理论力学A

变分：$\delta(\frac{dy}{dx}) = \frac{d}{dx}(\delta y)$　　$J[y] = \int_{x_0}^{x_1} F(x,y,y')dx$　　$\delta J = 0$

$\delta J[y] = \int_{x_0}^{x_1}(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y')dx = \int_{x_0}^{x_1}(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta\frac{dy}{dx})dx = \int_{x_0}^{x_1}(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\frac{d}{dx}(\delta y))dx$

$= \frac{\partial F}{\partial y'}\delta y\Big|_{x_0}^{x_1} + \int_{x_0}^{x_1}(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'})\delta y\,dx \Rightarrow$ E-L方程

外微分：判断是否为完整约束　　$\omega\wedge d\omega = 0$

$\omega = A dx + B dy + C dt$, $d\omega = (\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y})dx\wedge dy + (\frac{\partial C}{\partial y} - \frac{\partial B}{\partial t})dy\wedge dt + (\frac{\partial A}{\partial t} - \frac{\partial C}{\partial x})dt\wedge dx$

$\omega\wedge d\omega = 0 = [A(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial t}) + B(\frac{\partial A}{\partial t} - \frac{\partial C}{\partial x}) + C(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y})]\,dx\wedge dy\wedge dt = 0$

$\Rightarrow A(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial t}) + B(\frac{\partial A}{\partial t} - \frac{\partial C}{\partial x}) + C(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}) = 0$

eg. $y = \tan\theta\, x$　　$\omega = \tan\theta\, dx - dy$　　$d\omega = \frac{1}{\cos^2\theta}d\theta\wedge dx$

$\omega\wedge d\omega = -\frac{1}{\cos^2\theta}dy\wedge d\theta\wedge dx \neq 0$，$\Rightarrow$ 非完整约束

经典拉格朗日关系：$\frac{\partial \dot{\mathcal{U}}_i}{\partial \dot q_k} = \frac{\partial \mathcal{U}_i}{\partial q_k}$，　$\frac{d}{dt}(\frac{\partial \mathcal{U}_i}{\partial q_k}) = \frac{\partial \dot{\mathcal{U}}_i}{\partial q_k}$

证明：$\dot{\mathcal{U}}_i = \frac{\partial \mathcal{U}_i}{\partial q_k}\dot q_k + \frac{\partial \mathcal{U}}{\partial t}$　　$\frac{\partial \dot{\mathcal{U}}_i}{\partial q_k} = \frac{\partial^2 \mathcal{U}_i}{\partial q_k\partial q_j}\dot q_j + \frac{\partial^2 \mathcal{U}_i}{\partial q_k\partial t}$　　$\frac{d}{dt}(\frac{\partial \mathcal{U}_i}{\partial q_k}) = \frac{\partial^2 \mathcal{U}_i}{\partial q_j\partial q_k}\dot q_j + \frac{\partial^2 \mathcal{U}_i}{\partial t\partial q_k}$

达朗贝尔原理：$(\vec F_i + \vec I_i)\cdot\delta\vec r_i = 0$，　$\vec I_i = -m_i\ddot{\vec r}_i$

$\delta\vec r_i = \frac{\partial\vec r_i}{\partial q_k}\delta q_k$　　$(\vec F_i\frac{\partial\vec r_i}{\partial q_k} - m_i\ddot{\vec r}_i\frac{\partial\vec r_i}{\partial q_k})\delta q_k = 0$

$-m_i\ddot{\vec r}_i\frac{\partial\vec r_i}{\partial q_k} = -m_i\frac{d\dot{\vec r}_i}{dt}\frac{\partial\vec r_i}{\partial q_k} = -m_i\frac{d}{dt}(\dot{\vec r}_i\frac{\partial\vec r_i}{\partial q_k}) + m_i\dot{\vec r}_i\frac{d}{dt}(\frac{\partial\vec r_i}{\partial q_k})$

$= -m_i\frac{d}{dt}(\dot{\vec r}_i\frac{\partial\dot{\vec r}_i}{\partial\dot q_k}) + m_i\dot{\vec r}_i\frac{\partial\dot{\vec r}_i}{\partial q_k}$　　$Q_k = \vec F_i\frac{\partial\vec r_i}{\partial q_k}$，　$T = \frac12\sum m_i\dot{\vec r}_i^2$

$Q_k - \frac{d}{dt}\frac{\partial T}{\partial\dot q_k} + \frac{\partial T}{\partial q_k} = 0$　　令　$Q_k = -\frac{\partial V}{\partial q_k}$　　$V = V(\vec r, t)$

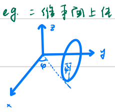
$-\frac{\partial V}{\partial q_k} + \frac{\partial T}{\partial q_k} - \frac{d}{dt}(\frac{\partial T}{\partial\dot q_k}) = 0$，　令 $L = T - V$，　则 $\frac{d}{dt}(\frac{\partial L}{\partial\dot q_k}) - \frac{\partial L}{\partial q_k} = 0$，

拉格朗日乘子法：有约束系统

$\delta S = \int_{t_1}^{t_2}dt\,[\frac{\partial L}{\partial q_k} - \frac{d}{dt}(\frac{\partial L}{\partial\dot q_k})]\,\delta q_k = 0$　　$g(q,t) = 0$，$\frac{\partial g}{\partial q_k}\delta q_k = 0$

$\frac{\partial L}{\partial q_k} - \frac{d}{dt}(\frac{\partial L}{\partial\dot q_k}) + \lambda\frac{\partial g}{\partial q_k} = 0$

eg. 二维平面上纯滚动的圆盘



约束 $\begin{cases}\dot x - R\dot\psi\cos\varphi = 0 \\ \dot y - R\dot\psi\sin\varphi = 0\end{cases}$　　$\delta x - R\cos\varphi\,\delta\psi = 0$　　$\delta y - R\sin\varphi\,\delta\psi = 0$

$L = \frac12 m(\dot x^2+\dot y^2) + \frac12\cdot\frac14 mR^2\dot\varphi^2 + \frac12\cdot\frac12 mR^2\dot\psi^2$

$\begin{cases} m\ddot x + \lambda_1 = 0 \\ m\ddot y + \lambda_2 = 0 \\ \ddot\varphi = 0 \\ \frac12 mR^2\ddot\psi - \lambda_1 R\cos\varphi - \lambda_2 R\sin\varphi = 0\end{cases}$　　$m\ddot x = mR\ddot\psi\cos\varphi - mR\sin\varphi\,\dot\varphi\dot\psi = -\lambda_1$，　$m\ddot y = mR\ddot\psi\cos\varphi + mR\cos\varphi\,\dot\varphi\dot\psi = -\lambda_2$

$\frac12 mR^2\ddot\psi + mR^2\ddot\psi = 0$，$\Rightarrow \ddot\psi = 0$　　平庸的解

eg. 将上述平面改为斜面，再次求解

$\begin{cases}\delta x - R\cos\varphi\,\delta\psi = 0 \\ \delta y - R\sin\varphi\,\delta\psi = 0\end{cases}$，　$L = \frac12 m(\dot x^2+\dot y^2) + \frac18 mR^2\dot\varphi^2 + \frac14 mR^2\dot\psi^2 + mgx\sin\theta$

$\begin{cases} m\ddot x - mg\sin\theta + \lambda_1 = 0 \\ m\ddot y + \lambda_2 = 0 \\ \ddot\varphi = 0 \\ \frac12 mR\ddot\psi - \lambda_1\cos\varphi - \lambda_2\sin\varphi = 0\end{cases}$　　$\ddot x = R\ddot\psi\cos\varphi - R\dot\varphi\dot\psi\sin\varphi = -\lambda_1 + g\sin\theta$，　$\ddot y = R\dot\varphi\dot\psi\cos\varphi + R\ddot\psi\sin\varphi = -\lambda_2$

$\frac12 R\ddot\psi + R\ddot\psi - g\sin\theta\cos\varphi = 0$，$\Rightarrow \ddot\psi = \frac{2g}{3R}\sin\theta\cos\varphi$　　$\varphi = \omega t + \varphi_0$

$\begin{cases}\psi = \psi_0 + \omega_2 t + \frac{g}{3R}\sin\theta\cos\varphi_0\, t^2 & \omega = 0 \\ \psi = \psi_0 + \omega t - \frac{2g}{3R\omega^2}\sin\theta\cos(\varphi_0 + \omega t) & \omega\neq 0\end{cases}$

$\begin{cases} x = x_0 + \omega_2 R\cos\varphi_0\, t + \frac{g}{3}\sin\theta\cos^2\varphi_0\, t^2 & \omega = 0 \\ x = -\frac{g}{6\omega^2}\sin\theta\cos(2\omega t + 2\varphi_0) + \frac{\omega_2}{\omega}R\sin(\omega t + \varphi_0) + x_0 & \omega\neq 0\end{cases}$

$\begin{cases} y = y_0 + \omega_2 R\sin\varphi_0\, t + \frac{g}{3}\sin\theta\cos\varphi_0\sin\varphi_0\, t^2 & \omega = 0 \\ y = y_0 - \frac{\omega_2}{\omega}R\cos(\omega t + \varphi_0) + \frac{g}{3\omega}\sin\theta\, t - \frac{g}{6\omega^2}\sin\theta\sin(2\omega t + 2\varphi_0) & \omega\neq 0\end{cases}$

广义力与广义势函数：$Q_k = \frac{d}{dt}(\frac{\partial U}{\partial\dot q_k}) - \frac{\partial U}{\partial q_k}$

$U(q,\dot q,t) = U_k(q,t)\dot q_k + U_0(q,t)$　　$Q_k(q,\dot q,t) = \Omega_{ki}(q,t)\dot q_i + \Omega_{k0}(q,t)$

$Q_k = \frac{dU_k}{dt} - \frac{\partial U_i}{\partial q_k}\dot q_i - \frac{\partial U_0}{\partial q_k} = \frac{\partial U_k}{\partial q_i}\dot q_i - \frac{\partial U_i}{\partial q_k}\dot q_i - \frac{\partial U_0}{\partial q_k} + \frac{\partial U_k}{\partial t}$

$\Omega_{ki} = \frac{\partial U_k}{\partial q_i} - \frac{\partial U_i}{\partial q_k}$，　$\Omega_{k0} = \frac{\partial U_k}{\partial t} - \frac{\partial U_0}{\partial q_k}$

$\omega = \sum_{k=0}^{s}U_k dq_k$　　$d\omega = -\frac12\sum_{k=0}^{s}\sum_{l=0}^{s}(\frac{\partial U_k}{\partial q_l} - \frac{\partial U_l}{\partial q_k})dq_k\wedge dq_l = -\frac12\sum_{k=0}^{s}\sum_{l=0}^{s}\Omega_{kl}dq_k\wedge dq_l$

存在广义势的条件：$\Omega_{kl} = -\Omega_{lk}$　　$\frac{\partial\Omega_{km}}{\partial q_l} + \frac{\partial\Omega_{lk}}{\partial q_m} + \frac{\partial\Omega_{ml}}{\partial q_k} = 0$

经典电磁学：$\vec F = e\vec E + e\vec v\times\vec B$，　$U(q,\dot q,t) = e\phi - e\vec v\cdot\vec A$

$\vec E = -\nabla\phi - \frac{\partial\vec A}{\partial t}$，　$\vec B = \nabla\times\vec A$

$\vec F = e(-\nabla\phi - \frac{\partial\vec A}{\partial t} + \vec v\times(\nabla\times\vec A)) = e(-\nabla\phi - \frac{\partial\vec A}{\partial t} + \nabla(\vec v\cdot\vec A) - (\vec v\cdot\nabla)\vec A)$

$= -\nabla(e\phi - e\vec v\cdot\vec A) + \frac{d}{dt}\nabla_v(e\phi - e\vec A\cdot\vec v)$

诺特定理：$L'(q,\dot q,t) = L(q,\dot q,t) + \frac{d}{dt}\delta\lambda(q,t)$　　$\delta\lambda = \Lambda^s\delta\alpha_s$

$t' = U_0(t,q,\alpha) + X_0^s\delta\alpha_s$　　$q_i' = U_i(t,q,\alpha) + X_i^s\delta\alpha_s$

则有 $P_iX_i^s - EX_0^s + \Lambda^s = const$

证明如下：　$S = \int L(q,\dot q,t)dt = \int L'(q',\dot q',t')dt'$

$L(q,\dot q,t) = L'(q',\dot q',t')\frac{dt'}{dt}$，　$L'(q,\dot q,t) = L(q,\dot q,t) + \frac{d}{dt}\delta\lambda$

$L(q,\dot q,t) = (L(q',\dot q',t') + \frac{d}{dt'}\delta\lambda(q',t'))(1+\frac{d\delta t}{dt}) \approx L(q',\dot q',t') + \frac{d}{dt}\delta\lambda + L(q,\dot q,t)\frac{d\delta t}{dt}$

$\Rightarrow \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial\dot q}\delta\dot q + \frac{\partial L}{\partial t}\delta t + \frac{d}{dt}\delta\lambda + L(q,\dot q,t)\frac{d\delta t}{dt} = 0$

$\delta\dot q = \frac{dq'}{dt'} - \frac{dq}{dt} = \frac{dq'}{dt}\frac{dt}{dt'} - \frac{dq}{dt} = (\frac{dq}{dt} + \frac{d\delta q}{dt})(1 - \frac{d\delta t}{dt}) - \frac{dq}{dt} \approx -\dot q\frac{d\delta t}{dt} + \frac{d\delta q}{dt}$

$\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial\dot q}(-\dot q\frac{d\delta t}{dt} + \frac{d\delta q}{dt}) + \frac{\partial L}{\partial t}\delta t + L(q,\dot q,t)\frac{d\delta t}{dt} + \frac{d}{dt}\delta\lambda$

$= \frac{d}{dt}(\frac{\partial L}{\partial\dot q})\delta q + \frac{\partial L}{\partial\dot q}\frac{d\delta q}{dt} - \dot q\frac{\partial L}{\partial\dot q}\frac{d\delta t}{dt} + \frac{\partial L}{\partial t}\delta t + L(q,\dot q,t)\frac{d\delta t}{dt} + \frac{d}{dt}\delta\lambda$

$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q}\dot q + \frac{\partial L}{\partial\dot q}\ddot q = \frac{\partial L}{\partial t} + \frac{d}{dt}(\frac{\partial L}{\partial\dot q})\dot q + \frac{\partial L}{\partial\dot q}\ddot q = \frac{\partial L}{\partial t} + \frac{d}{dt}(\frac{\partial L}{\partial\dot q}\dot q)$

$\Rightarrow \frac{d}{dt}(\frac{\partial L}{\partial\dot q}\delta q + (L - \frac{\partial L}{\partial\dot q}\dot q)\delta t + \delta\lambda) = 0$

$\Rightarrow$ 守恒量　$p\delta q - E\delta t + \delta\lambda = const$

eg. $L = \frac12(\dot x^2+\dot y^2) + \alpha(x\dot y - y\dot x) + \frac12\omega^2(x^2+y^2)$

(a) 求体系在无穷小旋转下具有对称性，并给出相应的守恒量

i) 极坐标，$x = r\cos\theta$，$y = r\sin\theta$，$x\dot y - y\dot x = r\cos\theta(\dot r\sin\theta + r\cos\theta\dot\theta) - r\sin\theta(\dot r\cos\theta - r\sin\theta\dot\theta) = r^2\dot\theta$

$L = \frac12\dot r^2 + \frac12 r^2\dot\theta^2 + \alpha r^2\dot\theta + \frac12\omega^2 r^2$，不含$\theta$，$\frac{\partial L}{\partial\dot\theta} = const = r^2\dot\theta + \alpha r^2$

ii) $L = \frac12\vec v^2 + \alpha(\vec r\times\vec v)\cdot\vec n_3 + \frac12\omega^2\vec r^2$

无穷小旋转：$\delta\vec r = \delta\phi(\vec r\times\vec n_3)$，　$\delta\vec v = \delta\phi(\vec v\times\vec n_3)$

$\delta L = \frac{\partial L}{\partial\vec r}\delta\vec r + \frac{\partial L}{\partial\vec v}\delta\vec v = \alpha(\vec v\times\vec n_3)\cdot\delta\vec r + \omega^2\vec r\cdot\delta\vec r + \vec v\cdot\delta\vec v + \alpha(\vec n_3\times\vec r)\cdot\delta\vec v$

$= \alpha\delta\phi(\vec v\times\vec n_3)\cdot(\vec r\times\vec n_3) + \alpha\delta\phi(\vec n_3\times\vec r)\cdot(\vec v\times\vec n_3) = 0$

$0 = \delta L = (\frac{\partial L}{\partial\vec r})\delta\vec r + (\frac{\partial L}{\partial\vec v})\delta\vec v = \frac{d}{dt}(\frac{\partial L}{\partial\vec v})\cdot\delta\vec r + (\frac{\partial L}{\partial\vec v})\frac{d}{dt}\delta\vec r = \frac{d}{dt}(\frac{\partial L}{\partial\vec v}\cdot\delta\vec r)$

$= \frac{d}{dt}(\vec v\cdot\delta\vec r + \alpha(\vec n_3\times\vec r)\cdot\delta\vec r) = \frac{d}{dt}\delta\phi(\vec v\cdot(\vec r\times\vec n_3) + \alpha(\vec n_3\times\vec r)\cdot(\vec r\times\vec n_3))$

$= -\frac{d}{dt}\delta\phi((\vec r\times\vec v)\cdot\vec n_3 + \alpha\vec r^2)$　　守恒量 $x\dot y - y\dot x + \alpha(x^2+y^2)$

(b) 证明系统在无穷小变换 $\begin{cases}x' = x - \delta\alpha\cos(\omega t) - \delta\beta\sin(\omega t) \\ y' = y + \delta\alpha\sin(\omega t) - \delta\beta\cos(\omega t)\end{cases}$ 下具有对称性，并给出守恒量

i) $\begin{cases}\delta x = -\cos\omega t\,\delta\alpha - \sin\omega t\,\delta\beta \\ \delta y = \sin\omega t\,\delta\alpha - \cos\omega t\,\delta\beta\end{cases}$　　$\begin{cases}\delta\dot x = \omega\sin\omega t\,\delta\alpha - \omega\cos\omega t\,\delta\beta \\ \delta\dot y = \omega\cos\omega t\,\delta\alpha + \omega\sin\omega t\,\delta\beta\end{cases}$

$\delta L = \dot x\delta\dot x + \dot y\delta\dot y + \alpha\dot y\delta x + \alpha x\delta\dot y - \alpha\dot x\delta y - \alpha y\delta\dot x + \omega^2 x\delta x + \omega^2 y\delta y$

$\alpha\delta y = \delta\dot x$，　$-\alpha\delta x = \delta\dot y$　　$\Rightarrow \delta L = 0$

$\delta L = (\frac{\partial L}{\partial\vec r})\delta\vec r + (\frac{\partial L}{\partial\vec v})\delta\vec v = \frac{d}{dt}(\frac{\partial L}{\partial\vec v})\delta\vec r + (\frac{\partial L}{\partial\vec v})\frac{d\delta\vec r}{dt} = \frac{d}{dt}(\frac{\partial L}{\partial\vec v}\cdot\delta\vec r)$

$= \frac{d}{dt}(\vec v\cdot\delta\vec r + \alpha(\vec n_3\times\vec r)\cdot\delta\vec r) = \frac{d}{dt}(\dot x\delta x + \dot y\delta y - \alpha y\delta x + \alpha x\delta y)$

$= \frac{d}{dt}\delta\alpha(-(\dot x - \alpha y)\cos\omega t + (\dot y + \alpha x)\sin\omega t) + \frac{d}{dt}\delta\beta(-(\dot x - \alpha y)\sin\omega t - (\dot y + \alpha x)\cos\omega t)$

$\begin{cases}(\dot x - \alpha y)\cos\omega t - (\dot y + \alpha x)\sin\omega t = C_1 \\ (\dot y + \alpha x)\cos\omega t + (\dot x - \alpha y)\sin\omega t = C_2\end{cases}$

ii) 利用诺特定理，$P_iX_i^s - EX_0^s + \Lambda^s = const$

$X_1^1 = \cos\omega t$　$X_2^1 = -\sin\omega t$　$X_1^2 = \sin\omega t$　$X_2^2 = \cos\omega t$

$P_1 = \dot x - \alpha y$，$P_2 = \dot y + \alpha x$，代入可得守恒量 $\begin{cases}(\dot x - \alpha y)\cos\omega t - (\dot y + \alpha x)\sin\omega t = C_1 \\ (\dot y + \alpha x)\cos\omega t + (\dot x - \alpha y)\sin\omega t = C_2\end{cases}$

(c) 求解该系统的运动

$\begin{cases}(\dot x - \alpha y)\cos\omega t - (\dot y + \alpha x)\sin\omega t = C_1 = 0 \\ (\dot y + \alpha x)\cos\omega t + (\dot x - \alpha y)\sin\omega t = C_2 = \alpha a\end{cases}$

$\Rightarrow \begin{cases}\dot x - \alpha y = C_1\cos\omega t + C_2\sin\omega t \\ \dot y + \alpha x = C_2\cos\omega t - C_1\sin\omega t\end{cases}$　　$\alpha\dot y = \ddot x + \alpha C_1\sin\omega t - \omega C_2\cos\omega t$

$\Rightarrow \ddot x + \alpha^2 x = 2C_2\omega\cos\omega t - 2C_1\omega\sin\omega t = 2\alpha a\cos\omega t$

$x = a(\cos\alpha t + \alpha t\sin\alpha t)$　　$y = a(\alpha t\cos\alpha t - \sin\alpha t)$

非惯性系力学：　K为惯性系，　k为非惯性系，　k相对K：$\vec V(t)$，$\vec\omega(t)$

$\vec v = \vec v_0 + \vec V + \vec\omega\times\vec r$，　$L = \frac12 m\vec v^2 = \frac12 m\vec V^2 + \frac12 m\vec v_0^2 + \frac12 m(\vec\omega\times\vec r)^2 + \underbrace{m\vec v_0\cdot\vec V}_{-m\frac{d\vec V}{dt}\cdot\vec r\ \text{(不影响E-L方程)}} + m\vec v_0\cdot(\vec\omega\times\vec r) + m\vec V\cdot(\vec\omega\times\vec r) - U$

$\frac{d}{dt}(\frac{\partial L}{\partial\vec v}) = \frac{d}{dt}(m\vec v) + m\frac{d\vec\omega}{dt}\times\vec r + m\vec\omega\times\vec v$

$\frac{\partial L}{\partial\vec r} = -m\vec\omega\times(\vec\omega\times\vec r) + m(\vec v\times\vec\omega) + m\vec V\times\vec\omega - \frac{\partial U}{\partial\vec r}$

$\Rightarrow m\frac{d\vec v}{dt} = -m\frac{d\vec V}{dt} - m\frac{d\vec\omega}{dt}\times\vec r - 2m\vec\omega\times\vec v - m\vec\omega\times\vec V - m\vec\omega\times(\vec\omega\times\vec r) - \frac{\partial U}{\partial\vec r}$

没有平动时，　$L = \frac12 m\vec v^2 + \frac12 m(\vec\omega\times\vec r)^2 + m\vec v\cdot(\vec\omega\times\vec r) - U$

$p = \frac{\partial L}{\partial\vec v} = m\vec v + m\vec\omega\times\vec r$　　$E = \vec p\cdot\vec v - L = \frac12 m\vec v^2 - \frac12 m(\vec\omega\times\vec r)^2 + U$

力学相似性与尺度变换：　$r' = \alpha r$，　$t' = \beta t$，　$V(\alpha r_1,\cdots,\alpha r_n) = \alpha^k V(r_1, r_2,\cdots, r_n)$

$L' = \frac{\alpha^2}{\beta^2}T - \alpha^k V$，　当 $\beta = \alpha^{1-\frac{k}{2}}$ 时，　$L' = \alpha^k L$，　不改变E-L方程，仅坐标改变

$\Rightarrow \frac{t'}{t} = (\frac{l'}{l})^{1-\frac{k}{2}}$　　e.g. 开三，$k = -1$，　$T\propto a^{3/2}$

位力定理：　$2T = \sum\vec p_a\cdot\vec v_a = \sum\vec p_a\cdot\frac{d\vec r_a}{dt} = \frac{d}{dt}(\sum\vec p_a\cdot\vec r_a) - \sum\vec r_a\cdot\frac{d\vec p_a}{dt}$

$= \frac{d}{dt}(\sum\vec p_a\cdot\vec r_a) + \sum_a\vec r_a\cdot\frac{\partial V}{\partial\vec r_a}$　　$\bar f = \lim_{T\to\infty}\frac1T\int_0^T f(t)dt$，　前一项平均为0

$\Rightarrow 2\langle T\rangle = \langle\sum_a\vec r_a\cdot\frac{\partial V}{\partial\vec r_a}\rangle = k\langle V\rangle$　　$\langle T\rangle = \frac{k}{k+2}\langle E\rangle$，　$\langle V\rangle = \frac{2}{k+2}\langle E\rangle$

最小作用量原理与相对论：　$S = -\int m\sqrt{1-\vec v^2}\,dt$，　$L = -m\sqrt{1-\vec v^2}$

$\vec p = \frac{\partial L}{\partial\vec v} = \frac{m\vec v}{\sqrt{1-\vec v^2}}$，　$E = \vec p\cdot\vec v - L = \frac{m}{\sqrt{1-\vec v^2}}$

e.g. 求弯曲时空下的测地线方程，　$S = -mc\int ds$，　$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$

$u^\mu = \frac{dx^\mu}{ds}$，　$(ds)^2 = g_{\mu\nu}dx^\mu dx^\nu$，　$2ds(\delta ds) = (\delta g_{\mu\nu})dx^\mu dx^\nu + g_{\mu\nu}d\delta x^\mu dx^\nu + g_{\mu\nu}dx^\mu d\delta x^\nu$

利用 $g_{\mu\nu} = g_{\nu\mu}$，　$2ds(\delta ds) = \delta g_{\mu\nu}dx^\mu dx^\nu + 2g_{\mu\nu}dx^\nu d\delta x^\mu$

$\delta ds = ds[\frac12\frac{\partial g_{\mu\nu}}{\partial x^\sigma}u^\mu u^\nu\delta x^\sigma + g_{\mu\nu}u^\nu d\delta x^\mu]$

$= ds[\frac12\frac{\partial g_{\mu\nu}}{\partial x^\sigma}u^\mu u^\nu\delta x^\sigma + \frac{d}{ds}(g_{\mu\nu}u^\nu\delta x^\mu) - \frac{d}{ds}(g_{\mu\nu}u^\nu)\delta x^\mu]$

$\delta S = \int\delta ds = -mc\int ds[\frac12\frac{\partial g_{\mu\nu}}{\partial x^\sigma}u^\mu u^\nu\delta x^\sigma + \frac{d}{ds}(g_{\mu\nu}u^\nu\delta x^\mu) - \frac{d}{ds}(g_{\mu\nu}u^\nu)\delta x^\mu]$　　端点固定

$= -mc\int ds[\frac12\frac{\partial g_{\mu\nu}}{\partial x^\sigma}u^\mu u^\nu\delta x^\sigma - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}u^\nu u^\sigma\delta x^\mu - g_{\mu\nu}\frac{du^\nu}{ds}\delta x^\mu]$　　$\underbrace{-mc\,g_{\mu\nu}u^\nu\delta x^\mu\Big|_a^b}_{=0}$

重命名指标：　$\frac{\partial g_{\mu\nu}}{\partial x^\sigma}u^\mu u^\nu\delta x^\sigma = \frac{\partial g_{\nu\sigma}}{\partial x^\mu}u^\nu u^\sigma\delta x^\mu$　　$\frac{\partial g_{\mu\nu}}{\partial x^\sigma}u^\nu u^\sigma = \frac{\partial g_{\mu\sigma}}{\partial x^\nu}u^\nu u^\sigma$

$\Rightarrow \frac12(\frac{\partial g_{\nu\sigma}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma} - \frac{\partial g_{\mu\sigma}}{\partial x^\nu})u^\nu u^\sigma - g_{\mu\nu}\frac{du^\nu}{ds} = 0$

$\Rightarrow \frac{d^2}{ds^2}x^\rho + \frac12 g^{\mu\rho}(\frac{\partial g_{\mu\nu}}{\partial x^\sigma} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\nu\sigma}}{\partial x^\mu})\frac{dx^\nu}{ds}\frac{dx^\sigma}{ds} = 0$

最小作用量原理与电磁学

e.g. 带电粒子在电磁场作用下的运动，考虑狭义相对论

$A^\mu = (\phi, \vec A)$，　$S = -\int mc\sqrt{1-\vec v^2}\,ds - \int e\,dx^\mu A_\mu$　　$L = -m\sqrt{1-\vec v^2} - e(\phi - \vec v\cdot\vec A)$

$\frac{\partial L}{\partial\vec x} = -e\nabla\phi + e\nabla(\vec v\cdot\vec A)$，　$\frac{d}{dt}(\frac{\partial L}{\partial\vec v}) = \frac{d}{dt}(\vec p + e\vec A) = \frac{d\vec p}{dt} + e\frac{\partial\vec A}{\partial t} + e(\vec v\cdot\nabla)\vec A$

$\vec v\times(\nabla\times\vec A) = \nabla(\vec v\cdot\vec A) - (\vec v\cdot\nabla)\vec A$　　$\Rightarrow \frac{d\vec p}{dt} = -e(\nabla\phi + \frac{\partial\vec A}{\partial t}) + e\vec v\times(\nabla\times\vec A)$

又 $\vec E = -\nabla\phi - \frac{\partial\vec A}{\partial t}$，$\vec B = \nabla\times\vec A$，　$\frac{d\vec p}{dt} = e\vec E + e\vec v\times\vec B$，　　注：$\vec p$并不是正则动量

$\delta S = -\int(m\,d\delta x^\mu\frac{dx_\mu}{ds} + e\,d\delta x^\mu A_\mu + e\,dx^\mu\delta A_\mu) = \int[(m\frac{du_\mu}{ds} + e\frac{dA_\mu}{ds})\delta x^\mu - e\frac{dx^\mu}{ds}\delta A_\mu]ds$

$dA_\mu = \frac{\partial A_\mu}{\partial x^\nu}dx^\nu$，　$\delta A_\mu = \frac{\partial A_\mu}{\partial x^\nu}\delta x^\nu$　　$\Rightarrow m\frac{d^2x^\mu}{ds^2} + e\frac{\partial A_\mu}{\partial x^\nu}\frac{dx^\nu}{ds} - e\frac{dx^\nu}{ds}\frac{\partial A_\nu}{\partial x^\mu} = 0$

$\Rightarrow m\frac{d^2x^\mu}{ds^2} = eF_{\mu\nu}\frac{dx^\nu}{ds}$　　$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}$

$F_{\mu\nu}$，电磁场张量，反对称，对角元为0

$F_{0i} = \frac{\partial A_i}{\partial t} - \frac{\partial A_0}{\partial x^i} = E_i$　　$F_{ij} = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} = -\epsilon_{ijk}B_k$

$m\frac{d^2x^\mu}{ds^2} = m\frac{d}{dt}(\frac{dx^\mu}{ds})\frac{dt}{ds} = -\frac{d\vec p}{dt}\frac{dt}{ds}$

$eF_{\mu\nu}\frac{dx^\nu}{ds} = eF_{i0}\frac{dt}{ds} + eF_{ij}\frac{dx^j}{dt}\frac{dt}{ds} = -(eE_i + e\epsilon_{ijk}v_jB_k)\frac{dt}{ds}$

$\Rightarrow m\frac{d\vec p}{dt} = e\vec E + e\vec v\times\vec B$

若μ取为时间方向，$m\frac{d^2x^0}{ds^2} = \frac{dE}{dt}\frac{dt}{ds}$，　$eF_{\mu\nu}\frac{dx^\nu}{ds} = eE_iv_i\frac{dt}{ds}$

$\Rightarrow \frac{dE}{dt} = e\vec E\cdot\vec v$

关于拉格朗日函数的改写：

$T = \frac12 m\dot r^2 + \frac12 mr^2\dot\theta^2$，　$V = V(r)$，　$M = mr^2\dot\theta = const$　　$L = \frac12 m\dot r^2 + \frac12 mr^2\dot\theta^2 - V(r)$

i) 直接用M替换$\dot\theta$，　$L = \frac12 m\dot r^2 + \frac{M^2}{2mr^2} - V(r) = L(r,\dot r)$，　$V_{eff} = V(r) - \frac{M^2}{2mr^2}$　(×)

ii) 推导能量，　$E = \dot q_k\frac{\partial L}{\partial\dot q_k} - L = \dot r m\dot r + mr^2\dot\theta^2 - L = \frac12 m\dot r^2 + \frac{M^2}{2mr^2} + V(r)$，　$V_{eff} = V(r) + \frac{M^2}{2mr^2}$　(√)

保证E-L方程与原先相同，　$L(q_1\cdots q_n, \dot q_{s+1}\cdots\dot q_n, \dot q_1\cdots\dot q_s) - \sum_{k=1}^{s}\dot q_k\frac{\partial L}{\partial\dot q_k} = L(q_1\cdots q_n) - \sum_{k=1}^{n}\dot q_k\frac{\partial L}{\partial\dot q_k}$

$\Rightarrow$ 有心问题的拉氏量：　$L = \frac12 m\dot r^2 - \frac{M^2}{2mr^2} - V(r)$

运动方程：　$\frac{\partial L}{\partial r} = \frac{M^2}{mr^3} - \frac{\partial V}{\partial r} = \frac{d}{dt}(\frac{\partial L}{\partial\dot r}) = m\ddot r$

比耐方程：$r = \frac1u$，　$M = mr^2\dot\theta = \frac{m}{u^2}\frac{d\theta}{dt}$，　$\Rightarrow dt = \frac{m}{u^2M}d\theta$