \delta x - \cos \varphi \cdot \delta y = 0$$

$$\text{给出约束/摩擦力: } F = -c(\dot{x} + \dot{y})$$

$$\text{给出动能: } T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I \dot{\varphi}^2$$

$$\text{由 } \left( \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}} \right) - \frac{\partial T}{\partial q} = Q_i \right)_{q_i=0} \text{ 得, 设 } LHS = \lambda r Ar \text{ 得}$$

$$\begin{cases} m \ddot{x} + c \dot{x} = \lambda \sin \varphi \\ m \ddot{y} + c \dot{y} = -\lambda \cos \varphi \end{cases} \text{ 再有: } \dot{x} \sin \varphi = \dot{y} \cos \varphi$$

$$I \ddot{\varphi} = 0, \varphi = \omega_0 t$$

$$\text{例作 15 } \frac{\dot{x}}{\cos \varphi} = K \Rightarrow \begin{cases} \dot{x} = K \cos \varphi \\ \dot{y} = K \sin \varphi \end{cases}$$

$$\begin{cases} m \left( \frac{dK}{dt} \cos \varphi - \sin \varphi \frac{d\varphi}{dt} K \right) = -c K \cos \varphi + \lambda \sin \varphi \quad (1) \\ m \left( \frac{dK}{dt} \sin \varphi + \cos \varphi \frac{d\varphi}{dt} K \right) = -c K \sin \varphi - \lambda \cos \varphi \quad (2) \end{cases}$$

$$(1) \cdot \cos \varphi + (2) \cdot \sin \varphi, \quad m \frac{dK}{dt} = -c K$$

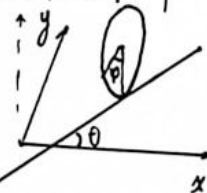
$$\Rightarrow K = C_1 e^{-\frac{c}{m} t}$$

$$\Rightarrow \begin{cases} \dot{x} = C_1 e^{-\frac{c}{m} t} \cos \omega_0 t \\ \dot{y} = C_2 e^{-\frac{c}{m} t} \sin \omega_0 t \end{cases} \text{ 代入 } \dot{x}(0) = v_0 \text{ 得}$$

$$C_1 = v_0$$

$$\Rightarrow \begin{cases} \varphi = \omega_0 t \\ x = \frac{m^2 v_0 \omega_0 \sin \omega_0 t - m v_0 c \cos \omega_0 t}{m^2 \omega_0^2 + c^2} e^{-\frac{c}{m} t} + \frac{m v_0 c}{m^2 \omega_0^2 + c^2} \\ y = -\frac{m^2 v_0 \omega_0 \cos \omega_0 t + m v_0 c \sin \omega_0 t}{m^2 \omega_0^2 + c^2} e^{-\frac{c}{m} t} + \frac{m^2 \omega_0 v_0}{m^2 \omega_0^2 + c^2} \end{cases}$$

(2) 作业 16 书例: 纯滚动盘.



取四个广义坐标  $x, y, \varphi, \theta$ .

(标号 1, 2, 3, 4, 方便约束时计算)

$$\text{给出约束: 沿直线: } \dot{x} \sin \theta = \dot{y} \cos \theta$$

$$\text{纯滚: } \dot{x} \cos \theta + \dot{y} \sin \theta = r \dot{\varphi}$$

$$\text{分别列出: } \begin{cases} A_{11} & A_{12} \\ \sin \theta \delta x - \cos \theta \delta y = 0 \\ A_{21} & A_{22} & A_{23} \\ \cos \theta \delta x + \sin \theta \delta y - r \delta \varphi = 0 \end{cases}$$

给出动能:

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} I_2 \dot{\varphi}^2$$





10. 列出拉格朗日系及配凑的方程:

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_\alpha} \right) - \frac{\partial T}{\partial q_\alpha} = Q_\alpha = \lambda r A_{\alpha\alpha}$$

$$\begin{cases} m\ddot{x} = \lambda_1 A_{11} + \lambda_2 A_{21} \quad ① \\ m\ddot{y} = \lambda_1 A_{12} + \lambda_2 A_{22} \quad ② \\ I_2 \ddot{\varphi} = \lambda_2 A_{23} = -\lambda_2 r \dot{\varphi} \quad ③ \\ I_1 \ddot{\theta} = 0 \quad ④ \end{cases} \Rightarrow \begin{cases} \dot{x} \sin \theta - y \cos \theta = 0 \quad ⑤ \\ \dot{x} \cos \theta + y \sin \theta = r \dot{\varphi} \quad ⑥ \end{cases}$$

④:  $\theta = \omega t + \theta_0$ ;

$$\begin{cases} m\ddot{x} = \lambda_1 \sin \theta + \lambda_2 \cos \theta \quad ① \\ m\ddot{y} = -\lambda_1 \cos \theta + \lambda_2 \sin \theta \quad ② \\ I_2 \ddot{\varphi} = -\lambda_2 r \quad ③ \end{cases}$$

$$① \cdot \sin \theta + ② \cdot \cos \theta: m\ddot{x} \cos \theta + m\ddot{y} \sin \theta = \lambda_1 \quad (A)$$

$$① \cdot \sin \theta - ② \cdot \cos \theta: m\ddot{x} \sin \theta - m\ddot{y} \cos \theta = \lambda_2 \quad (B)$$

$$(A): m \frac{d}{dt} (\dot{x} \sin \theta - \dot{y} \cos \theta) - m (\dot{x} \cos \theta + \dot{y} \sin \theta) \dot{\theta} = \lambda_1$$

$$\Rightarrow -m r \dot{\varphi} \dot{\theta} = \lambda_1$$

$$(B): m \frac{d}{dt} (\dot{x} \cos \theta + \dot{y} \sin \theta) - m (-\dot{x} \sin \theta + \dot{y} \cos \theta) \dot{\theta} = \lambda_2$$

$$\Rightarrow \lambda_2 = m r \ddot{\varphi}$$

$$\Rightarrow (I_2 - m r^2) \ddot{\varphi} = 0, \quad \ddot{\varphi} = 0 \Rightarrow \varphi = c t + \varphi_0$$

$$\dot{x} = r \varphi \cos \theta = r C \cos (\omega t + \theta_0) \quad \text{同理得 } y$$

$$\Rightarrow \frac{(x-x_0)^2 + (y-y_0)^2}{\omega^2} = \frac{r^2 C^2}{\omega^2}$$

总结: 约束条件  
- 洛某方向:  $\dot{x} \sin \theta = \dot{y} \cos \theta$   
- 绳长:  $\dot{x} \cos \theta + \dot{y} \sin \theta = r \dot{\varphi}$

$A_{\alpha\alpha}$  标清, 记得求和

难度: 较难

(※. 难度: 5)

5. 广势体系下 Lagrangian Equations:

举2例. 来自作业.

$$1. (T_1) \quad U = V(\vec{r}) + \vec{\sigma} \cdot \vec{J}$$

$$\text{由 } \vec{J} = \vec{r} \times \vec{p} = m \vec{r} \times \vec{v} = -m \vec{v} \times \vec{r}$$

$$广势力 F_1' = Q_{\alpha_1} = -\frac{\partial U}{\partial r} = -\frac{\partial V}{\partial r} + m \vec{\sigma} \times \vec{v}$$

$$F_2' = Q_{\alpha_2} = \frac{d}{dt} \frac{\partial U}{\partial \dot{v}} = \frac{d}{dt} m \vec{\sigma} \times \vec{r} = m \vec{\sigma} \times \vec{v}$$

$$\therefore F_r = -\frac{\partial V}{\partial r} + 2m \vec{v} \times \vec{\sigma} = -\nabla V + 2m \vec{v} \times \vec{\sigma}$$

$$\text{Lagrangian Equation 给出 } \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_\alpha} \right) - \frac{\partial T}{\partial q_\alpha} = Q_\alpha$$

$$\text{即 } m \ddot{\vec{r}} - 2m (\vec{\sigma} \times \vec{v}) = -\nabla V$$

// 忠告: 写出 (\*) 式代换很重要. 否则容易 = 次犯错误

2. (T<sub>1</sub>) 稳定均匀磁场中:

$$\vec{F} = e\vec{E} + e\vec{v} \times \vec{B}$$

$$\text{代换 } \begin{cases} \vec{B} = \nabla \times \vec{A} \\ \vec{E} = -\nabla \varphi - \frac{\partial \vec{A}}{\partial t} \end{cases} \rightarrow \left\{ \begin{array}{l} \text{这些需要掌握} \end{array} \right.$$

$$U = e(\varphi - \vec{v} \cdot \vec{A}). \quad (\text{实在不行记这个})$$

$$\therefore L = T - U = \frac{m}{2} v^2 - e(\varphi - \vec{v} \cdot \vec{A})$$

$$h = \sum_\alpha p_\alpha q_\alpha - L = \frac{m}{2} v^2 + e\varphi = \text{const}$$

(难度: 说不准)

第一章至此结束.

# Remark: 几道课堂例题又作业题.

$$(作业 T_3.) \quad \begin{cases} y = y(x) \\ z = z(x) \end{cases} \quad \begin{cases} F_r = \phi(x) \\ F_\varphi = \psi(x) \\ F_z = -mg \end{cases}$$

$$T = \frac{1}{2} m (\dot{x}^2 + y'^2 \dot{x}^2 + z'^2 \dot{x}^2) = \frac{1}{2} m \dot{x}^2 (1 + y'^2 + z'^2)$$

$$\therefore \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}} \right) = m \ddot{x} (1 + y'^2 + z'^2) + 2m \dot{x}^2 (y'y'' + z'z'')$$

$$(\text{利用 } \dot{y}' = y'' \dot{x}, \quad \dot{z}' = z'' \dot{x})$$

地址: 中国北京市海淀区颐和园路5号

邮编: 100871 www.pku.edu.cn

第3小问有  $\ddot{x} = \frac{gx}{4c} \Rightarrow x = \sqrt{\frac{g}{4c}}$

$x = -2c + ce^{-\alpha t} + ce^{\alpha t}$

1. 例题. 圆摆



$L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$

$ml^2\ddot{\theta} + mgl\sin\theta = 0 \Leftrightarrow$

$\left[ \ddot{\theta} + \omega^2 \sin\theta = 0 \quad (\omega = \sqrt{\frac{g}{l}}) \right]$

取初积分.  $\frac{1}{2}l\dot{\theta}^2 - g\cos\theta = \text{const}$

设振幅为  $\theta_0$ . 则  $\text{const} = -g\cos\theta_0$

$\Rightarrow \frac{1}{2}l\dot{\theta}^2 + g(\cos\theta_0 - \cos\theta) = 0$

$\Rightarrow \dot{\theta}^2 + \frac{2g}{l}(-\sin^2\frac{\theta_0}{2} + \sin^2\frac{\theta}{2}) = 0$

$\Leftrightarrow \frac{d\theta}{\sqrt{2\sin^2\frac{\theta_0}{2} - 2\sin^2\frac{\theta}{2}}} = \omega dt$

$\int_0^u \frac{du'}{\sqrt{(1-u'^2)(1-k^2u'^2)}} = \omega t + \delta \quad (k = \sin\frac{\theta_0}{2})$

$k \ll 1$  时  $\frac{t}{\sqrt{2\pi/g}} = 1 + \frac{k^2}{4} + \frac{9k^4}{64} \dots$

圆摆摆  $\begin{cases} x = b(1 + \sin\theta) \\ y = b(1 - \cos\theta) \end{cases} \quad T \text{ 与 } \theta \text{ 无关}$

第2章 有心运动

1. 考点:

1. 求轨道方程/微分方程
2. 行星探测器轨道转移
3. 特殊势场讨论轨道稳定性及周期
4. 卢瑟福公式. 散射问题.

1. 基本方程推导如下:

平方反比引力下,  $T = \frac{1}{2}mr^2 + \frac{1}{2}mr^2\dot{\varphi}^2$  (取  $\mu \approx m$ )

$V = -\frac{GMm}{r} \Rightarrow L = T - V = \frac{1}{2}mr^2 + \frac{1}{2}mr^2\dot{\varphi}^2 + \frac{GMm}{r}$

$\frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi} = \text{const}$ . 记  $r^2\dot{\varphi} = h$ , 可有时取  $mr^2\dot{\varphi} = J$

记  $E = T + V = \frac{1}{2}mr^2 + \frac{1}{2}mr^2\dot{\varphi}^2 - \frac{GMm}{r} = \frac{1}{2}mr^2 + \frac{mh^2}{2r^3} - \frac{GMm}{r}$

E 守恒. 得. 在此方程下

$\frac{1}{2}mr^2 = E - \frac{mh^2}{2r^3} + \frac{GMm}{r}$

$\frac{dr}{\sqrt{\frac{2}{m}(E - \frac{mh^2}{2r^3} + \frac{GMm}{r})}} = dt \Leftrightarrow$  利用  $\frac{d\varphi}{dt} = \frac{h}{r^2}$

$\Leftrightarrow \frac{h dr}{r^2 \sqrt{\frac{2}{m}(E - \frac{mh^2}{2r^3} + \frac{GMm}{r})}} = d\varphi$

$\Rightarrow r = \frac{\frac{h^2}{GM}}{1 + \sqrt{\frac{1 - \frac{2Eh^2}{GM^2m}}{\cos(\varphi + C)}}$

其中  $h, E, C$  由初条件给出

(一般方便写作  $\rho = \frac{\frac{h^2}{GM}}{1 + \sqrt{\frac{1 - \frac{2Eh^2}{GM^2m}}{\cos(\varphi + C)}}$ )

例1. (作  $\perp T_1$ )

$V = \sqrt{\frac{GM}{4R}}$   
得  $h = r^2\dot{\varphi} = \sqrt{\frac{GM R}{8}}$   
而  $E = \frac{1}{2}mv^2 - \frac{GMm}{R} = -\frac{7GMm}{8R}$

代入得  $\rho = \frac{\frac{R}{8}}{1 + \sqrt{\frac{1 - \frac{(-\frac{7}{8})^2}{GM^2m}}{\cos(\varphi + C)}}}$   
 $= \frac{R}{8 + 5\sqrt{2}\cos(\varphi + C)}$

代入  $\rho = R$  时.  $\varphi = 0 \Leftrightarrow C = -\arcsin\frac{\sqrt{2}}{10}$





北京大学  
PEKING UNIVERSITY

慈诚

例2. (作业T2)



已知  $V_{se} = \sqrt{\frac{GM}{R}}$   
( $M = M_{sun}$ ,  $R = 1 \text{ AU}$ )

如图, 取日心-轨道近地点连线方向为极轴.

则  $V_p = -(\sqrt{2}-1)\sqrt{\frac{GM}{R}}$ ,  $V_{se} = V_p = \sqrt{\frac{GM}{R}}$

$\Rightarrow h = r^2\dot{\varphi} = \sqrt{GM}R$ ;  $E = \frac{1}{2}m(v^2 + V_{se}^2) - \frac{GMm}{R}$

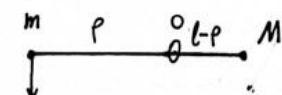
$\Rightarrow E = \frac{1}{2}m(\frac{GM}{R} + (3-2\sqrt{2})\frac{GM}{R}) - \frac{GMm}{R} = (1-\sqrt{2})\frac{GMm}{R}$

代入  $\rho = \frac{\frac{h^2}{GM}}{1 + \frac{2Eh^2}{G^2M^2m} \cos(\varphi+C)} = \frac{R}{1 + (\sqrt{2}-1)\cos(\varphi+C)}$

由于极轴向取的是如图所求的连线方向

$\Rightarrow \rho = R$  时,  $\cos(\varphi+C) = 0$ , 即  $C = \pm \frac{\pi}{2}$

例3. (作业T4)



如图, 记  $l_{mo} = \rho$ ,  $l_{mm} = l$

首先  $h = \rho_0 v_0$

$T = \frac{1}{2}(M+m)\dot{\rho}^2 + \frac{mh^2}{2\rho^2} = E$

$V_{off} = \frac{mh^2}{2\rho^2}$

(QS): 微分方程给出:  $\frac{h d\rho}{\sqrt{\frac{2}{M+m}(E - \frac{mh^2}{2\rho^2})}} = d\varphi$  (此处  $\mu = M+m$ )

由  $E = \frac{mv_0^2}{2}$ ,  $h = \rho_0 v_0$

$\Rightarrow \frac{\rho_0 v_0 d\rho}{\sqrt{\frac{2}{M+m}(\frac{mv_0^2}{2} - \frac{m\rho_0^2 v_0^2}{2\rho^2})}} = d\varphi$  (积分3.0.1分)

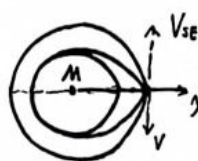
更简洁的方式:

$(M+m)\ddot{\rho} - \frac{mh^2}{\rho^3} = 0$

$\frac{M+m}{m}\ddot{\rho} - \frac{h^2}{\rho^3} = 0$  记  $u = \frac{1}{\rho}$

$\Rightarrow \frac{m+M}{m} \frac{d^2 u}{d\varphi^2} + u = 0$

2. (往年题又作业T3)



考察“地球-金星”轨道转移.

(初态分析) 设  $v_0$  是满足条件的发射速

则  $V^2 = v_0^2 - v_2^2$ , 其中  $v_2$  是地球第一宇宙速

依题有:  $V = V_{se} - v$  满足.

$\frac{1}{2}mV^2 - \frac{GMm}{R} = -\frac{GMm}{2a}$  ( $E_{初} = -\frac{GMm}{2a}$ )

$\Rightarrow V^2 = \frac{2GM}{R}(1 - \frac{R}{2a})$  ( $2a = R_{地} + R_{金}$ )

$\therefore v_0^2 = v_2^2 + \left[ \sqrt{\frac{GM}{R}} - \sqrt{\frac{2GM}{R}(1 - \frac{R}{2a})} \right]^2$

$\approx 133.14 \text{ (km/s)} \Rightarrow$

$v_0 \approx 11.5 \text{ km/s}$

3. (作业题T6)

$V = -\frac{ke^{-a\rho}}{\rho}$

首先  $L = T \cdot V = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\varphi}^2 - k\frac{e^{-a\rho}}{\rho}$

此处记  $J = m\rho^2\dot{\varphi} \Rightarrow L = \frac{1}{2}m\dot{\rho}^2 + \frac{J^2}{2m\rho^2} - k\frac{e^{-a\rho}}{\rho}$

$V_{eff} = \frac{J^2}{2m\rho^2} - k\frac{e^{-a\rho}}{\rho}$

$\frac{\partial V_{eff}}{\partial \rho} = \frac{-J^2}{m\rho^3} - \frac{k}{\rho^2}(-ae^{-a\rho}\rho - e^{-a\rho})$

利用  $\rho = \rho_0$  时,  $\frac{\partial V_{eff}}{\partial \rho} = 0 \Rightarrow$

$\frac{J_0^2}{m\rho_0^3} = \frac{k}{\rho_0^2} e^{-a\rho_0} (a\rho_0 + 1) \Rightarrow J_0^2 = km\rho_0 e^{-a\rho_0} (a\rho_0 + 1)$

地址: 中国北京市海淀区颐和园路5号

邮编: 100871 www.pku.edu.cn



代入  $V_{eff}$  有:

$$V_{eff} = \frac{J_0^2}{2m\rho_0^2} - \frac{ke^{-a\rho}}{\rho_0}$$

$$= \frac{km\rho_0 e^{-a\rho_0}(a\rho_0+1)}{2m\rho_0^3} - \frac{ke^{-a\rho}}{\rho_0}$$

$$= \frac{ke^{-a\rho}}{2\rho_0^2}(a\rho_0+1) \leq 0 \quad (\text{利用已知结论}).$$

即  $a\rho < 1$ :

分析  $\frac{\partial^2 V_{eff}}{\partial \rho^2} \Big|_{\rho_0}$ :

根据公式只需比较  $\frac{F(\rho_0)}{F'(\rho_0)} + \frac{3}{\rho_0}$  是否  $> 0$ .

(由于  $V = \frac{-ke^{-a\rho}}{\rho}$  可得  $F \cdot F'$ )

中题中直接算 = 所算为有效:

$$\frac{\partial V_{eff}}{\partial \rho} = \frac{-J^2}{2m\rho^3} + \frac{k}{\rho^2} e^{-a\rho}(a\rho+1)$$

$$\therefore \frac{\partial^2 V_{eff}}{\partial \rho^2} = \frac{3J^2}{m\rho^4} + k \frac{(-ae^{-a\rho}(a\rho+1) + ae^{-a\rho})\rho^2 - 2\rho e^{-a\rho}(a\rho+1)}{\rho^4}$$

$$\frac{\partial^2 V_{eff}}{\partial \rho^2} \Big|_{\rho_0} = \frac{3km\rho_0 e^{-a\rho_0}(a\rho_0+1)}{m\rho_0^4} + \frac{k}{\rho_0^3} e^{-a\rho_0} [\rho_0(a - a(a\rho_0+1)) - 2(a\rho_0+1)]$$

$$= \frac{k}{\rho_0^3} e^{-a\rho_0} [-a^2\rho_0^2 + a\rho_0 + 1]$$

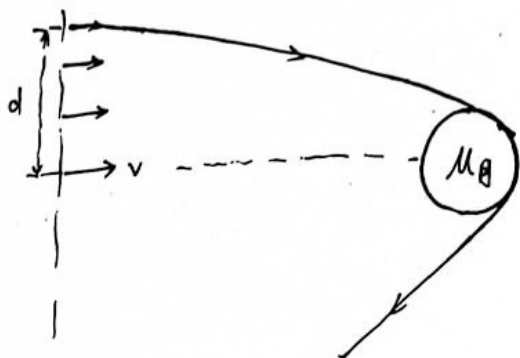
$$= \frac{k}{\rho_0^3} e^{-a\rho_0} [a\rho_0(1-a\rho_0) + 1] > 0$$

故轨道稳定.

$$\tau = \sqrt{\frac{2\pi}{\frac{\partial^2 V_{eff}}{\partial \rho^2} \Big|_{\rho_0}}} = \sqrt{\frac{2\pi}{\frac{k}{\rho_0^3} e^{-a\rho_0} [a\rho_0(1-a\rho_0) + 1]}}$$

#### 4. 浅举两例

1. 作业 75. 柱截面面积.



分析能量守恒:

$$\frac{1}{2}m v_0^2 = \frac{1}{2}m v^2 - \frac{GMm}{R}$$

角动量守恒:  $dV_d = R \cdot v$

$$\text{可知 } v = \sqrt{v_d^2 + \frac{2GM}{R}}$$

$$\therefore d = \frac{RV}{V_d} = R \sqrt{1 + \frac{2GM}{RV_d^2}}$$

$$\Rightarrow S = \pi d^2 = \pi R^2 \left(1 + \frac{2GM}{RV_d^2}\right)$$

2. (1) 在题中只考虑例题).  $V = \frac{k}{r}$ , 推导卢瑟福公式.

首先有  $V = \frac{k}{r}$  ( $k > 0$ ):

$$\text{代入 } p = \frac{P}{\cos\varphi' - 1} \quad (\text{双曲线方程})$$

$$\text{其中 } p = \frac{mk^2}{K}; \quad e = \sqrt{1 + \frac{2Eh^2 m}{K^2}}$$

(小技巧:  $k \ll GMm$  代入已知的公式)

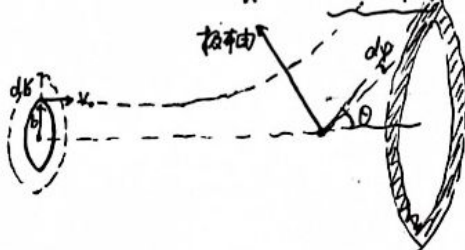
$$\text{有: } \theta = \pi - 2\arccos \frac{1}{e}$$

$$\Rightarrow \cot \frac{\theta}{2} = \sqrt{e^2 - 1} = \sqrt{\frac{2Eh^2 m}{K^2}}$$

$$\text{有: } h = b v_0, \quad E = \frac{1}{2} m v_0^2$$

$$\therefore \cot \frac{\theta}{2} = \frac{m v_0^2 b}{K} = \frac{2Eb}{K} \quad (*)$$

(如图):



列出微分方程:

$$2\pi b db \cdot I = \sigma I d\Omega = \sigma I \int_0^{2\pi} \sin\theta d\theta d\varphi = 2\pi\sigma I \sin\theta d\theta$$

$$\Leftrightarrow b db = \sigma \sin\theta d\theta$$

$$\Rightarrow \sigma = \frac{b db}{\sin\theta d\theta} \quad (\sigma = \frac{dS}{d\Omega})$$

$$\text{代入 } (*): \quad \sigma(\theta) = \left| \frac{k}{4E} \right|^2 \frac{1}{\sin^4 \frac{\theta}{2}} \quad \checkmark$$

$$\text{而 } \int_{\Omega} \sigma d\Omega = \int dS, \quad \text{由 } \frac{\sum l_{\theta}}{\sum} dS = \frac{dN}{N} \left[ \frac{1}{\Omega} \right]$$

$$\Rightarrow dS = \frac{dN}{N} \cdot \frac{1}{\Omega} \quad \text{可得}$$

$$\text{得 } S = \int_0^\pi 2\pi \sigma \sin\theta d\theta = +\infty \quad \text{③ 截面微分推}$$

$$\sigma = \frac{b db}{\sin\theta d\theta}$$

(四步走: ①  $p = \frac{P}{\cos\varphi - 1}$  代入  $p = \frac{mk^2}{K}$ ;  $e = \sqrt{1 + \frac{2Eh^2 m}{K^2}}$  ② 代入 ② 中  $\frac{db}{d\theta}$  关系.

② 求解  $\theta = \pi - 2\arccos \frac{1}{e}$  得  $\cot \frac{\theta}{2} = \frac{2Eb}{K}$