理论力学A

$\ddot r = \frac{d^2}{dt^2}r = \frac{d}{dt}(\frac{dr}{dt}) = \frac{d}{dt}(\frac{d(\frac1u)}{m}\frac{u^2M}{d\theta}) = \frac{d}{dt}(-\frac{Mdu}{md\theta}) = -\frac{u^2M^2}{m^2}\frac{d^2u}{d\theta^2}$

$\Rightarrow \frac{u^2M^2}{m}\frac{d^2u}{d\theta^2} + \frac{M^2}{m}u^3 = -f(\frac1u) = \frac{M^2u^2}{m}(\frac{d^2u}{d\theta^2}+u)$

开普勒问题，$f(r) = -\frac{k}{r^2}$，代入比耐方程，$\frac{d^2u}{d\theta^2}+u = \frac{mk}{M^2}$ $u = \frac{mk}{M^2} + A\cos\theta$

$r = \frac{1}{A\cos\theta + \frac{mk}{M^2}} = \frac{p}{1+e\cos\theta}$，$p = \frac{M^2}{mk}$，$e = pA$ ⇒ 椭圆、双曲线、抛物线

周期：f为掠面速度，$M \propto f \propto \sqrt p \propto a^{-\frac12}b$ $T \propto \frac{ab}{f} \propto \frac{ab}{a^{-\frac12}b} \propto a^{\frac32}$

LRL矢量：$\vec A = \vec p\times\vec M - mk\frac{\vec r}{r} = \text{const}$

证明守恒：$\frac{d\vec A}{dt} = \frac{d\vec p}{dt}\times\vec M + \vec p\times\frac{d\vec M}{dt} - mk\frac{d}{dt}(\frac{\vec r}{r})$

第一项，$\frac{d\vec p}{dt} = -\frac{k}{r^3}\vec r$，$\vec M = m\vec r\times\dot{\vec r}$，$-\frac{k}{r^3}\vec r\times(\vec r\times\dot{\vec r}) = -\frac{k}{r^3}((\vec r\cdot\dot{\vec r})\vec r - r^2\dot{\vec r}) = \frac{k}{r}\dot{\vec r} - \frac{k}{r^3}(\vec r\cdot\dot{\vec r})\vec r$

第二项，M守恒，显然为0 第三项，$-mk\frac{d}{dt}(\frac{\vec r}{r}) = -mk\frac{\dot{\vec r}}{r} + \frac{mk}{r^2}\dot r\vec r = -mk\frac{\dot{\vec r}}{r} + \frac{mk}{r^3}(\vec r\cdot\dot{\vec r})\vec r$

LRL矢量的方向：i) 垂直于$\vec M$，ii) 平行于$\vec r_{min}$

i) $\vec A\cdot\vec M = (\vec p\times\vec M - mk\frac{\vec r}{r})\cdot\vec M = \overset{0}{\overset{\|}{(\vec p\times\vec M)}}\cdot\vec M - \frac{m^2k}{r}\vec r\cdot\overset{0}{\overset{\|}{(\vec r\times\vec v)}} = 0$

ii) 考虑近日点处，$\vec A = \frac{M^2}{r_{min}^2}\vec r_{min} - \frac{mk}{r_{min}}\vec r_{min}$ // $\vec r_{min}$

LRL矢量的大小：$\vec A\cdot\vec r = Ar\cos\theta = (\vec p\times\vec M)\cdot\vec r - mkr = \vec M\cdot(\vec r\times\vec p) - mkr = M^2 - mkr$

$\Rightarrow r = \frac{M^2}{mk + A\cos\theta} = \frac{\frac{M^2}{mk}}{1+\frac{A}{mk}\cos\theta}$ $\Rightarrow A = mke$

LRL矢量的对称性：基于势能的形式(1/r)的对称性，而非时空的基本对称性

引入无穷小变换：$\vec r' = \vec r + [2\vec p(\vec r\cdot\delta\vec a) - \vec r(\vec p\cdot\delta\vec a) - \delta\vec a(\vec r\cdot\vec p)] = \vec r + [\delta\vec a\times(\vec p\times\vec r) + \vec r\times(\vec p\times\delta\vec a)]$

证明具有形式不(协)变性：$\frac{\partial L}{\partial q_k}\delta q_k + \frac{\partial L}{\partial\dot q_k}\frac{d}{dt}(\delta q_k) = \frac{d}{dt}\delta\lambda(q,t)$ $L = \frac12 m\dot{\vec r}^2 - V$

第一项：$-\frac{k}{r^3}\vec r\cdot[\delta\vec a\times(\vec p\times\vec r) + \vec r\times(\vec p\times\delta\vec a)] = -\frac{k}{r^3}(\vec p\times\vec r)\times\vec r\cdot\delta\vec a$

$= mk(\frac{\dot{\vec r}}{r} - \frac{(\vec r\cdot\dot{\vec r})}{r^3}\vec r)\cdot\delta\vec a = \frac{d}{dt}(mk\frac{\vec r}{r})\cdot\delta\vec a$

第二项：$m\dot{\vec r}\cdot\frac{d}{dt}[\delta\vec a\times(\vec p\times\vec r) + \vec r\times(\vec p\times\delta\vec a)] = \vec p\cdot[\delta\vec a\times(\dot{\vec p}\times\vec r) + \vec r\times(\dot{\vec p}\times\delta\vec a)]$

$= [(\vec p\times\vec r)\times\dot{\vec p} + (\vec p\times\vec r)\times\dot{\vec p}]\cdot\delta\vec a = \frac{d}{dt}(-\vec M\times\vec p)\cdot\delta\vec a = \frac{d}{dt}(\vec p\times\vec M)\cdot\delta\vec a$

$\Rightarrow -\delta\lambda = (\vec p\times\vec M + mk\frac{\vec r}{r})\delta\vec a$，由诺特定理，$\vec p\cdot\delta\vec r + \delta\lambda = \text{const}$

$\vec p\cdot[\delta\vec a\times(\vec p\times\vec r) + \vec r\times(\vec p\times\delta\vec a)] = 2(\vec p\times\vec r)\times\vec p\cdot\delta\vec a = 2\vec p\times\vec M\cdot\delta\vec a$ $\Rightarrow \vec p\times\vec M - mk\frac{\vec r}{r} = \text{const}$

无穷小变换的由来：$\delta q_i = -\delta a_j[A_j, q_i] = -\delta a_j\frac{\partial A_j}{\partial p_k}\delta_{ik} = -\delta a_j(2p_iq_j - q_ip_j - (\vec p\cdot\vec q)\delta_{ij})$

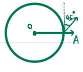
$\delta p_i = -\delta a_j[A_j, p_i]$，哈密顿量不变，$\delta a_j[A_j, H] = 0$，⇒ A为守恒量，$[A, H] = 0$

有心力下的质点运动轨道：$M = mr^2\dot\varphi$，$E = \frac12 m\dot r^2 + \frac{M^2}{2mr^2} + V(r)$

$dt = \frac{mr^2}{M}d\varphi$ ： $dt = \frac{dr}{\sqrt{\frac2m(E-V(r)) - \frac{M^2}{m^2r^2}}}$ $\Rightarrow \int d\varphi = -\int\frac{\frac{M}{m}du}{\sqrt{\frac2m(E-V(u)) - \frac{M^2}{m^2}u^2}}$

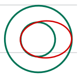
e.g. (a) 在地球赤道处以第一宇宙速度的一半和仰角45°发射一物体，求该物体以OA为极轴的轨道方程。

$\frac{mv_1^2}{R_E} = \frac{\alpha_E}{R_E^2}$ ⇒ $v_1 = \sqrt{\frac{\alpha_E}{mR_E}}$，$M = mR_E\cdot\frac12 v_1\cdot\frac{\sqrt2}{2} = \frac{\sqrt2}{4}\sqrt{\alpha_E mR_E} = \sqrt{\alpha_E mp}$ ⇒ $p = \frac18 R_E$



$E = \frac12 m(\frac12 v_1)^2 - \frac{\alpha_E}{R_E} = -\frac{7\alpha_E}{8R_E} = -\frac{\alpha_E}{2a}$ ⇒ $a = \frac47 R_E$ $b = \frac{\sqrt{14}}{14}R_E$，$c = \frac{5\sqrt2}{14}R_E$ ⇒ $e = \frac{5\sqrt2}{8}$

$r = \frac{\frac18 R_E}{1 - \frac{5\sqrt2}{8}\sin(\theta+\varphi_0)}$ ⇒ $\varphi_0 = \sin^{-1}\frac{7\sqrt2}{10}$



(b) 地球、金星轨道看做同一平面内的圆，半径分别为R和0.7R，现从地面发射一颗人造卫星去金星，发射速度最小为多少？ $m\frac{V_E^2}{R} = \frac{\alpha_S}{R^2}$

脱离地球：$\frac12 mv^2 - \frac{\alpha_E}{R_E} = \frac12 mv'^2$ 太阳系中，$\vec v'' = \vec v' + \vec v_E$，$v_E = \sqrt{\frac{\alpha_S}{mR}}$

$2a = 1.7R$ $E = -\frac{\alpha_S}{1.7R} = -\frac{10\alpha_S}{17R} = \frac12 mv''^2 - \frac{\alpha_S}{R}$ ⇒ $v'' = \sqrt{\frac{14}{17}\frac{\alpha_S}{mR}}$

$v_{min} = \sqrt{\frac{2\alpha_E}{mR_E} + (\sqrt{\frac{14}{17}} - 1)^2\frac{\alpha_S}{mR}} = \sqrt{\frac{2GM_E}{R_E} + 0.0086\frac{GM_S}{R}}$

(c) 在地球上以第三宇宙速度v_3正对太阳发射一星体（假设地球速度矢量指向太阳），求轨道方程。

$\frac12 m|\vec v' + \vec v_E|^2 - \frac{\alpha_S}{R} = 0$，$E = \frac12 m(v'^2 + v_E^2) - \frac{\alpha_S}{R} = -mv'v_E = -\frac{\alpha_S}{2a} = -(\sqrt2-1)\frac{\alpha_S}{R}$，⇒ $a = \frac{\sqrt2+1}{2}R$

$J = \sqrt{m\alpha_S R} = \sqrt{mp\alpha_S}$ ⇒ $p = R$，$e = \sqrt2 - 1$

伯特兰定理：有心力场下质点运动具有稳定闭合轨道的条件

$\frac{d^2u}{d\varphi^2} + u = J(u) = -f(\frac1u)\frac{m}{M^2u^2}$ 圆周运动，$u_0 = J(u_0)$，

考虑微扰，$\delta u = u - u_0 = \xi$ $\frac{d^2\xi}{d\varphi^2} + \xi + u_0 = J(u_0) + J'(u_0)\xi + O(\xi^2)$

$\Rightarrow \frac{d^2\xi}{d\varphi^2} + (1 - J'(u_0))\xi = 0$，$\beta^2 = 1 - J'(u_0)$

i) $\beta^2 > 0$，轨道稳定在某个值附近 ii) $\beta^2 < 0$，指数衰减或增长

若轨道闭合，β需为有理数，$J(u) = -f(\frac1u)\frac{m}{M^2u^2}$ $u_0 = -f(\frac{1}{u_0})\frac{m}{M^2u_0^2}$

$J'(u_0) = f(\frac{1}{u_0})\frac{m}{M^2}\frac{2}{u_0^3} - \frac{m}{M^2u_0^2}\frac{df(\frac1u)}{du}\Big|_{u=u_0} = -2 + \frac{u_0}{f(\frac{1}{u_0})}\frac{df(\frac1u)}{du}\Big|_{u=u_0}$

$\beta^2 = 3 - \frac{u_0}{f(\frac{1}{u_0})}\frac{df(\frac1u)}{du}\Big|_{u=u_0}$ 稳定闭合条件对不同u_0成立，⇒ β^2为常数

$\frac{df}{dr} = (\beta^2-3)\frac{f}{r}$ ⇒ $f(r) = -kr^{\beta^2-3}$ 有心力幂次大于-3

考虑J(u)的更高阶项对轨道的影响

$\frac{d^2\xi}{d\varphi^2} + \beta^2\xi = \frac12 J''(u_0)\xi^2 + \frac16 J'''(u_0)\xi^3 + O(\xi^4)$

$\xi = \xi_0 + \xi_1\cos\beta\varphi + \xi_2\cos2\beta\varphi + \xi_3\cos3\beta\varphi$，代入方程，比较系数，$\xi_1$是一阶小量

$\cos^3\beta\varphi = \cos\beta\varphi\frac{\cos2\beta\varphi+1}{2} = \frac34\cos\beta\varphi + \frac14\cos3\beta\varphi$

$$\begin{cases}\cos0\beta\varphi: & \beta^2\xi_0 = \frac12 J''(u_0)\xi_1^2\cdot\frac12\\ \cos\beta\varphi: & 0 = \frac12 J''(u_0)(2\xi_0\xi_1 + \xi_1\xi_2) + \frac16 J'''(u_0)\cdot\frac34\xi_1^3\\ \cos2\beta\varphi: & -3\beta^2\xi_2 = \frac12 J''(u_0)\xi_1^2\cdot\frac12\end{cases}$$

ξ_0、ξ_2为二阶小量，ξ_3为三阶小量

$J(u) = -f(\frac1u)\frac{m}{M^2u^2} = \frac{km}{M^2}u^{1-\beta^2}$，$J(u_0) = \frac{km}{M^2}u_0^{1-\beta^2} = u_0$ ⇒ $J(u) = u_0^{\beta^2}u^{1-\beta^2}$

$J'(u) = (1-\beta^2)u_0^{\beta^2}u^{-\beta^2}$，$J''(u) = -\beta^2(1-\beta^2)u_0^{\beta^2}u^{-\beta^2-1}$，$J'''(u) = \beta^2(1-\beta^2)(1+\beta^2)u_0^{\beta^2}u^{-\beta^2-2}$

代入方程组：$\xi_0 = -\frac{(1-\beta^2)}{4u_0}\xi_1^2$，$\xi_2 = \frac{(1-\beta^2)}{12u_0}\xi_1^2$ 代入中间式子，$\beta^2(1-\beta^2)(4-\beta^2) = 0$

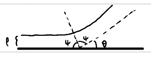
$\Rightarrow \begin{cases}\beta^2=1 & \text{万有引力}\\ \beta^2=4 & \text{球对称谐振子}\end{cases}$ $J'(u_0) = (1-\beta^2)$，$J''(u_0) = \frac{-\beta^2(1-\beta^2)}{u_0}$，… 每一项均为0

力学相似性与尺度变换，椭圆轨道

伯特兰定理：质点在有心力运动和球对称谐振子两种情况下有稳定闭合轨道

微分散射截面：$2\pi\sin\theta\,\sigma(\theta)d\theta = 2\pi\rho d\rho$ ⇒ $\sigma(\theta) = \frac{\rho}{\sin\theta}|\frac{d\rho}{d\theta}|$

总散射截面 $\sigma_{tot} = 2\pi\int_0^\pi\sigma(\theta)\sin\theta\,d\theta$



$\theta(\rho) = \pi - 2\psi = \pi - 2\int_0^{u_{max}}\frac{\frac{M}{m}du}{\sqrt{\frac2m(E-V) - \frac{M^2}{m^2}u^2}}$ $M = mv_0\rho$，$E = \frac12 mv_0^2$

$= \pi - 2\int_0^{u_{max}}\frac{\rho du}{\sqrt{1 - \frac VE - \rho^2u^2}}$

e.g. 开普勒问题库仑排斥力下的散射截面

$V = -ku$，$E = \frac12 mv_0^2 = \frac{k}{2a}$ $M = mv_0\rho = \sqrt{pmk}$ $p = \frac{mv_0^2\rho^2}{k} = \frac{2E\rho^2}{k} = (e^2-1)a = \frac{k}{2E}(e^2-1)$

$\Rightarrow e = \sqrt{1 + (\frac{2E\rho}{k})^2}$ $\cos\psi = \cos(\frac\pi2 - \frac\theta2) = \sin\frac\theta2 = \frac1e = \frac{1}{\sqrt{1+(\frac{2E\rho}{k})^2}}$，$\cot\frac\theta2 = \frac{2E\rho}{k}$

$\Rightarrow \sigma(\theta) = \frac14(\frac{k}{2E})^2\frac{1}{\cos^4\frac\theta2}$ $\sigma_{tot} = 2\pi\int\rho d\rho = 2\pi\int_0^\pi\sigma(\theta)\sin\theta d\theta = \infty$

e.g. 势能满足 $V = \begin{cases}0 & r>a\\ +\infty & r<a\end{cases}$ 下的散射截面

$\sigma(\theta) = \pi - 2\int_0^{u_{max}}\frac{\rho du}{\sqrt{1-\frac VE - \rho^2u^2}} = \pi - 2\int_0^{u_{max}}\frac{\rho du}{\sqrt{1-\rho^2u^2}} = \pi - 2\arcsin(\frac\rho a)$

$\cos\frac\theta2 = \frac\rho a$ $-\frac12\sin\frac\theta2 d\theta = \frac{d\rho}{a}$ $\Rightarrow \sigma(\theta) = \frac{a^2}{4}$ $\sigma_{tot} = \int_0^\pi 2\pi\sin\theta\,\sigma(\theta)d\theta = \pi a^2$

e.g. 汤川势 $U(r) = -\alpha\frac{e^{-\frac ra}}{r}$，a>0，α>0，考虑α是个小量

(a) 写出质点运动方程 $\begin{cases}J = mr^2\dot\varphi\\ E = \frac12 m\dot r^2 + \frac{J^2}{2mr^2} - \alpha\frac{e^{-r/a}}{r}\end{cases}$

(b) 给出等效一维有效势，描述在不同的能量和角动量下，运动轨道的特性。

$V_{eff} = \frac{J^2}{2mr^2} - \frac{\alpha e^{-r/a}}{r}$

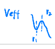
$\frac{dV_{eff}}{dr} = -\frac{J^2}{mr^3} - \frac{\alpha e^{-\frac ra}}{r^2}(-\frac1a r - 1) = \frac{1}{r^3}[-\frac{J^2}{m} + \alpha e^{-\frac ra}(r + \frac{r^2}{a})]$

$F(r) = \alpha e^{-\frac ra}(r + \frac{r^2}{a})$ $F'(r) = \alpha e^{-\frac ra}(-\frac1a(r+\frac{r^2}{a}) + 1 + \frac{2r}{a}) = \alpha e^{-\frac ra}(-\frac{r^2}{a^2} + \frac ra + 1)$ ⇒ $r = \frac{1+\sqrt5}{2}a$

F(r) ，极大值点在 $r_0 = \frac{1+\sqrt5}{2}a$ 取到 $F(r_0)$

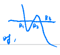
1*) J较大，$J > \sqrt{mF(r_0)}$，$V'_{eff} \le 0$，V_{eff} ↓，只能在 r>R 处运动，$\frac{J^2}{2mR^2} - \frac{e^{-\frac Ra}}{R} = E$

2*) ，$J < \sqrt{mF(r_0)}$，r_1, r_2 是方程 $F(r) = \frac{J^2}{m}$ 的两根，$r_1<r_2$



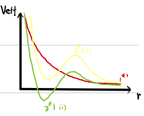
i) $V_{eff}(r_1) > 0$，则当 $E > V_{eff}(r_2)$ 或 $E < V_{eff}(r_1)$ 时，粒子只能在 r>R 区域内运动

$V_{eff}(r_1) < E < V_{eff}(r_2)$ 时，



可以在 $R_1<r<R_2$ 处有界运动，或 $r>R_3$ 处运动 $R_1<R_2<R_3$为3个根

ii) $V_{eff}(r_1) < 0$，当 $E > V_{eff}(r_2)$ 时，只能在 r>R 内运动



当 $0 < E < V_{eff}(r_2)$ 时，$R_1<r<R_2$ 内做有界运动，$r>R_3$处做无界运动

当 E ≤ 0 时，只能在 $R_1<r<R_2$ 内做有界运动

(c) 假设近似圆轨道，求周期进动角。

$m\ddot r = \frac{1}{r^3}[\frac{J^2}{m} - \alpha e^{-\frac ra}(r + \frac{r^2}{a})]$

$\frac{J^2}{m} = \alpha e^{-\frac{r_0}{a}}(r_0 + \frac{r_0^2}{a}) = m\omega_\theta^2r_0^4$ $\omega_\theta^2 = \frac{\alpha e^{-\frac{r_0}{a}}}{mr_0^3}(1 + \frac{r_0}{a})$

$\omega_r^2 = -\frac1m\{\frac{1}{r^3}[\frac{J^2}{m} - \alpha e^{-\frac ra}(r + \frac{r^2}{a})]\}' = \frac1m[\frac{J^2}{m}\frac{3}{r_0^4} - \alpha e^{-\frac{r_0}{a}}(\frac{1}{a^2r_0} + \frac{2}{ar_0^2} + \frac{2}{r_0^3})] = \frac{\alpha e^{-\frac{r_0}{a}}}{mr_0^3}(1 + \frac{r_0}{a} - \frac{r_0^2}{a^2})$

$\frac{\Delta\omega}{\omega} = \frac{r_0^2}{2a^2}$ ⇒ $\Delta\phi = 2\pi\cdot\frac{r_0^2}{2a^2} = \pi\frac{r_0^2}{a^2}$

(d) 考虑质点在汤川势下发生散射，求散射截面微分公式

$\theta = \pi - 2\int_0^{u_{max}}\frac{\rho du}{\sqrt{1 - \frac VE - \rho^2u^2}} = \pi - 2\int_0^{u_{max}}\frac{\rho du}{\sqrt{1 + \frac{\alpha}{E}ue^{-\frac{1}{au}} - \rho^2u^2}}$ $J = mv_0\rho = \sqrt{2mE}\rho$

$E = \frac12 mv_0^2 = \frac12 m(\frac Jm u_{max})^2 - \alpha u_{max}e^{-\frac{1}{au_{max}}} = E\rho^2u_{max}^2 - \alpha u_{max}e^{-\frac{1}{au_{max}}}$

$\Rightarrow u_{max} = u_0 + O(a^{-1})$ $\theta = \pi - 2\int_0^{u_0 + O(a^{-1})}\frac{\rho du}{\sqrt{1 + \frac{\alpha u}{E} - \rho^2u^2 - \frac{\alpha}{Ea}}}$

? u→0时，小量展开不成立

题目建议改成微弱均匀外场下，微分散射截面的改变 $\frac{\sigma'}{\sigma} = 1 + \frac{\delta V}{E}$

$\frac\pi2 - \frac\theta2 = \arcsin\frac{\frac{\alpha}{2\rho E}}{\sqrt{(1+\frac{\alpha}{2\rho E})^2 - \frac{\alpha}{Ea}}} + \arcsin\frac{\sqrt{(1+\frac{\alpha}{2\rho E})^2 + O(a^{-1})}}{\sqrt{(1+\frac{\alpha}{2\rho E})^2 - \frac{\alpha}{Ea}}}$

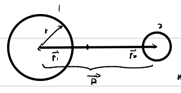
后一项：$\sin^{-1}(\square) = \frac\pi2 + \Delta$，$\square = \cos\Delta$，Δ无一阶项

$\Rightarrow \sin\frac\theta2 = \frac{\frac{\alpha}{2\rho E}}{\sqrt{(1+\frac{\alpha}{2\rho E})^2 - \frac{\alpha}{Ea}}}$ $k^2 = (1 + k^2 - \frac{\alpha}{Ea})\sin^2\frac\theta2$，$k^2 = \tan^2\frac\theta2(1 - \frac{\alpha}{Ea})$

$\frac{\alpha}{2\rho E} = \tan\frac\theta2(1 - \frac{\alpha}{2Ea})$ ⇒ $\rho = \frac{\alpha}{2E}\cot\frac\theta2(1 + \frac{\alpha}{2Ea})$

$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E^2}\sin^{-4}\frac\theta2(1 + \frac{\alpha}{Ea})$

潮汐现象



公转角速度 $w^2 = \frac{G(m_1+m_2)}{R^3}$

$m_2w^2r_2 = w^2\frac{m_1m_2}{m_1+m_2}R = \frac{Gm_1m_2}{R^2}$

单位质量势能：$U(r) = -\frac{Gm_1}{r} - \frac{Gm_2}{|\vec R - \vec r|} - \frac12 w^2(\vec r_1 + \vec r)^2$

考虑 R>>r，$\frac{1}{|\vec R - \vec r|} = (R^2 + r^2 - 2\vec R\cdot\vec r)^{-1/2} = \frac1R(1 + \frac{r^2}{R^2} - \frac{2\hat R\cdot\vec r}{R})^{-1/2} = \frac1R(1 - \frac{r^2}{2R^2} + \frac{\hat R\cdot\vec r}{R} - \frac32\frac{(\hat R\cdot\vec r)^2}{R^2})$

消去

$\Rightarrow U(r) = -\frac{Gm_1}{r} - \frac{Gm_2}{R}(1 - \frac{r^2}{2R^2} + \frac{\hat R\cdot\vec r}{R} - \frac32\frac{(\hat R\cdot\vec r)^2}{R^2}) - \frac12 w^2(r_1^2 + r^2) - w^2\vec r_1\cdot\vec r$

$= -\frac{Gm_1}{r} - \frac{Gm_2}{R} - \frac12 w^2(r_1^2 + r^2) + \frac{Gm_2}{2R^3}(r^2 + 3(\hat R\cdot\vec r)^2)$

i) 一天之中，由于地球自转，有两次潮汐 $(\hat R\cdot\vec r)^2$ 有两次取到最值

ii) 潮汐的影响正比于m_2，反比于R^3，与受地球公转的影响

近日点进动：$\Delta\phi = 2\int_{r_{min}}^{r_{max}}\frac{M/r^2\,dr}{\sqrt{2m(E-V) - \frac{M^2}{r^2}}}$

考虑 $V(r) = V^0(r) + \delta V(r)$，$\Delta\phi = 2\pi + \delta\phi$

$\delta\phi = 2\int_{r_{min}}^{r_{max}}\frac{\partial}{\partial V}\frac{M/r^2\,dr}{\sqrt{2m(E-V) - \frac{M^2}{r^2}}}\Big|_{V=V_0}\delta V = 2\int_{r_{min}}^{r_{max}}2m\cdot\frac{M}{r^2}\frac{\partial}{\partial(\frac{M^2}{r^2})}\frac{dr}{\sqrt{2m(E-V) - \frac{M^2}{r^2}}}\Big|_{V=V_0}\delta V$

$= 2m\int_{r_{min}}^{r_{max}}\frac{\partial}{\partial M}\frac{\delta V(r)\,dr}{\sqrt{2m(E-V_0) - \frac{M^2}{r^2}}}$

e.g. 水星进动问题，模型简化：质量均匀分布在环带上

$\delta V(r) = -\int_{-\pi}^{\pi}\frac{GM_im}{2\pi}\frac{d\theta}{\sqrt{a_i^2 + r^2 - 2a_ir\cos\theta}} = -\int_{\pi}^{\pi}\frac{GM_im}{2\pi a_i}(1 + \frac{r^2}{a_i^2} - 2\frac{r}{a_i}\cos\theta)^{-1/2}d\theta$ $a_i >> r$

$= -\frac{GM_im}{a_i}(1 + \frac14\frac{r^2}{a_i^2} + \cdots)$ $\delta V(r) = -\frac{GM_im}{4a_i^3}r^2$

$\delta\phi = 2m\frac{\partial}{\partial M}\int_{r_{min}}^{r_{max}}\frac{-\frac{GM_im}{4a_i^3}r^2dr}{\sqrt{2m(E + \frac kr) - \frac{M^2}{r^2}}} = -\sqrt{\frac{2m}{|E|}}\frac{GM_im}{4a_i^3}\frac{\partial}{\partial M}\int_{r_{min}}^{r_{max}}\frac{r^3dr}{\sqrt{(r - r_{min})(r_{max} - r)}}$

令 $a = \frac{r_{min} + r_{max}}{2}$，$c = \frac{r_{max} - r_{min}}{2}$ $I(r) = \int_{-c}^{c}\frac{(r+a)^3dr}{\sqrt{c^2 - r^2}} = \frac{\pi a^3}{2}(2 + 3e^2)$

$E = -\frac{k}{2a}$ $M = \sqrt{pkm} = \sqrt{a(1-e^2)km} = \sqrt{-\frac{mk^2}{2E}(1-e^2)}$ ⇒ $e = \sqrt{1 + \frac{2M^2E}{mk^2}}$

$\frac{\partial e^2}{\partial M} = \frac{4EM}{mk^2} = \frac{4E}{mk^2}\sqrt{-\frac{mk^2}{2E}(1-e^2)}$，$\delta\phi = -\sqrt{\frac{2m}{-E}}\frac{GM_im}{4a_i^3}\cdot\frac{3\pi a^3}{2}\frac{4E}{mk^2}\sqrt{-\frac{mk^2}{2E}}\sqrt{1-e^2}$

$\Rightarrow \delta\phi = \frac{3\pi}{2}\sqrt{1-e^2}(\frac{a}{a_i})^3(\frac{M_i}{M_\odot})$，$a_i$为水外行星轨道半径，$M_i$为质量，$M_\odot$为太阳质量

e.g. 相对论效应造成的近日点进动

$L = m\sqrt{1 - \frac{v^2}{c^2}} + \frac kr$ ⇒ $L = \frac12 mv^2(1 + \frac14\frac{v^2}{c^2} + \cdots) + \frac kr$ $v^2 = \dot r^2 + r^2\dot\varphi^2$

$M = P_\varphi = \frac{\partial L}{\partial\dot\varphi} = \frac{\partial L}{\partial v}\frac{\partial v}{\partial\dot\varphi} = mr^2\dot\varphi(1 + \frac12\frac{v^2}{c^2})$ $P_r = \frac{\partial L}{\partial\dot r} = m\dot r(1 + \frac12\frac{v^2}{c^2})$

$E = \dot rP_r + \dot\varphi P_\varphi - L = \frac12 mv^2(1 + \frac34\frac{v^2}{c^2}) - \frac kr$

$\frac12 mv^2 = \frac12 m(\dot r^2 + r^2\dot\varphi^2) = \frac12 m((\frac{dr}{d\varphi})^2 + r^2)\dot\varphi^2 = \frac12 m((\frac{dr}{d\varphi})^2 + r^2)\frac{M^2}{m^2r^4}(1 - \frac{v^2}{c^2}) = \frac{M^2}{2m}((\frac{du}{d\varphi})^2 + u^2)(1 - \frac{v^2}{c^2})$

$= \frac{M^2}{2m}((\frac{du}{d\varphi})^2 + u^2) - \frac12 m\frac{v^4}{c^2}$ ⇒ $E = \frac{M^2}{2m}((\frac{du}{d\varphi})^2 + u^2) - \frac18 m\frac{v^4}{c^2} - ku$

零级近似，$\frac12 mv^2 = E + ku$ ⇒ $-\frac18 m\frac{v^4}{c^2} = -\frac{(E+ku)^2}{2mc^2}$

$\overset{E'}{E(1 + \frac{E}{2mc^2})} + \overset{k'}{k}u(1 + \frac{E}{mc^2}) = \frac{M^2}{2m}((\frac{du}{d\varphi})^2 + u^2(1 - \frac{k^2}{M^2c^2}))$

$d\varphi = \frac{du}{\sqrt{(E' + k'u)\frac{2m}{M^2} - (1 - \frac{k^2}{M^2c^2})u^2}} = \frac{Mdu}{\sqrt{2m(E' + k'u) - M^2u^2 + \frac{k^2}{c^2}u^2}}$ 令 $A = 2m(E' + k'u) - M^2u^2$

$= \frac{Mdu}{\sqrt A} - \frac12\frac{u^2k^2}{c^2}\frac{Mdu}{A\sqrt A} = \frac{Mdu}{\sqrt A} - \frac{k^2}{2c^2}\frac{\partial}{\partial M}\frac{du}{\sqrt A}$

$\int\frac{Mdu}{\sqrt A} = 2\pi$，$\int\frac{du}{\sqrt A} = \frac{2\pi}{M}$ $\frac{\partial}{\partial M}\int\frac{du}{\sqrt A} = -\frac{2\pi}{M^2}$ ⇒ $\delta\varphi = \frac{-k^2}{2c^2}\frac{-2\pi}{M^2} = \frac{\pi k^2}{M^2c^2}$

小振动：$L = \frac12 m_{ij}\dot x_i\dot x_j - \frac12 k_{ij}x_ix_j$ m_{ij}是正定的。

运动方程：$m_{ij}\ddot x_j + k_{ij}x_j = 0$ $x_j(t) = A_j\exp(i\omega t)$，假设共享同一种频率ω。

$(-m_{ij}\omega^2 + k_{ij})A_j = 0$ $A = \begin{pmatrix}A_1\\A_2\\\vdots\\A_f\end{pmatrix}$ $(k - \omega^2M)A = 0$ 有非零解，$\det(k - \omega^2M) = 0$

向量 $A^{(\alpha)}$ 共有f个，对应f个 ω_α，每个的解可写为 $\begin{pmatrix}\Delta_{i1}^{(\alpha)}\\\Delta_{i2}^{(\alpha)}\\\vdots\\\Delta_{if}^{(\alpha)}\end{pmatrix}$ 其中 Δ_{ij} 为代数余子式