

第一章. 基本原理概述

Law 1. 牛顿: $\vec{F} = \frac{d\vec{p}}{dt}$, \vec{p} 为物体线动量.

(注: 在 S-R 中, \vec{p} 为四维 p^μ 的空间分量, \vec{F} 为四维力)

Law 2. 动量守恒: 若合外力 $\vec{F} = \vec{0}$ 则 $\dot{\vec{p}} = \vec{0}$, 线动量守恒.
(线动量, 一般为 $\vec{p} = m\vec{v}$) \downarrow
 $K^\mu = 0$.

Law 3. 角动量守恒: 选定参考点 O, 相对 O 矢径为 \vec{r} , 动量为 \vec{p}

Def. 角动量: $\vec{L} := \vec{r} \times \vec{p}$ (Att. \vec{L} 为相对某参考点)

Def. 力矩: $\vec{N} := \vec{r} \times \vec{F}$ (同上 Att.)

在相对 O 点 $\vec{N} = \vec{0}$ 时, 相对 O 点的角动量 \vec{L} 守恒

$$\vec{N} = \vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt} = \frac{d(\vec{r} \times \vec{p})}{dt} - \frac{d\vec{r}}{dt} \times \vec{p}, \vec{p} \parallel \vec{v} \text{ 因而 } \vec{N} = \frac{d\vec{L}}{dt}, \vec{N} = \vec{0} \Leftrightarrow \dot{\vec{L}} = \vec{0}$$

Extra. 参考系变换时质点系动量与角动量变化.

$$\text{定义质心: } \vec{r}_c = \frac{\sum m_i \vec{r}_i}{\sum m_i}, \text{ 质心速度 } \vec{v}_c = \frac{\sum m_i \vec{v}_i}{\sum m_i}, \text{ 总质量 } M = \sum m_i$$

1° 参考系变换时(不同速, 因为动量不需多考虑)

$$\text{总动量原: } \vec{p} = \sum m_i \vec{v}_i = \sum \vec{p}_i = M \vec{v}_c \quad (\text{相当于质心动量})$$

$$\text{总动量后: } \vec{p}' = \sum m_i (\vec{v}_i - \vec{v}_0) = M(\vec{v}_c - \vec{v}_0) \quad (\text{相当于质心的变换后动量})$$

$$\text{Especially: 质心系中 } \vec{p}_c = \sum m_i (\vec{v}_i - \vec{v}_c) = 0. \quad (\text{质心系中动量为零, 也称为动量中心系})$$

2° 参考系变换时.

总角动量: $\vec{L} = \sum m_i \vec{r}_i \times \vec{v}_i = \sum m_i \vec{r}_i \times (\vec{v}_i - \vec{v}_c) + M \vec{r}_c \times \vec{v}_c = \vec{L}_c + \vec{L}_r$ (质心角动量 + 相对质心角动量)

总角动量: $\vec{L}' = \sum m_i (\vec{r}_i + \vec{R}) \times \vec{v}_i = \vec{L} + M \vec{R} \times \vec{v}_c$ ($\vec{R} = \vec{O} \vec{O}'$) (相对于质心的相对 O' 角动量 + 相对 O)

Especially: 以质心为参考系, $\vec{L}_c = \vec{L}_c + \vec{L}_r = \vec{L}$ 且在受力时, 若相对质心的力矩和为零 (重力, 惯性力, \vec{L} 仍守恒.

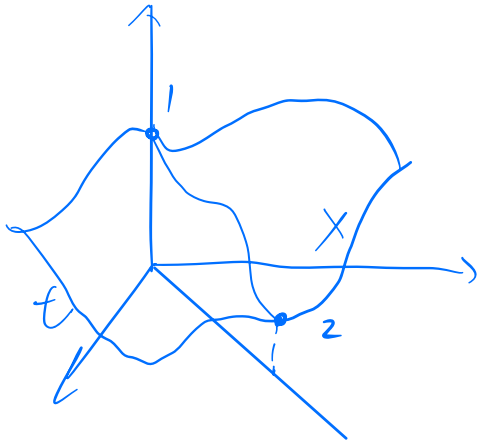
Law 4. 功能原理: 外力做功等于动能变化, 即 $\int_1^2 \vec{F} \cdot d\vec{v} = W_2 - W_1$ (S-R 中也正确)

Def. 保守场: 场力做功与路径无关

Tip: 此时场力 \vec{F} 表为 $-\nabla V(\vec{r})$.

Law 5. 机械能守恒 (由于损耗不守恒, 在 S-R 中一直成立; 若只有保守力做功, 则

$(T + V) = \text{const}$ (V 为势能, T 为动能) (After: V 与 t 无关)



过程 $1 \rightarrow 2$ 中.

$$\Delta V = \int_1^2 \left(\frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial x} dx \right) \quad (\text{第一型曲线积分})$$

At: $\int_1^2 \frac{\partial V}{\partial t} dt$ 非做功, $\int_1^2 \frac{\partial V}{\partial x} (x, t) dx$ 才做功.

$$\text{但 若将曲线表示成 } \begin{cases} x = x(s) \\ t = t(s) \end{cases} \Rightarrow \stackrel{T_2 - T_1}{W} = \int_1^2 \frac{\partial V}{\partial x} (x(s), t(s)) x'(s) ds \neq V_2 - V_1$$

因而 $T + V \neq \text{const}$

§2 质点力学

Def. 内力与外力: 质点 i 受力为 $\vec{F}_i = \vec{F}_i^{(e)} + \left[\sum_j \vec{F}_{ji} \right]$ (j 对 i 作用力, $\vec{F}_{ii} = 0$)

$$\vec{F}_i = \vec{F}_i = \vec{F}_i^{(e)} + \sum_j \vec{F}_{ji}$$

↓ ↓
外力 内力

Thm. 质心运动定理 (牛顿第二), 对任意系和, 若有牛三, 则 $\vec{F}_{ji} + \vec{F}_{ij} = 0$

$$\sum_i \vec{p}_i = \frac{d}{dt}(\sum_i \vec{p}_i) = \sum_i \vec{F}_i^{(e)} + \sum_{i,j} \vec{F}_{ji} = \sum_i \vec{F}_i^{(e)}$$

$$\sum_i \vec{p}_i = M \vec{v}_c \quad \text{记} \quad \sum_i \vec{F}_i^{(e)} = \vec{F}^{(e)} \Rightarrow \vec{F}^{(e)} = \frac{d(M \vec{v}_c)}{dt} = M \frac{d\vec{v}_c}{dt} = M \vec{a}_c \quad (\text{非 S-R})$$

Att: 例如质心不动牛三, $\sum_{ij} \vec{F}_{ij} = \vec{F}_{\text{ext}} \neq 0$

Extra Thm: 质心位置量守恒: $\vec{F}^{(e)} = 0$.

Extra Thm: 根据牛三, 有 $\vec{N}_{ij} + \vec{N}_{ji} = 0$ 因而 $\vec{N}^{(e)} = \frac{d\vec{L}}{dt}$ (相对任意系)

Att: 质心-角动量定理, 仍成立

Law: 牛三: = 物间相互作用力等值反向

角动量守恒定律不仅要求牛三, 还要求力有心

质心量守恒 只要求牛三.

但牛三并非一直成立

Thm. 能量守恒: $E_k := \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i (\vec{v}_c + \vec{v}_{ri})^2 = \sum_i \frac{1}{2} m_i (v_c^2 + v_{ri}^2) + \sum_i m_i \vec{v}_c \cdot \vec{v}_{ri} = E_{kc} + E_{kr}$

$$E_{kc} := \sum_i \frac{1}{2} m_i v_c^2 = \frac{1}{2} M v_c^2, \quad E_{kr} := \sum_i \frac{1}{2} m_i v_{ri}^2$$

若质心间作用力全为保守力, 且外力全为保守力, 则系统总势能可写作

$$V = \sum_i V_i + \frac{1}{2} \sum_{i,j} V_{ij} \quad (V_i \text{ 为第 } i \text{ 个质点在外力场中势, } V_{ij} \text{ 为 } i, j \text{ 相互作用势, } V_{ij} = V_{ij}(\vec{r}_i - \vec{r}_j) = V_{ij}(\vec{r}_{ij}))$$

此时 $E_k + V = \text{const.}$

$$W = \sum_i \int_1^2 \vec{F}_i \cdot d\vec{S}_i = \sum_i \int_1^2 (\vec{F}_i^{(e)} + \sum_{j \neq i} \vec{F}_{ji}) \cdot d\vec{S}_i = \sum_i \int_1^2 \vec{F}_i^{(e)} \cdot d\vec{S}_i + \sum_{i,j} \int_1^2 \vec{F}_{ji} \cdot d\vec{S}_i$$

若外力保守, 则 $\vec{F}_i^{(e)} = -\nabla V_i$. 若内力保守.

$$\vec{F}_{ji} = -\nabla_i V_{ij}(\vec{r}_i - \vec{r}_j) = -\nabla_{ij} V_{ij}(\vec{r}_{ij}) = \nabla_j V_{ij}(\vec{r}_i - \vec{r}_j) = \nabla_j V_{ji}(\vec{r}_j - \vec{r}_i) = -\vec{F}_{ij}$$

$$\text{则 } \vec{F}_{ji} = -\nabla_{ij} V_{ij}(\vec{r}_{ij}), \quad \int_1^2 \vec{F}_{ji} \cdot d\vec{S}_i + \vec{F}_{ij} \cdot d\vec{S}_j = \int_1^2 \vec{F}_{ji} \cdot (d\vec{S}_i - d\vec{S}_j) = \int_1^2 \vec{F}_{ji} \cdot d\vec{r}_{ij} = -\int_1^2 \nabla_{ij} V_{ij}(\vec{r}_{ij}) \cdot d\vec{r}_{ij}$$

$$\therefore \sum_{i,j} \int_1^2 \vec{F}_{ji} \cdot d\vec{S}_i = -\frac{1}{2} \sum_{i,j} V_{ij} \Big|_1^2$$

$$\text{综上, } W = -\left(\sum_i V_i + \frac{1}{2} \sum_{i,j} V_{ij} \right) \Big|_1^2, \quad \text{又有 } W = E_k \Big|_1^2 \quad \text{因而 } E_k + V = \text{const.}$$

Att. 刚体中, $\vec{r}_{ij} = \text{const.}$ 则 $d\vec{r}_{ij} \cdot \vec{r}_{ij} = 0$. 若有内力则, 仍 $\sum_{i,j} \vec{F}_{ij} \cdot d\vec{r}_{ij} = 0$. 因而无内力做功

§3. 约束

Def. 约束分类:

1° 若子表为 $f(\vec{r}_i, t) = 0$, 则称为完全约束 (再加上子约束)

Ex. 刚体: $(\vec{r}_i - \vec{r}_j)^2 = c_{ij}$ or 在曲线上的运动 约束即为用坐标表示的曲线方程

2° 非完全约束: 不能表示成如上的约束

Ex: $|\vec{r}_i| \leq a \quad \frac{d\vec{r}_i}{dt} = \frac{x}{y}$ 等.

OR 1° 可反约束: 约束方程显含时

2° 不可反约束: 不显含时的约束. (显含: 不通过坐标与时间相联)

OR 1° 双白约束: 约束方程为等式

2° 单白约束: 约束方程含不等式

Intro: 约束力学解问题带束缚

困难 I 运动方程不再全部独立, 一般系统运动方程数量也少于自由度
需加上约束方程才求解

困难 II. 一些约束力不能事先给出, 因为从人为 \rightarrow 运动 \Rightarrow 部分力 + 部分运动 \rightarrow 部分运动 + 部分力

I: 为了简化方程, 在 完全约束条件 下, 第一个困难可靠如下方法解决.

1° 因约束 $\begin{cases} f_1(\vec{r}_i, t) = 0 \\ \vdots \\ f_k(\vec{r}_i, t) = 0 \end{cases}$ 消去 k 个自由度, 以剩下 n 作为自由度

2° 引入 $3N-k$ (N 个质点) 个独立变量, 由于约束相当于给定位形空间内的 $3N-k$ 维曲面,

在一般情形下, 由 $3N-k$ 个独立变量完全描述曲面 因而可将 \vec{r}_i 表为 $(3N-k)$

$\vec{r}_i = \vec{r}_i(q_1, q_2, \dots, q_f, t)$ ($i=1, 2, \dots, N$), 这是一组将 $\{\vec{r}_i\} \rightarrow \{q_i\}$ 的一组变换方程.

且一般来讲, 是可逆的.

Att. 广义坐标的选取限制极少, 量纲随意.

若是 非完全约束,

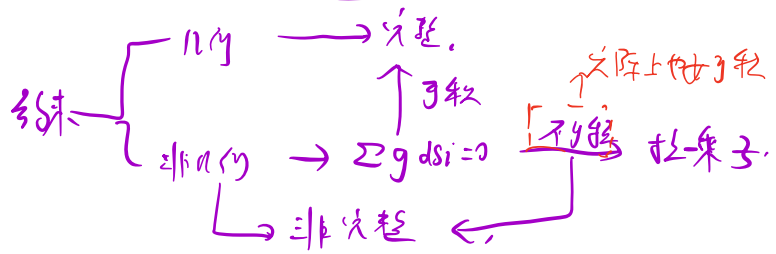
例如 $\sum g_i(\vec{r}_i) d\vec{r}_i = 0$ 的约束

① 非几何的 非完全 约束 可能是完全约束.

例如 将 x_i, y_i, z_i 表为 s_i $\sum q_i(s_i) ds_i = 0$ 的非完全约束.

若 $\exists f(\{s_i\})$, s.t. $\forall i, j \quad \frac{\partial f(s)}{\partial s_j} = \frac{\partial f(s)}{\partial s_i}$ 则 g 为 $\sum f g_i ds_i$ 看成果 $dF(s) = 0$ 即 $F(s) = 0$ 为完整约束

② y 通过 拉-乘子法 求解。 \rightarrow 条件极值 (在作用量中体现为泛函的条件极值问题)



II. 利用不同约束力的公式阐述 (需约束力作功为零)

§4. 达朗贝尔原理与拉格朗日方程.

Def. 虚位移: 是系统位形的无限小变化, 用 $\delta \vec{r}_i$ 表示, 受制于约束与力

与实位移区别在 它不改变力与约束 (相当于作用量及分点内某一点上的变分)
包含内力与一些外力 (仅技术大小的一些力)

Thm. 虚功原理: 若约束力的虚功为零 ($\sum \vec{f}_i^c \cdot \delta \vec{r}_i = 0$)
或系统保持平衡, 虚功均为零力. \rightarrow g 的自由扣除一些已知虚功为零的外力

则系统保持平衡的必要条件是外力的虚功为零. $\sum \vec{F}_i^e \cdot \delta \vec{r}_i = 0$

Tip. 实际上是所有力的虚功和为零, 因为 $\vec{F}_i^c = 0 \Rightarrow \sum \vec{F}_i^c \cdot \delta \vec{r}_i = 0 \Rightarrow \sum \vec{F}_i^e \cdot \delta \vec{r}_i + \sum \vec{f}_i^c \cdot \delta \vec{r}_i = 0$

Att. 由 $\sum \vec{F}_i^e \cdot \delta \vec{r}_i = 0 \Rightarrow \vec{F}_i^e = 0$, 因为 $\delta \vec{r}_i$ 毕竟非独立且受约束.

Thm. 达朗贝尔原理: 根据 $\vec{F}_i = \vec{p}_i \Rightarrow \sum (\vec{F}_i - \vec{p}_i) \cdot \delta \vec{r}_i = 0$.

若从中扣除约束力, 则有 $\sum (\vec{F}_i^e - \vec{p}_i) \cdot \delta \vec{r}_i = 0$.

Thm. 达朗贝尔—拉格朗日方程 (力学框架下的 E-L 方程): $T = \sum_i \frac{1}{2} m_i v_i^2$, $\vec{r}_i = \vec{r}_i(q_1, \dots, q_n, t)$, $Q_i = \sum_j \vec{F}_j \cdot \frac{\partial \vec{r}_i}{\partial \dot{q}_j}$

则有 $\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j$ (D-L 方程)

根据达朗贝尔原理, 并略去外力 (非约束力) 上的 e 标.

$$\sum_i (\vec{F}_i - \vec{f}_i) \cdot \delta \vec{r}_i = 0.$$

第一取 $\sum_i \vec{F}_i \cdot \delta \vec{r}_i$, 根据 $\delta \vec{r}_i = \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j \xrightarrow{\uparrow \delta t=0} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j = 0$

记 $Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$, 则有 $Q_j \delta q_j = \sum_i \vec{F}_i \cdot \delta \vec{r}_i$. (Q_j 是广义力, 但 $Q_j \delta q_j$ 是广义功)

第二取 $\sum_i \vec{f}_i \cdot \delta \vec{r}_i = \sum_i m_i \ddot{\vec{r}}_i \cdot \left(\sum_j \frac{\partial \vec{r}_i}{\partial \dot{q}_j} \delta \dot{q}_j \right) = \sum_j \left(\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial \dot{q}_j} \right) \delta \dot{q}_j$

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial \dot{q}_j} = \sum_i \frac{d}{dt} \left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial \dot{q}_j} \right) - m_i \dot{\vec{r}}_i \cdot \frac{d}{dt} \left(\frac{\partial \vec{r}_i}{\partial \dot{q}_j} \right)$$

$$\frac{d}{dt} \left(\frac{\partial \vec{r}_i}{\partial \dot{q}_j} \right) = \frac{\partial \vec{r}_i}{\partial q_k \partial \dot{q}_j} \dot{q}_k + \frac{\partial \vec{r}_i}{\partial t \partial \dot{q}_j}$$

$$\Rightarrow \frac{\partial \left(\frac{\partial \vec{r}_i}{\partial \dot{q}_j} \right)}{\partial q_j} = \frac{\partial \left(\frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \vec{r}_i}{\partial t} \right)}{\partial q_j} = \frac{d}{dt} \left(\frac{\partial \vec{r}_i}{\partial \dot{q}_j} \right)$$

于是 $-m_i \ddot{\vec{r}}_i \cdot \frac{d}{dt} \left(\frac{\partial \vec{r}_i}{\partial \dot{q}_j} \right) = -m_i \ddot{\vec{r}}_i \cdot \frac{\partial \left(\frac{\partial \vec{r}_i}{\partial \dot{q}_j} \right)}{\partial q_j} = - \frac{\partial \left(\frac{1}{2} m_i v_i^2 \right)}{\partial q_j}$

$$\Rightarrow \frac{\partial \vec{r}_i}{\partial \dot{q}_j} = \frac{\partial \left(\frac{\partial \vec{r}_i}{\partial \dot{q}_k} \dot{q}_k + \frac{\partial \vec{r}_i}{\partial t} \right)}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial \dot{q}_j}$$

于是 $m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial \dot{q}_j} = \frac{\partial \left(\frac{1}{2} m_i v_i^2 \right)}{\partial q_j}$

$$\sum_i \vec{f}_i \cdot \delta \vec{r}_i = \sum_j \left(\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right) \delta q_j$$

代入一、二式有 $\sum_j \left(\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j \right) \delta q_j = 0$. 此时 δq_j 独立, 因而每项系数为 0

即得 D-L 方程 $\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j$

杨氏

Tip. 若 $\vec{F}_i = -\nabla_i V(\vec{r}_1, \dots, \vec{r}_k, \dots)$ 则 $Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i -\frac{\partial V}{\partial \vec{r}_i} \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j}$

而 \vec{r}_i 只与 q_k 有关, 则 V 与 q_k 无关, $\frac{\partial V}{\partial q_k} = 0$

因而原式可写作 $\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_j}) - \frac{\partial L}{\partial q_j} = 0$, $L = T - V$ (在这里, 力学问题中有如此开式)

即为 E-L 方程, 且 L 中任意加入某个 $F(q, t)$ 对时间全微分 $\frac{dF(q, t)}{dt}$, 不影响方程

$$\left. \begin{aligned} \frac{dF(q, t)}{dt} &= \frac{\partial F}{\partial q_i} \dot{q}_i + \frac{\partial F}{\partial t}, & \frac{d}{dt} \left(\frac{\partial F}{\partial \dot{q}_j} \right) &= \frac{d}{dt} \left(\frac{\partial F}{\partial \dot{q}_j} \right) = \frac{\partial^2 F}{\partial q_j \partial q_i} \dot{q}_i + \frac{\partial^2 F}{\partial t \partial q_j} \end{aligned} \right\} \text{相消, 相成为零.}$$

$$\frac{\partial (\frac{dF}{dt})}{\partial \dot{q}_j} = \frac{\partial^2 F}{\partial q_i \partial q_j} \dot{q}_i + \frac{\partial^2 F}{\partial t \partial q_j}$$

§5. 与速度相关的势和耗散函数

Intro(I): 处理技巧: 倘若势与速度有关, 但若 Q_j 表为如下开式

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt} \left(\frac{\partial U}{\partial \dot{q}_j} \right) \quad \text{则仍有 E-L 方程成立且 } L = T - U$$

此处的 U 称为 **广义势**.

Ex. 求电磁场的广义势

不妨选取 x, y, z 作为广义坐标.

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}). \quad \text{代入 } \vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla \phi, \quad \vec{B} = \nabla \times \vec{A}$$

$$\vec{F} = q \left(-\frac{\partial \vec{A}}{\partial t} - \nabla \phi + \vec{v} \times (\nabla \times \vec{A}) \right)$$

$$F_x = q \left[-\frac{\partial A_x}{\partial t} - \frac{\partial \phi}{\partial x} + (\partial_x A_y - \partial_y A_x) v_y - (\partial_z A_x - \partial_x A_z) v_z \right]$$

$$= q \left[-\frac{\partial}{\partial x} (\phi - A_y v_y - A_z v_z) - \frac{\partial A_x}{\partial t} - v_y \partial_y A_x - v_z \partial_z A_x \right]$$

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + (\nabla A_x) \cdot \vec{v} \Rightarrow -\frac{\partial A_x}{\partial t} - v_y \partial_y A_x - v_z \partial_z A_x = -\frac{dA_x}{dt} + v_x \partial_x A_x$$

$$\Rightarrow F_x = q \left[-\frac{\partial}{\partial x} (\phi - \vec{A} \cdot \vec{v}) - \frac{dA_x}{dt} \right] = q \left[-\frac{\partial}{\partial x} (\phi - \vec{A} \cdot \vec{v}) + \frac{d}{dt} \left(\frac{\partial (\phi - \vec{A} \cdot \vec{v})}{\partial v_x} \right) \right]$$

$$\therefore F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt} \left(\frac{\partial U}{\partial v_x} \right), \quad U = q\phi - q\vec{A} \cdot \vec{v}$$

Thm. 电磁场中带电粒子的拉氏函数: $\mathcal{L} = T - U = T - q\phi + q\vec{A} \cdot \vec{v}$

Intro (II): 我们将 \mathcal{L} 表为 $T - V \rightarrow V$ 为保守势

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j} \right) - \frac{\partial \mathcal{L}}{\partial q_j} = Q_j \rightarrow \text{非保守力的广义力}$$

Ex. 库仑耗散力

仍选取 x, y, z 为广义坐标, 且让库仑力 $\propto v$.

$$f_{ix} = -k_x v_x, \quad f_{iy} = -k_y v_y, \quad f_{iz} = -k_z v_z.$$

$$\text{定义 } \mathcal{F} = \frac{1}{2} \sum_i (k_x v_{ix}^2 + k_y v_{iy}^2 + k_z v_{iz}^2)$$

$$\vec{f} = -\nabla \mathcal{F}, \quad dW_f = \sum_i \vec{f}_i \cdot d\vec{r}_i = -\sum_i \vec{f}_i \cdot \vec{v}_i dt = -\sum_i (k_x v_{ix}^2 + k_y v_{iy}^2 + k_z v_{iz}^2) dt = -2\mathcal{F} dt.$$

$$\text{于是能量耗散率为 } \frac{dW_f}{dt} = -2\mathcal{F}$$

$$\text{对于广义力, } Q_j = \sum_i \vec{f}_i \cdot \frac{\partial \vec{r}_i}{\partial \dot{q}_j} = \sum_i \vec{f}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j} = -\sum_i (k_x v_{ix} \frac{\partial v_{ix}}{\partial \dot{q}_j} + k_y v_{iy} \frac{\partial v_{iy}}{\partial \dot{q}_j} + k_z v_{iz} \frac{\partial v_{iz}}{\partial \dot{q}_j}) = -\frac{\partial \mathcal{F}}{\partial \dot{q}_j}$$

$$\text{此时有 } \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j} \right) - \frac{\partial \mathcal{L}}{\partial q_j} + \frac{\partial \mathcal{F}}{\partial \dot{q}_j} = 0$$

Att. 此时若写本 EOM, 则需得到 $\mathcal{L}(q, \dot{q}, t)$ 与 $\mathcal{F}(q, \dot{q}, t)$

§6. 拉格朗日表述的简单应用

Thm 1. 动能函数 T 的性质: $T = T_0 + T_1 + T_2$.

T_0 与 \dot{q} 无关, T_1 是 \dot{q} 线性函数, T_2 是 \dot{q} 的二次项

且若 $\vec{r} = \vec{r}(q)$ (即坐标反演不含时间) 则 $T_0 = T_1 = 0$, T 是 \dot{q} 的二次齐次函数

$$T = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2} \sum_i m_i \left(\frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \vec{r}_i}{\partial t} \right)^2 = \frac{1}{2} \sum_i m_i \left(\frac{\partial \vec{r}_i}{\partial t} \right)^2 + \sum_j \left(\sum_i m_i \frac{\partial \vec{r}_i}{\partial t} \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right) \dot{q}_j + \frac{1}{2} \sum_{jk} \left(\sum_i m_i \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{\partial \vec{r}_i}{\partial q_k} \right) \dot{q}_j \dot{q}_k$$

$$\text{记为 } T = M_0 + \sum_j M_j \dot{q}_j + \frac{1}{2} \sum_{jk} M_{jk} \dot{q}_j \dot{q}_k$$

$$M_0 = \frac{1}{2} \sum_i m_i \left(\frac{\partial \vec{r}_i}{\partial t} \right)^2, \quad M_j = \sum_i m_i \left(\frac{\partial \vec{r}_i}{\partial t} \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right), \quad M_{jk} = \sum_i m_i \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{\partial \vec{r}_i}{\partial q_k} \quad (M_{jk} = M_{kj})$$

而 $\vec{r} = \vec{r}(q, t)$, 则 M_0, M_j, M_{jk} 均与 q 无关, 因而有 $T = T_0 + T_1 + T_2$.

$$\text{若 } \frac{\partial \vec{r}}{\partial t} = 0 \text{ 则 } T_0 = T_1 = 0, \quad T = T_2 = \frac{1}{2} \sum_{ij} M_{ij} \dot{q}_i \dot{q}_j$$