理论力学A

通解: $X = \sum_\alpha A^{(\alpha)}\xi^{(\alpha)}$ $\begin{pmatrix}x_1\\x_2\\\vdots\\x_f\end{pmatrix} = \begin{pmatrix}A_1^{(1)} & \cdots & A_1^{(f)}\\A_2^{(1)} & \cdots & A_2^{(f)}\\\vdots & & \vdots\\A_f^{(1)} & \cdots & A_f^{(f)}\end{pmatrix}\begin{pmatrix}\xi^{(1)}\\\xi^{(2)}\\\vdots\\\xi^{(f)}\end{pmatrix}$ $X = \bar A\xi$ $\bar A = (A^{(1)}A^{(2)}\cdots A^{(f)})$

$\bar A$ 可以用来使 M, K 矩阵同时对角化, $\begin{pmatrix}[A^{(1)}]^T\\ [A^{(2)}]^T\\ \vdots\\ [A^{(f)}]^T\end{pmatrix} M (A^{(1)} A^{(2)} \cdots A^{(f)}) = \tilde M = \begin{pmatrix}m_1 & & 0\\ & m_2 & \\ 0 & & \ddots m_f\end{pmatrix}$

$\tilde M_{\alpha\beta} = [A^{(\alpha)}]^T M A^{(\beta)}$, 由运动方程, 有 $KA^{(\beta)} = \omega_\beta^2 MA^{(\beta)}$, $[A^{(\alpha)}]^T KA^{(\beta)} = \omega_\beta^2[A^{(\alpha)}]^T MA^{(\beta)}$

交换 α, β: $KA^{(\alpha)} = \omega_\alpha^2 MA^{(\alpha)}$, $[A^{(\beta)}]^T KA^{(\alpha)} = \omega_\alpha^2[A^{(\beta)}]^T MA^{(\alpha)}$, 取转置, 注意 K, M 是对称矩阵

$\Rightarrow [A^{(\alpha)}]^T KA^{(\beta)} = \omega_\alpha^2[A^{(\alpha)}]^T MA^{(\beta)}$, 若 $\omega_\alpha \neq \omega_\beta$, 则 $\tilde M_{\alpha\beta} = [A^{(\alpha)}]^T MA^{(\beta)}$

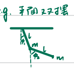
若 $\omega_\alpha = \omega_\beta$, 则有重根, 需要对 $A^{(1)} \cdots A^{(f)}$ 作施密特正交化. 对 K 同理, 且有 $k_\alpha = \omega_\alpha^2 m_\alpha$

用 ξ 来写拉格朗日函数, $L = \frac12\dot x^T M\dot x - \frac12 x^T Kx = \frac12\dot\xi^T\tilde M\dot\xi - \frac12\xi^T\tilde K\xi$ $X = \bar A\xi$

则 ξ 即为简正坐标, 定义 $Q_\alpha = \sqrt{m_\alpha}\,\xi^{(\alpha)}$, $L = \frac12\dot Q_\alpha\dot Q_\alpha - \frac12\omega_\alpha^2 Q_\alpha Q_\alpha$

$\bar A^T MX = \bar A^T M\bar A\xi = \tilde M\tilde M^{-\frac12}Q = \tilde M^{\frac12}Q$ $\Rightarrow Q_\alpha = m_\alpha^{-\frac12}A_i^{(\alpha)}m_{ij}x_j$

e.g. 平面双摆 $T = \frac12 ml^2\dot\phi_1^2 + \frac12 ml^2[(-\sin\phi_1\dot\phi_1 - \sin\phi_2\dot\phi_2)^2 + (\cos\phi_1\dot\phi_1 + \cos\phi_2\dot\phi_2)^2]$



$= ml^2\dot\phi_1^2 + \frac12 ml^2\dot\phi_2^2 + ml^2\cos(\phi_1-\phi_2)\dot\phi_1\dot\phi_2$, $V = -mgl(2\cos\phi_1 + \cos\phi_2)$

ϕ_1, ϕ_2 小量, $L = \frac12 ml^2(2\dot\phi_1^2 + \dot\phi_2^2 + 2\dot\phi_1\dot\phi_2) - \frac12 mgl(2\phi_1^2 + \phi_2^2)$

$M = ml^2\begin{pmatrix}2 & 1\\1 & 1\end{pmatrix}$, $K = mgl\begin{pmatrix}2 & 0\\0 & 1\end{pmatrix}$ $\begin{vmatrix}2\omega_0^2-2\omega^2 & -\omega^2\\-\omega^2 & \omega_0^2-\omega^2\end{vmatrix} = 0$

$\Rightarrow \omega^2 = (2\pm\sqrt2)\omega_0^2$, $\omega_0 = \sqrt{\frac gl}$

对于 ω_+, $\begin{pmatrix}-2-2\sqrt2 & -2-\sqrt2\\-2-\sqrt2 & -1-\sqrt2\end{pmatrix}A_+ = 0 \Rightarrow A_+ = \begin{pmatrix}1\\-\sqrt2\end{pmatrix}$

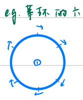
对于 ω_-, $\begin{pmatrix}2\sqrt2-2 & -2+\sqrt2\\-2+\sqrt2 & \sqrt2-1\end{pmatrix}A_- = 0 \Rightarrow A_- = \begin{pmatrix}1\\\sqrt2\end{pmatrix}$

$\bar A = \begin{pmatrix}1 & 1\\-\sqrt2 & \sqrt2\end{pmatrix}$, $\tilde M = \bar A^T M\bar A = ml^2\begin{pmatrix}1 & -\sqrt2\\1 & \sqrt2\end{pmatrix}\begin{pmatrix}2 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\-\sqrt2 & \sqrt2\end{pmatrix} = ml^2\begin{pmatrix}4-2\sqrt2 & 0\\0 & 4+2\sqrt2\end{pmatrix}$

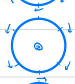
$\Rightarrow Q = \tilde M^{-\frac12}\bar A^T MX = ml^2\tilde M^{-\frac12}\begin{pmatrix}1 & -\sqrt2\\1 & \sqrt2\end{pmatrix}\begin{pmatrix}2 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}\phi_1\\\phi_2\end{pmatrix} = ml^2\tilde M^{-\frac12}\begin{pmatrix}(\sqrt2-1)(\sqrt2\phi_1 - \phi_2)\\(\sqrt2+1)(\sqrt2\phi_1 + \phi_2)\end{pmatrix}$

不要求归一化, 则简正坐标为 $\sqrt2\phi_1 - \phi_2$, $\sqrt2\phi_1 + \phi_2$

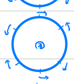
e.g. 苯环的六种简正模式:



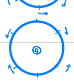
①: $\omega_1 = 0$,
$A_1 : A_2 : A_3 : A_4 : A_5 : A_6 = 1:1:1:1:1:1$



②: $T = 4\cdot\frac12 m\dot x^2$, $V = 4\cdot\frac12 kx^2$, $\omega_2 = \sqrt{\frac km}$
i) $A_1 : A_2 : A_3 : A_4 : A_5 : A_6 = 0:1:1:0:-1:-1$
ii) $A_1 : A_2 : A_3 : A_4 : A_5 : A_6 = -1:-1:0:1:1:0$
对应 2 种简正模式, 第三种由 i), ii) 相加得到



③: $T = 6\cdot\frac12 m\dot x^2$, $V = 6\cdot\frac12 k(2x)^2$, $\omega_3 = 2\sqrt{\frac km}$
$A_1 : A_2 : A_3 : A_4 : A_5 : A_6 = 1:-1:1:-1:1:-1$



④: $T = 4\cdot\frac12 m\dot x^2$, $V = 4\cdot\frac12 kx^2 + 2\cdot\frac12 k(2x)^2 = 6kx^2$, $\omega_4 = \sqrt{\frac{3k}{m}}$
i) $A_1 : A_2 : A_3 : A_4 : A_5 : A_6 = 0:1:-1:0:1:-1$
ii) $A_1 : A_2 : A_3 : A_4 : A_5 : A_6 = 1:0:1:-1:0:1$
对应 2 种简正模式, 第三种由 i), ii) 相加得到

e.g. N 个质量为 m 的质点构成的环, 求特征频率

$L = \sum_{n=1}^N \frac12 m\dot x_n^2 - \sum_{n=1}^N \frac12 k(x_n - x_{n-1})^2$, 周期性边界条件: $x_{n+N} = x_n$

$\Rightarrow \ddot x_n = \omega_0^2(x_{n+1} + x_{n-1} - 2x_n)$, 其中 $\omega_0^2 = \frac km$, $x_n = A_n e^{-i\omega t}$

$\Rightarrow -\omega^2 A_n = \omega_0^2(A_{n+1} + A_{n-1} - 2A_n)$

特征方程: $\omega_0^2 r^2 - (2\omega_0^2 - \omega^2)r + \omega_0^2 = 0$, $r_1 r_2 = 1$, $r_1 + r_2 = \frac{2\omega_0^2 - \omega^2}{\omega_0^2}$

$A_n = \alpha r_1^n + \beta r_2^n = \alpha r_1^{n+N} + \beta r_2^{n+N}$, 对任意 n 成立, $\Rightarrow r_1^N = r_2^N = 1$

$\Rightarrow r_1 + r_2 = 2\cos(\frac{2\pi l}{N})$ $\Rightarrow \omega^2 = 2\omega_0^2(1 - \cos\frac{2\pi l}{N})$

N=6 时, $\omega^2 = 2\omega_0^2(1-1,\ 1-\frac12,\ 1+\frac12,\ 1+1,\ 1+\frac12,\ 1-\frac12) = 0, \omega_0^2, 3\omega_0^2, 4\omega_0^2$ 其中 $\omega_0^2, 3\omega_0^2$ 为重根

受迫振动: $L = \frac12 m\dot x^2 - \frac12 kx^2 + xF(t)$ $m\ddot x + kx = F(t)$, $\ddot x + \omega^2 x = \frac{F(t)}{m}$

F(t) 如果也是周期性函数, $F(t) = f\cos(\gamma t + \beta)$.

通解+特解: $x = a\cos(\omega t + \alpha) + \frac{f}{m(\omega^2 - \gamma^2)}\cos(\gamma t + \beta)$

改写前一项: $x = b\cos(\omega t + \alpha) + \frac{f}{m(\omega^2 - \gamma^2)}[\cos(\gamma t + \beta) - \cos(\omega t + \beta)]$

$\xrightarrow{\gamma\to\omega}$ $b\cos(\omega t + \alpha) + \frac{f}{2m\omega}t\sin(\omega t + \beta)$ ← 共振的运动方程

对于一般的 F(t), $\ddot x + \omega^2 x = \frac{F(t)}{m}$, 令 $\xi = \dot x + i\omega x$, $\dot\xi = \ddot x + i\omega\dot x$, $\Rightarrow \dot\xi - i\omega\xi$

考虑 $a = i\omega$, $\dot\xi - i\omega\xi = \frac{F(t)}{m}$, 使得齐次解: $\xi = A(t)e^{i\omega t}$

$\Rightarrow \dot A(t) = \frac{F(t)}{m}e^{-i\omega t}$ $\Rightarrow \xi = (\int_0^t \frac{F(t)}{m}e^{-i\omega t}dt + A_0)e^{i\omega t}$ $\xi = \dot x + i\omega x$, $\xi = B(t)e^{-i\omega t}$

$\Rightarrow x(t) = e^{-i\omega t}[\int_0^t e^{2i\omega t}(A_0 + \int_0^t \frac{F(\tau)}{m}e^{-i\omega\tau}d\tau) + B(0)]$

系统能量, $E = \frac m2(\dot x^2 + \omega^2 x^2) = \frac m2\xi\xi^* = \frac m2|\xi|^2$ $|\xi(t)| = |\int_0^t \frac{F(t)}{m}e^{-i\omega t}dt + A(0)|$

阻尼振动: $m\ddot x = -kx - \alpha\dot x$, 引入 $\frac{\alpha}{m} = 2\lambda$, $\frac km = \omega_0^2$, $\ddot x + 2\lambda\dot x + \omega_0^2 x = 0$

通解: $x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, $r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega_0^2}$

$\lambda \ll \omega_0$, $E \propto e^{-2\lambda t}$ $(\frac12 m\dot x^2 + \frac12 kx^2)$, $x_0 = a\cos(\omega t + \phi)$, $E = E_0 e^{-2\lambda t}$

$\lambda > \omega_0$, 运动趋于平衡位置, 不再振动. $\lambda = \omega_0$, $x = (a_1 + a_2 t)e^{-\lambda t}$ 临界阻尼

阻尼+强迫: $\ddot x + 2\lambda\dot x + \omega_0^2 x = \frac fm\cos\gamma t = \frac fm e^{i\gamma t}$, 特解: $x(t) = Be^{i\gamma t}$

$B = \frac{f}{m(\omega_0^2 - \gamma^2 + 2i\lambda\gamma)} = be^{i\delta}$ $b = \frac{f}{m\sqrt{(\omega_0^2-\gamma^2)^2 + 4\lambda^2\gamma^2}}$, $\tan\delta = \frac{2\lambda\gamma}{\gamma^2 - \omega_0^2}$

方程的解 $x(t) = g(t)e^{-\lambda t} + b\cos(\gamma t + \delta)$, g(t) 形式由 λ 与 ω_0 大小关系决定

e.g. 天空是蓝色的? 云是白色的?

天空: $\omega_0 \gg \gamma$, $b = \frac{f}{m\omega_0^2}$, $P \propto b^2\gamma^4 \propto \gamma^4$, 高频率散射大, 瑞利散射

云: $\gamma \gg \omega_0$, $b = \frac{f}{m\gamma^2}$, $P \propto b^2\gamma^4 \propto \frac{f^2}{m^2}$, 与 γ 无关, 米氏散射

参变共振问题: $\ddot x + \omega(t)^2 x = 0$

假设 $\omega(t)^2 = \omega_0^2(1 + f(t))$, $f(t) = 4h\cos(2\gamma t)$, $0 < h \ll 1$, $\gamma = \omega_0 + \varepsilon$, $\varepsilon \ll \omega_0$

$\ddot x + \omega_0^2(1 + 4h\cos(2\gamma t))x = 0$ $x = a(t)\cos\gamma t + b(t)\sin\gamma t$

$\ddot x = (2\dot b\gamma - a\gamma^2)\cos\gamma t - (2\dot a\gamma + b\gamma^2)\sin\gamma t$

$\Rightarrow [(\omega_0^2 - \gamma^2)a + 2\dot b\gamma + 2\omega_0^2 ah]\cos\gamma t + [(\omega_0^2 - \gamma^2)b - 2\dot a\gamma - 2\omega_0^2 bh]\sin\gamma t = 0$

考虑 $a = a_0 e^{st}$, $b = b_0 e^{st}$, 则有 $\begin{cases}(\omega_0^2 - \gamma^2 + 2\omega_0^2 h)a_0 + 2\gamma s b_0 = 0\\ -2\gamma s a_0 + (\omega_0^2 - \gamma^2 - 2\omega_0^2 h)b_0 = 0\end{cases}$

$\Rightarrow (\omega_0^2 - \gamma^2)^2 - 4\omega_0^4 h^2 = -4\gamma^2 s^2$, $s^2 = \frac{4\omega_0^4 h^2}{4\gamma^2} - \frac{(\omega_0^2 - \gamma^2)^2}{4\gamma^2} \approx \omega_0^2 h^2 - \varepsilon^2$

在以 ω_0 为中心, $\varepsilon = \omega_0 h$ 范围内都会发生

e.g. 杆长为 l, 悬点在竖直方向内快速振动, $y_0 = a\cos\omega t$, y 轴正方向竖直向下.

求摆角摆动的振动情况 (小角度)

$a = \ddot y_0 = -\omega^2 a\cos\omega t$, 重力修正为 $mg + m\omega^2 a\cos\omega t = mg(1 + 4h\cos\omega t)$, $h = \frac{\omega^2 a}{4g}$

$\ddot\theta + \omega_0^2(1 + 4h\cos\omega t)\theta = 0$, $\omega_0 = \sqrt{\frac gl}$

$\theta = A_0\cos\Omega t + A_1\cos(\omega - \Omega)t + A_2\cos(\omega + \Omega)t$

$\ddot\theta = -\Omega^2 A_0\cos\Omega t - A_1(\omega - \Omega)^2\cos(\omega - \Omega)t - A_2(\omega + \Omega)^2\cos(\omega + \Omega)t$

代入运动方程, $[-\Omega^2 A_0 + \omega_0^2 A_0 + 2h\omega_0^2(A_1 + A_2)]\cos\Omega t + [(\omega_0^2 - (\omega - \Omega)^2)A_1 + 2h\omega_0^2 A_0]\cos(\omega - \Omega)t$

$+ [(\omega_0^2 - (\omega + \Omega)^2)A_2 + 2h\omega_0^2 A_0]\cos(\omega + \Omega)t = 0$, 其他频率项省略, 仅对 A 有约束

$\begin{vmatrix}\omega_0^2 - \Omega^2 & 2h\omega_0^2 & 2h\omega_0^2\\ 2h\omega_0^2 & \omega_0^2 - (\omega - \Omega)^2 & 0\\ 2h\omega_0^2 & 0 & \omega_0^2 - (\omega + \Omega)^2\end{vmatrix} = 0$ [illegible]

略去高阶, 得 $\frac12\gamma^2\omega^4 - \omega^2\Omega^2 = 0$ $h = \frac{\omega^2 a}{4g} = \frac12\gamma(\frac{\omega}{\omega_0})^2$

$\Rightarrow \Omega^2 = \omega_0^2 + \frac12\gamma^2\omega^2$, 若悬点在上方, $\omega_0^2 \to -\omega_0^2$, $\Omega^2 = \frac12\gamma^2\omega^2 - \omega_0^2$

当 $\gamma\omega > \sqrt2\omega_0$ 时, 原来可以在上方稳定做单摆运动.

快速变化场中质点的运动: $f(q,t) = h(q)\cos(\omega t + \beta)$

$m\ddot q = -\frac{dV}{dq} + f$ $q(t) = X(t) + \xi(t)$, $X(t) = \bar q(t)$, ξ(t) 是快速振动

$m\ddot X + m\ddot\xi = -\frac{dV}{dq}|_{q=X} - \xi\frac{d^2V}{dq^2}|_{q=X} + f(X,t) + \xi\frac{\partial f}{\partial q}|_{q=X}$

$m\ddot\xi = -\xi\frac{d^2V}{dq^2}|_{q=X} + f(X,t) + \xi\frac{\partial f}{\partial q}|_{q=X}$ ← 高阶小量

$m\omega_0^2 = \frac{d^2V}{dq^2}|_{q=X}$, $\ddot\xi + \omega_0^2\xi = \frac1m f(X,t) = \frac hm\cos(\omega t + \beta)$ $\Rightarrow \xi = \frac{f(X,t)}{m(\omega_0^2 - \omega^2)}$

$m\ddot X = -\frac{dV}{dq}|_{q=X} - \frac{1}{m(\omega^2 - \omega_0^2)}\overline{f\frac{\partial f}{\partial X}} = -\frac{dV_{eff}}{dX}$ $V_{eff} = V + \frac{1}{2m(\omega^2 - \omega_0^2)}\overline{f^2(X,t)}$

代入上述例子, $f(X,t) = -m\omega^2 a\cos\omega t\,\theta = -m\gamma\omega^2\cos\omega t\,\theta$

$V_{eff} = -\frac12 m\omega_0^2\theta^2 + \frac{1}{2m\omega^2}m^2\gamma^2\omega^4\theta^2\cdot\frac12$, $\Omega^2 = \frac{\gamma^2\omega^2}{2} - \omega_0^2 = \frac12\gamma^2\omega^2 - \omega_0^2$

e.g. 小球约束在旋转的马鞍面上, 平衡位置附近, $V(x,y) = \frac12 m\omega^2(x^2 - y^2)$. 不稳定平衡, 旋转使其稳定

其转化 $\begin{cases}x' = x\cos\omega t + y\sin\omega t\\ y' = x\sin\omega t + y\cos\omega t\end{cases}$ $V(x,y) = \frac12 m\omega^2[(x^2 - y^2)\cos\omega t + 2xy\sin\omega t]$

$\vec f = -m\omega^2(x\cos\omega t + y\sin\omega t,\ -y\cos\omega t + x\sin\omega t)$ $\overline{f^2} = m^2\omega^4(x^2 + y^2)$

$V_{eff} = \frac{m^2\omega^4}{2m\omega^2}(x^2 + y^2) = \frac{m\omega^2}{2}(x^2 + y^2)$ ⇒ 振动频率 ω

粒子运动方程 $\begin{cases}m\ddot x = m\omega^2 x - m\omega^2 x + 2m\omega\dot y\\ m\ddot y = -m\omega^2 y + m\omega^2 y - 2m\omega\dot x\end{cases}$ $2\omega\dot y = \ddot x - \omega^2 x + \omega^2 x$, $2\omega\dot y = \dddot x - \omega^2\dot x + \omega^2\dot x$

$\Rightarrow \ddddot x + 2\omega^2\ddot x + (\omega^4 - \omega_0^4)x = 0$, $\Rightarrow \lambda = \pm i\omega \pm i\frac{\omega_0^2}{2\omega}$

$\Rightarrow \begin{cases}x = A\cos\frac{\omega_0^2}{2\omega}t\cos\omega t\\ y = A\cos\frac{\omega_0^2}{2\omega}t\sin\omega t\end{cases}$ (慢尚有的相幅), 即圆周运动半径在以 $\frac{\omega_0^2}{2\omega}$ 振动

哈密顿力学: $L(q,\dot q,t) = L(q,\dot q(q,p,t),t) = \bar L(q,p,t)$, $p_k = \frac{\partial L}{\partial\dot q_k}$

$\begin{cases}\frac{\partial\bar L}{\partial q_k} = \frac{\partial L}{\partial q_k} + \frac{\partial L}{\partial\dot q_i}\frac{\partial\dot q_i}{\partial q_k} = \frac{\partial L}{\partial q_k} + p_i\frac{\partial\dot q_i}{\partial q_k} & \text{把 p, q 视作独立变量}\\ \frac{\partial\bar L}{\partial p_k} = \frac{\partial L}{\partial\dot q_i}\frac{\partial\dot q_i}{\partial p_k} = p_i\frac{\partial\dot q_i}{\partial p_k}\end{cases}$ $p_i\frac{\partial\dot q_i}{\partial q_k} = \frac{\partial}{\partial q_k}(p_i\dot q_i)$, $p_i\frac{\partial\dot q_i}{\partial p_k} = \frac{\partial}{\partial p_k}(p_i\dot q_i) - \dot q_k$

代入得, $\begin{cases}-\frac{\partial\bar L}{\partial q_k} = -\frac{d}{dt}(\frac{\partial L}{\partial\dot q_k}) = \frac{\partial}{\partial q_k}(p_i\dot q_i - \bar L) = -\dot p_k = \frac{\partial H}{\partial q_k}\\ \frac{\partial}{\partial p_k}(p_i\dot q_i) - \dot q_k = \frac{\partial\bar L}{\partial p_k} \Rightarrow \frac{\partial}{\partial p_k}(p_i\dot q_i - \bar L) = \dot q_k = \frac{\partial H}{\partial p_k}\end{cases}$

最小作用量原理: $\delta\int_{t_1}^{t_2}[p_i\dot q_i - H(q,p,t)]dt = 0$

$\Rightarrow \int_{t_1}^{t_2}(p_i\delta\dot q_i + \dot q_i\delta p_i - \frac{\partial H}{\partial q_i}\delta q_i - \frac{\partial H}{\partial p_i}\delta p_i)dt = 0$

$\int_{t_1}^{t_2}p_i\delta\dot q_i dt = \int_{t_1}^{t_2}p_i\frac{d}{dt}(\delta q_i)\cdot dt = p_i\delta q_i\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\dot p_i\delta q_i dt$

$\Rightarrow \int_{t_1}^{t_2}[(\dot q_i\delta p_i - \frac{\partial H}{\partial p_i}\delta p_i) - (\frac{\partial H}{\partial q_i}\delta q_i + \dot p_i\delta q_i)]dt = 0$, δq_i, δp_i 任意, 故 $\begin{cases}\frac{\partial H}{\partial q_i} = -\dot p_i\\ \frac{\partial H}{\partial p_i} = \dot q_i\end{cases}$

用正则方程得到 p_k 的定义: $\frac{\partial L}{\partial\dot q_k} = \frac{\partial}{\partial\dot q_k}[p_i\dot q_i - H(q,p(q,\dot q,t),t)] = \dot q_i\frac{\partial p_i}{\partial\dot q_k} + p_k - \frac{\partial H}{\partial p_i}\frac{\partial p_i}{\partial\dot q_k} = p_k$

勒让德变换: $dL(q,\dot q,t) = Adq + \frac{\partial L}{\partial\dot q}d\dot q + Cdt$, 令 $p = \frac{\partial L}{\partial\dot q}$, $d(p\dot q) = \dot qdp + pd\dot q$

$d(p\dot q - L(q,\dot q,t)) = \dot qdp - Adq - Cdt$

泊松括号: $\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_k}\dot q_k + \frac{\partial f}{\partial p_k}\dot p_k = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial H}{\partial q_k} = \frac{\partial f}{\partial t} + [f,H]$

$[f,g] = \frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q_k}$, 容易得到 $[q_k,q_l] = 0$, $[p_k,p_l] = 0$, $[q_k,p_l] = \delta_{kl}$

雅可比恒等式: $[f,[g,h]] + [g,[h,f]] + [h,[f,g]] = 0$

$[f,h] = (\frac{\partial f}{\partial q_k}\frac{\partial}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial}{\partial q_k})h = D_f h$, 找到 h 的二阶偏导项

$[f,[g,h]] - [g,[f,h]] = D_fD_gh - D_gD_fh = (D_fD_g - D_gD_f)h = 0$, 偏导可交换.

同理, f, g, h 二阶偏导均为 0 ⇒ 原式成立

泊松定理: 如果 f, g 是守恒量, 则 [f,g] 也是守恒量

$[f,H] = -\frac{\partial f}{\partial t}$, $[g,H] = -\frac{\partial g}{\partial t}$, $[f,[g,H]] + [g,[H,f]] + [H,[f,g]] = 0$

$-[f,\frac{\partial g}{\partial t}] + [g,\frac{\partial f}{\partial t}] + [H,[f,g]] = 0 = -\frac{\partial}{\partial t}[f,g] + [H,[f,g]]$ $\Rightarrow [[f,g],H] = -\frac{\partial}{\partial t}[f,g]$

e.g. 验证中心势场下 M_i 与 H 相容, M_i 与 $T = p^2/2m$ 相容, M_i 与 V 相容, $[M_i,M_j]$ 构成的新守恒量

$M_i = \varepsilon_{ijk}q_jp_k$, $[M_i,H] = [\varepsilon_{ijk}q_jp_k,H] = \varepsilon_{ijk}p_k\frac{\partial H}{\partial p_j} - \varepsilon_{ijk}q_j\frac{\partial H}{\partial q_k} = \varepsilon_{ijk}p_k\dot q_j + \varepsilon_{ijk}q_j\dot p_k = 0$

$[M_i,\sum p_l^2] = [\varepsilon_{ijk}q_jp_k,p_lp_l] = \varepsilon_{ijk}p_k\cdot2p_j = 2\varepsilon_{ijk}p_jp_k = 0$

$[M_i,V] = [\varepsilon_{ijk}q_jp_k,V] = -\varepsilon_{ijk}q_j\frac{\partial V}{\partial q_k} = -\frac{dV}{dr}\frac1r\varepsilon_{ijk}q_jq_k = 0$, 其中 $\frac{\partial r}{\partial q_k} = \frac1r q_k$

$[M_i,M_j] = [\varepsilon_{iuv}q_up_v,\varepsilon_{jtw}q_tp_w] = \varepsilon_{iuv}p_v\varepsilon_{jtu}q_t - \varepsilon_{iuv}q_u\varepsilon_{jvw}p_w = \varepsilon_{ijk}M_k$

此处涉及转换符号相关内容, (Levi-Civita 符号), 重要结论 $\varepsilon_{abc}\varepsilon_{def} = \begin{vmatrix}\delta_d^a & \delta_e^a & \delta_f^a\\ \delta_d^b & \delta_e^b & \delta_f^b\\ \delta_d^c & \delta_e^c & \delta_f^c\end{vmatrix}$

下面给出分析证明: i) $\varepsilon_{abc}\varepsilon_{abc} = 6$, 总共有 6 个 ε_{abc} 非零.

ii) $\varepsilon_{abc}\varepsilon_{abf} = 2\delta_f^c$, $c \neq f$ 时, 显然为 0. $c = f$ 时, 由于 c 是固定的, 只有 (a,b), (b,a) 可取, 故为 2

iii) $\varepsilon_{abc}\varepsilon_{aef} = \begin{vmatrix}\delta_e^b & \delta_f^b\\ \delta_e^c & \delta_f^c\end{vmatrix}$, 若 $b = e = \mu$, 相当于 ii) 中确定了一半的情况, 故有 $\delta_e^b\delta_f^c$

若 $b = f = \mu$, $\delta_{a\mu c}\delta_{ae\mu} = -\delta_{a\mu c}\delta_{a\mu e}$, 其中 μ 固定, a 求和, 也是 ii) 中一半的情况, 故有 $-\delta_f^b\delta_e^c$

iv) $\varepsilon_{abc}\varepsilon_{def}$ 分 $a = d$, $a = e$, $a = f$ 三类讨论, 再套用 iii) 结论, $\Rightarrow \varepsilon_{abc}\varepsilon_{def} = \begin{vmatrix}\delta_d^a & \delta_e^a & \delta_f^a\\ \delta_d^b & \delta_e^b & \delta_f^b\\ \delta_d^c & \delta_e^c & \delta_f^c\end{vmatrix}$

根据以上公式, 计算 $[M_i,M_j]$, 需注意 i, j 是固定的数, 不求和

$[M_i,M_j] = [\varepsilon_{iuv}q_up_v,\varepsilon_{jtw}q_tp_w] = \varepsilon_{iuv}p_v\varepsilon_{jtu}q_t - \varepsilon_{iuv}q_u\varepsilon_{jvw}p_w$

$\varepsilon_{iuv}\varepsilon_{jtu} = \begin{vmatrix}\delta_j^i & \delta_t^i & \delta_u^i\\ \delta_j^u & \delta_t^u & \delta_u^u\\ \delta_j^v & \delta_t^v & \delta_u^v\end{vmatrix} = \begin{vmatrix}0 & \delta_t^i & 0\\ 0 & 0 & 1\\ \delta_j^v & \delta_t^v & 0\end{vmatrix} = \delta_t^i\delta_j^v$ $[M_i,M_j] = p_jq_i - q_jp_i = \varepsilon_{kij}(q_ip_j - q_jp_i)\varepsilon_{ijk}$

$\varepsilon_{iuv}\varepsilon_{jvw} = \begin{vmatrix}\delta_j^i & \delta_v^i & \delta_w^i\\ \delta_j^u & \delta_v^u & \delta_w^u\\ \delta_j^v & \delta_v^v & \delta_w^v\end{vmatrix} = \begin{vmatrix}0 & 0 & \delta_w^i\\ \delta_j^u & 0 & \delta_w^u\\ 0 & 1 & 0\end{vmatrix} = \delta_w^i\delta_j^u$ $= \varepsilon_{ijk}M_k$

e.g. LRL 矢量 $A_i = \varepsilon_{ijk}p_jM_k - mk\frac{q_i}{r}$, 定为 $\vec D = \frac{\vec A}{\sqrt{2m|E|}}$, 验证 $[D_i,M_j] = \varepsilon_{ijk}D_k$

$[\varepsilon_{iuv}p_uM_v - \frac{mk}{r}q_i, M_j] = \varepsilon_{iuv}p_u[M_v,M_j] + \varepsilon_{iuv}M_v[p_u,M_j] - [\frac{mk}{r}q_i, M_j]$

$[M_i,M_j] = \varepsilon_{iuv}\varepsilon_{jmn}[q_up_v, q_mp_n] = \varepsilon_{iuv}\varepsilon_{jmn}(p_vq_m\delta_u^n - q_up_n\delta_m^v) = \varepsilon_{ikj}\varepsilon_{jik}p_jq_i - \varepsilon_{ijk}\varepsilon_{jki}q_jp_i$

$= q_ip_j - q_jp_i = \varepsilon_{ijk}\varepsilon_{kij}(q_ip_j - q_jp_i) = \varepsilon_{ijk}M_k$

$[p_i,M_j] = [p_i,\varepsilon_{jmn}q_mp_n] = -\varepsilon_{jmn}p_n\delta_m^i = -\varepsilon_{jik}p_k = \varepsilon_{ijk}p_k$

$[\frac{mk}{r}q_i,M_j] = [\frac{mk}{r}q_i, \varepsilon_{jmn}q_mp_n] = \frac{mk}{r}\varepsilon_{jmn}q_m\delta_n^i - \frac{mkq_i}{r^2}\frac{q_n}{r}\varepsilon_{jmn}q_m = \frac{mk}{r}\varepsilon_{jki}q_k = \varepsilon_{ijk}\frac{mk}{r}q_k$

$[\varepsilon_{iuv}p_uM_v - \frac{mk}{r}q_i, M_j] = \varepsilon_{iuv}p_u[M_v,M_j] + \varepsilon_{iuv}M_v[p_u,M_j] - [\frac{mk}{r}q_i, M_j]$