



## ch 1. Tensor analysis

The physical quantity.

↓  
Geometrical.

scalar, vector, tensor

↓      ↳ magnitude, and direction

physical q. having only magnitude, no direction.

\* Def. Linear space

eg. P. scalar in 3D space.

↓  
not scalar in 4D Minkowski spaceConsider  $V$  of  $d$ -dim. characterised by  $d$  orthogonal bases $\hat{e}_1, \dots, \hat{e}_d$ 

Describe

$$\vec{A} = \sum_{i=1}^d A^i \hat{e}_i \quad A^i: \text{the } i^{\text{th}} \text{ component of the}$$

Contravariant vector

↳ 逆变矢量

$$\vec{r} = x \hat{e}_x + y \hat{e}_y + z \hat{e}_z \quad \left\{ \begin{array}{l} r^1 = x \\ r^2 = y \\ r^3 = z \end{array} \right.$$

 $\Rightarrow$  Tensor    Bases:  $\hat{e}_i \hat{e}_j, \dots, \hat{e}_1 \hat{e}_d$ 

$$\vdots \\ \hat{e}_d \hat{e}_1, \dots, \hat{e}_d \hat{e}_d$$

$$\Rightarrow \hat{T}^{(2,0)} = \sum_{ij} T^{ij} \hat{e}_i \hat{e}_j$$

Generally: Rank- $n$ .

$$\hat{T}^{(n,0)} = \sum T^{i_1 i_2 \dots i_n} \hat{e}_{i_1} \dots \hat{e}_{i_n}$$

Dual Space of  $V \rightarrow V^*$

The bases  $\hat{e}^1, \dots, \hat{e}^d$

$$\vec{A} = \sum A_i \hat{e}^i$$

Tensor  $T^{(n,m)} = \sum T_{i_1 \dots i_m} \hat{e}^{i_1} \dots \hat{e}^{i_m}$

Generic rank  $(n,m)$  Tensor

$$T^{(n,m)} = \sum T^{i_1 i_2 \dots i_n}_{j_1 \dots j_m} \hat{e}^{i_1} \dots \hat{e}^{i_n} \hat{e}_{j_1} \dots \hat{e}_{j_m}$$

Eg.

Kronecker notation

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{else} \end{cases}$$

Levi-Civita notation (in 3D)

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is even permutation of } (1,2,3) \\ -1 & \text{if } (i,j,k) \text{ is odd permutation of } (1,2,3) \\ 0 & \text{otherwise} \end{cases}$$

(For  $g_{\mu\nu} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$ , only in 3D Euclidean space)

1)  $\epsilon_{ijk} = -\epsilon_{jik} = \epsilon^{ijk}$

2)  $\sum_{\text{cyclic}} \epsilon_{ijk} \epsilon_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}$

Metric Tensor  $\rightarrow$  Fundamental tensor  $g^{\mu\nu}, g_{\mu\nu}$

Tensors in  $V$  and  $V^*$  are connected by  $g^{\mu\nu}, g_{\mu\nu}$

$$\begin{cases} A^\mu = \sum g^{\mu\nu} A_\nu \\ A_\mu = \sum g_{\mu\nu} A^\nu \end{cases} \quad T^{\mu\nu} = \sum g^{\mu\rho} g^{\nu\sigma} T_{\rho\sigma}$$

eg. In the 3D space with Cartesian coordinates  $g_{ij} = \delta_{ij}$

4D Minkowski space  $g_{\mu\nu} = \eta_{\mu\nu} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$   $\det(g_{\mu\nu}) = -1$

Classical:  $g_{\mu\nu} = g_{\nu\mu}$

In quantum,  $[x^\mu, x^\nu] \neq 0 \Rightarrow x^\mu = i\hbar \partial_\mu + \mathcal{A}^\mu(\vec{k})$   
 $\Downarrow$   
 $[dx^\mu, dx^\nu] \neq 0$

Algebra

1) Scalar product

$$\vec{A} \cdot \vec{A} = \sum g^{\mu\nu} A_\mu A_\nu$$

2) Tensor product  $T = \vec{A} \otimes \vec{B}$

$$T^{\mu\nu} = A^\mu B^\nu \quad / \quad T_{\mu\nu} = A_\mu B_\nu \quad \{(2,0) \text{ or } (0,2) \text{ Tensor}\}$$

Examples 1) vector product in 3D

$$\vec{A} \times \vec{B} = \sum \epsilon_{ijk} A^j B^k \hat{e}_i$$

wedge.

\ actual def \  $\vec{A} \wedge \vec{B} = \sum A_i B_j \underbrace{\hat{e}_i \wedge \hat{e}_j}_{\substack{\text{---} \\ T_{ij}}}$

$$\Rightarrow \begin{cases} T_{ij} = -T_{ji} \\ T_{ij} = A_i B_j - A_j B_i \end{cases} \Rightarrow \text{only 3 Independent} \Rightarrow \text{Redef } C_i = \frac{1}{2} \sum_{j,k} \epsilon_{ijk} T^{jk}$$

$$\begin{cases} (\vec{A} \otimes \vec{B}) \cdot \vec{C} = \vec{A} (\vec{B} \cdot \vec{C}) \\ \vec{C} \cdot (\vec{A} \otimes \vec{B}) = (\vec{C} \cdot \vec{A}) \vec{B} \end{cases}$$

# Differential Calculations $\rightarrow$ Tensor fields

## Operation of geometric operators on the tensor fields

### 1) del-operator

$$a) \text{ 3D } \quad \nabla = \frac{\partial}{\partial x} \hat{e}^1 + \frac{\partial}{\partial y} \hat{e}^2 + \frac{\partial}{\partial z} \hat{e}^3 = \sum e^i \frac{\partial}{\partial x^i}$$

$$b) \text{ 4D } \quad \partial_\mu \rightarrow \left( \frac{1}{c} \frac{\partial}{\partial t}, \nabla \right) \leftarrow \frac{\partial}{\partial x^\mu}$$

$$\partial^\mu = \eta^{\mu\nu} \partial_\nu = \left( -\frac{1}{c} \frac{\partial}{\partial t}, \nabla \right)$$

$$\text{gradient of } a(x^\mu) \quad \text{In 3D. } \vec{\nabla} a = \sum_i e^i A_i, \quad A_i = \partial_i a$$

$$\text{4D. } \vec{\nabla} a = \sum_i e^i A_i, \quad A_\mu = \partial_\mu a. \quad (A^0 = -A_0)$$

$$\text{divergence } \nabla \cdot \vec{A} = \sum \frac{\partial}{\partial x^i} A^i$$

$$\text{The Laplace } \nabla^2 a = \nabla \cdot (\nabla a)$$

$$\text{The d'Alembert } \text{4D. } \sum_\mu \partial^\mu (\partial_\mu a) = \left[ -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \nabla^2 \right] a = \square a$$

$$\square = -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \nabla^2$$

$$\text{The curl. } \nabla \times \vec{A} = \sum \epsilon_i^{jk} \partial_j A_k \hat{e}^i$$

$$\text{4D: } \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\nabla \times (\nabla a) = 0$$

for smooth  $a(x^\mu)$  field

$$\nabla \cdot (\nabla \times \vec{A}) = 0$$

{ In 4D:  $\neq 0$  }

$$\{\text{why 4D } \neq 0? \quad \epsilon^{ijk} \text{ not } \epsilon^{\mu\nu\rho\sigma}\}$$

Einstein Sum: 省略  $\sum$  号

eg.

$$f(r) = |\vec{r} - \vec{r}'|$$

$$\nabla f = \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|}$$

$$f = \frac{1}{|\vec{r} - \vec{r}'|}$$

$$\nabla f = - \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}$$

$$\nabla^2 f = -4\pi \delta^3(\vec{r} - \vec{r}')$$

$$4D. \quad \varepsilon_{\mu\nu\rho\sigma} = \begin{cases} +1, & \text{if } (\mu\nu\rho\sigma) \text{ is even permutation of } (0123), \\ -1, & \text{--- odd ---} \\ 0, & \text{otherwise} \end{cases}$$

$$\varepsilon^{\mu\nu\rho\sigma} = \begin{cases} -1, & \text{even} \\ +1, & \text{odd} \\ 0, & \end{cases}$$

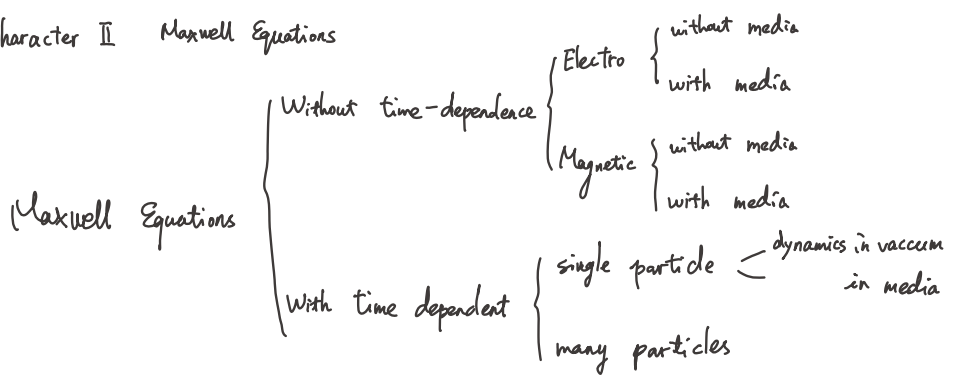
$$\text{More generally, } \varepsilon_{i_1 \dots i_n} = \begin{cases} +1, & \text{even} \\ -1, & \text{odd} \\ 0, & \end{cases}$$

identity.

$$\varepsilon_{i_1 \dots i_n} \varepsilon_{j_1 \dots j_n} = \begin{vmatrix} \delta_{i_1 j_1} & \dots & \delta_{i_1 j_n} \\ \vdots & & \vdots \\ \delta_{i_n j_1} & \dots & \delta_{i_n j_n} \end{vmatrix}$$

$$(\det g = \pm 1).$$

## Character II Maxwell Equations



Eqn of  $\vec{E}$ ,  $\vec{B}$  fields

$$\begin{cases} \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} & \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 & \nabla \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j} \end{cases}$$

$\epsilon_0$  permittivity ;  $\mu_0$  permeability.

(Book. : "" indicates using SI)

The Eqn of motion for  $\vec{E}$  field.

$$\nabla \times (\nabla \times \vec{E}) + \frac{\partial}{\partial t} (\nabla \times \vec{B}) = 0$$

$\Downarrow$

$$\nabla \left( \frac{\rho}{\epsilon_0} \right) - \nabla^2 \vec{E}$$

$$\Rightarrow -\nabla^2 \vec{E} + \nabla \left( \frac{\rho}{\epsilon_0} \right) + \frac{\partial}{\partial t} (\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{j}) = 0$$

$$\Rightarrow \nabla^2 \vec{E} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \nabla \left( \frac{\rho}{\epsilon_0} \right) + \mu_0 \frac{\partial \vec{j}}{\partial t}$$

If vacuum without  $\rho$ ,  $\vec{j}$ .  $\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0$

Wave eq.  $\vec{E} = \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$  (similar for  $\vec{B}$ )

Dually,  $\vec{E} \rightarrow \vec{B}$  ;  $\vec{B} \rightarrow -\mu_0 \epsilon_0 \vec{E}$  , the Eqs are invariant  
( in vacuum  $\rho = \vec{j} = 0$  )

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S.I units  $\rightarrow$  Gaussian Units (c.g.s).

$$[\vec{E}] = [\vec{B}] \left\{ \begin{array}{l} \nabla \cdot \vec{E} = 4\pi \rho \quad , \quad \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0 \quad , \quad \nabla \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c} \vec{j} \end{array} \right.$$

Take  $\overset{\text{SI}}{\vec{E}'} = \alpha \vec{E}$  ,  $\vec{B}' = \beta \vec{B}$  ,  $\rho' = \gamma \rho$  ,  $\vec{j}' = \delta \vec{j}$

{ Find out  $\alpha, \beta, \gamma, \delta$  ? }

$$\left\{ \begin{array}{l} \text{Maxwell Eqn.} \\ \text{Coulomb Interaction} \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \gamma = 4\pi \epsilon_0 \alpha \\ \beta = \alpha/c \\ \delta = \frac{4\pi \alpha}{c^2} \cdot \frac{1}{\mu_0} \end{array} \right.$$

$\Downarrow$

$$\gamma = \sqrt{4\pi \epsilon_0}.$$

$$\delta = \gamma = \sqrt{4\pi \epsilon_0} \quad ; \quad \alpha = \frac{1}{\sqrt{4\pi \epsilon_0}} \quad ; \quad \beta = \sqrt{\frac{\mu_0}{4\pi}}$$


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Maxwell Eqn in Gaussian Units

$$\nabla \cdot \vec{E} = 4\pi \rho \quad , \quad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0 \quad , \quad \nabla \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c} \vec{j}$$



# Macroscopic media

$$\rho \rightarrow \rho_t = \rho_f + \rho_b$$

$$\vec{J} \rightarrow \vec{J}_t = \vec{J}_f + \vec{J}_{me} \Rightarrow \text{effect of media}$$

$\vec{P}$ : electric polarization

$\vec{M}$ : magnetization

$$\left\{ \begin{array}{ll} \vec{D} & \text{electric displacement} \\ \vec{H} & \text{magnetic field} \end{array} \right. \quad \left\{ \begin{array}{ll} \vec{E} & \text{electric field} \\ \vec{B} & \text{magnetic induction} \end{array} \right.$$

relation

~~$$\vec{D} = \vec{E} + 4\pi \vec{P}$$~~
~~$$\vec{H} = \vec{B} - 4\pi \vec{M}$$~~

$$\left\{ \begin{array}{ll} \nabla \cdot \vec{D} = 4\pi \rho_f & \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0 & \nabla \times \vec{H} - \frac{1}{c} \frac{\partial \vec{D}}{\partial t} = \frac{4\pi}{c} \vec{J}_f \end{array} \right.$$

another set of relation

$$D' = \sqrt{\frac{\epsilon_0}{4\pi}} D, \quad P' = \sqrt{4\pi \epsilon_0} P$$

$$H' = \frac{1}{\sqrt{4\pi \mu_0}} H, \quad M' = \sqrt{\frac{4\pi}{\mu_0}} M$$

$$\nabla \cdot \vec{P} = -\rho_b, \quad \nabla \times \vec{M} = \frac{1}{c} (\vec{J}_t - \vec{J}_f - \vec{J}_b) = \frac{1}{c} \vec{J}_m$$

分子电流 molecular current density  $\vec{J}_m$

$$\vec{J}_b = \frac{\partial \vec{P}}{\partial t}$$

Linear media & isotropic media

The relation between  $(\vec{D}, \vec{P})$  and  $\vec{E}$  is linear, similar for  $(\vec{H}, \vec{M})$

1) Constant susceptibilities

$$\vec{D} = \epsilon \vec{E}, \quad \vec{B} = \mu \vec{H}$$

$$\text{if, } \rho_f = \vec{J}_f = 0$$

$$\Rightarrow \nabla^2 \vec{E} - \frac{\mu\epsilon}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0.$$

$$\left\{ \begin{array}{l} \nabla^2 \vec{B} - \frac{\mu\epsilon}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = 0. \end{array} \right.$$

$$\Rightarrow v_p = \frac{c}{\sqrt{\mu\epsilon}} = \frac{c}{\underline{n}}.$$

$n = \sqrt{\mu\epsilon}$  the refractive index

2) Dispersive media

$$\vec{P}(t) = \int_{-\infty}^{+\infty} \chi(t-t') \cdot \vec{E}(t') dt'$$

Similarly,

$$\vec{M}(t) = \int_{-\infty}^t \chi_m(t-t') \vec{H}(t') dt'$$

$$\chi_{e,m} \begin{cases} \neq 0, & \text{for } t' \leq t \\ = 0, & \text{for } t' > t \end{cases}$$

Fourier trans to frequency domain

$$\vec{P} = \int \vec{P}(t) e^{i\omega t} dt$$

$$\vec{P}(t) = \frac{1}{2\pi} \int \vec{P}(\omega) e^{-i\omega t} d\omega$$

$$\Rightarrow \vec{P}(\omega) = \int dt \int dt' \chi(t-t') \vec{E} e^{-i\omega t}$$

$$\int dt \chi(t-t') e^{i\omega(t-t')} = \chi(\omega)$$

$$\Rightarrow \int dt' \chi(\omega) \vec{E}(t') e^{-i\omega t'} = \chi(\omega) \vec{E}(\omega)$$

$$\text{So, } \vec{P}(\omega) = \chi(\omega) \vec{E}(\omega),$$

$$\vec{M}(\omega) = \chi_m(\omega) \vec{H}(\omega).$$

We get that,

$$\vec{D} = \vec{E} + 4\pi \vec{P}$$

$$\vec{D}(\omega) = \underbrace{[1 + 4\pi \chi(\omega)]}_{\rightarrow \epsilon(\omega) = 1 + 4\pi \chi(\omega)} \vec{E}(\omega)$$

$$\rightarrow \epsilon(\omega) = 1 + 4\pi \chi(\omega).$$

$$\text{Also, } \mu(\omega) = 1 + 4\pi \chi_m(\omega)$$

$$\text{then } n = \sqrt{\mu(\omega) \epsilon(\omega)}, \quad v_p = \frac{c}{n(\omega)}$$

$$\text{Dispersion relation } k = \frac{\omega}{v_p} = \frac{\omega}{c} n(\omega)$$

$$\nabla^2 E - \frac{n^2(\omega)}{c^2} \frac{\partial^2 E}{\partial t^2} = 0$$

$$\vec{E}(\vec{r}, t) = \sum_{\vec{k}} \vec{E}_{\vec{k}} e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

assume a small range of  $\vec{k} \in [k_0 - \delta k, k_0 + \delta k]$

$$\vec{E}(\vec{r}, t) = \sum_{\delta k} \vec{E}_{\delta k} e^{i(\delta \vec{k} \cdot \vec{r} - \omega t)} e^{i(\vec{k}_0 \cdot \vec{r} - \omega t)} = \vec{\tilde{E}}(\vec{r}, t) e^{i(\vec{k}_0 \cdot \vec{r} - \omega t)}$$

$$\vec{E}(\vec{r}, t) = \sum_{\vec{k}} \vec{E}_{\vec{k}} e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$v_p = \frac{\omega_0}{k} = \frac{c}{n(\omega_0)}$$

$$v_g = \left. \frac{\partial \omega}{\partial k} \right|_{dk \rightarrow 0} \longrightarrow \left. \frac{d\omega}{dk} \right|_{k=k_0} \longrightarrow \left( \frac{n}{c} + \frac{\omega}{c} \frac{dn}{d\omega} \right)^{-1}$$

If  $\frac{dn}{d\omega} > 0$ ,  $v_g < v_p$  normal disp

if  $n + \omega \frac{dn}{d\omega} > 1$ ,  $v_g < c$

$\gg 1$ ,  $\ll c \Rightarrow$  slow light.

strongly dispersive media

$$E_p \nabla \cdot \vec{r}$$

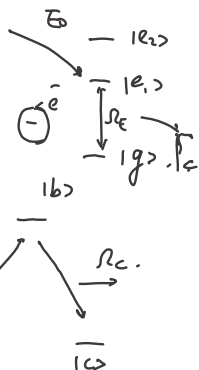
harmonic oscillators  $\hookleftarrow$

透明

Electromagnetically induced transparency (E.I.T)

Large dispersion, no dissipation.

Rabi Frequency for  $E_p$ .



a stat. in form

$$|\psi_D\rangle = \frac{\Omega_c}{\Omega} |b\rangle - \frac{\Omega_p}{\Omega} |c\rangle$$

$$\Omega = \sqrt{|\Omega_p|^2 + |\Omega_c|^2}$$

$\Rightarrow |4_b\rangle$ ; "dark-state".

$$\Omega_p, \Omega_b: \langle 4_b | 4_a \rangle = 0.$$

a small detuning  $\delta = \omega_{bb} - \omega_p$

$$\Rightarrow |\tilde{4}_b\rangle \rightarrow \frac{\delta}{\Omega} |a\rangle, \quad |\tilde{4}_b\rangle = \frac{\Omega_c}{\Omega} |b\rangle - \frac{\Omega_p}{\Omega} |c\rangle - 2 \frac{\Omega_p \Omega_c \delta}{\Omega^2} |a\rangle$$

The polarization of a single atom

$$p_c = e C a C_b \delta_c = - \frac{2 \Omega_c^2 \Omega_p \delta}{\Omega^4} e \delta_c$$

for  $N_A$ ,

$$P = - \frac{2 N_A \Omega_c^2 \Omega_p \delta}{\Omega^4} e \delta_c \quad \text{if } |\Omega_c| \gg |\Omega_p|, \Omega_c = \Omega.$$

then

$$p = \frac{2 \Omega_p}{\Omega} e \delta_c (\omega_p - \omega_{bb}) N_A$$

$$\text{from QM} \quad \Omega_p = \frac{1}{2} \frac{\vec{d} \cdot \vec{E}_p}{\hbar} = \frac{1}{2} \frac{e d_c E_p}{\hbar}$$

$$\Rightarrow p = \frac{2 e^2 d_c^2 E_p}{\hbar \Omega_c^2} (\omega_p - \omega_{bb}) N_A = \epsilon_0 \chi(\omega) E_p$$

$$\therefore \chi(\omega) = \frac{e^2 d_c^2}{\hbar \Omega_c^2 \epsilon_0} N_A (\omega_p - \omega_{bb}) \quad \{ \omega_p \rightarrow \omega_{bb}, \chi \rightarrow 0 \}$$

For  $\chi \ll 1$ ,

$$n = 1 + \frac{1}{2} \chi(\omega), \quad v_g = \frac{c}{1 + \frac{1}{2} \frac{d\chi}{d\omega}}$$

$$\Rightarrow v_g = \frac{c}{1 + \frac{\omega}{2} \frac{e^2 \epsilon_c^2}{\hbar \epsilon_c^2 \epsilon_0} N_A} = \frac{c}{1 + g^2 \frac{N_A}{\epsilon_c^2}},$$

$$g = \sqrt{\frac{\omega}{2\hbar \epsilon_0}} e \epsilon_c$$

if  $g^2 N_A \gg \epsilon_c^2$ .  $v_g \ll c$ .  $\Rightarrow$  slow light

if  $\omega \frac{dn}{d\omega} < 0$ ,  $v_g > c$ . superluminal.

Kramer - Kronig relation

$$n = n_1 + i n_2, \quad \epsilon(\omega) = \epsilon_1 + i \epsilon_2, \quad \chi = \chi_1 + i \chi_2$$

$$\epsilon(\omega) = 1 + 4\pi \chi(\omega)$$

$$= 1 + 4\pi \int_0^{+\infty} \chi(\tau) e^{i\omega\tau} d\tau$$

$$= 1 + 4\pi \int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{\epsilon(\omega') - 1}{4\pi} e^{-i\omega'\tau + i\omega\tau} d\omega' d\tau = 1 + \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega' \int_0^{+\infty} d\tau (\epsilon - 1) e^{-i(\omega' - \omega)\tau}$$

assume that the integral is analytic in the upper plane

take  $\omega \rightarrow \omega + i\delta$

$$\Rightarrow \epsilon(\omega) = 1 + \frac{1}{2\pi} \int d\omega' \int d\tau [\epsilon(\omega') - 1] e^{-i(\omega' - \omega)\tau}$$

$$= 1 + \frac{1}{2\pi} \int d\omega' \frac{\epsilon(\omega') - 1}{i(\omega' - \omega - i\delta')} = 1 + \frac{1}{2\pi i} \oint_{\text{upper}} \frac{\epsilon(\omega') - 1}{\omega' - (\omega + i\delta)}$$

we have

$$\oint \frac{f(\omega')}{\omega' - \omega - i\delta} = \overset{\text{principle}}{P} \oint \frac{1}{\omega' - \omega} + i\pi \int \delta(\omega' - \omega) f(\omega')$$

$$\Rightarrow \xi(\omega) = 1 + \frac{1}{\pi i} P \oint - \dots$$

$$\therefore \operatorname{Re}[\xi(\omega)] = 1 + \frac{1}{\pi} P \int \frac{\operatorname{Im}(\xi(\omega') - 1)}{\omega' - \omega} d\omega'$$

$$\operatorname{Im}[\xi(\omega)] = -\frac{1}{\pi} P \int \frac{\operatorname{Re}(\xi(\omega') - 1)}{\omega' - \omega} d\omega'$$

$$\Rightarrow \begin{cases} \chi_1(\omega) = \frac{1}{\pi} P \int \frac{\chi_2(\omega')}{\omega' - \omega} d\omega' \\ \chi_2(\omega) = -\frac{1}{\pi} P \int \frac{\chi_1(\omega')}{\omega' - \omega} d\omega' \end{cases}$$

The boundary conditions

$$\vec{n} \cdot (\vec{D}_2 - \vec{D}_1) = 4\pi \sigma_f \quad \vec{n} \times (\vec{E}_2 - \vec{E}_1) = 0$$

$\frac{\epsilon_2, \mu_2}{\epsilon_1, \mu_1}$

$$\vec{n} \cdot (\vec{B}_2 - \vec{B}_1) = 0$$

$$\vec{n} \times (\vec{H}_2 - \vec{H}_1) = \frac{4\pi}{c} \vec{K}_f$$

Conservation Law.

1) charge conservation

$$\nabla \cdot \vec{E} = 4\pi \rho_t \Rightarrow \frac{\partial}{\partial t} \nabla \cdot \vec{E} = 4\pi \frac{\partial \rho_t}{\partial t}$$

$$\nabla \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} = 4\pi \vec{J}_t$$

$$\Rightarrow \nabla \cdot \vec{J}_t + \partial_t \rho_t = 0.$$

$$\Rightarrow \partial_\mu J^\mu = 0 \quad , \quad J^\mu = (c\rho, \vec{J}).$$

2) the energy conservation

$$\begin{aligned} \frac{dw}{dt} &= q \vec{v} \cdot \vec{E} \\ &= \int_V \vec{j} \cdot \vec{E} d\vec{r} \\ &\Rightarrow \frac{dw}{dt} = \frac{1}{4\pi} \int_V \left( c \nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} \right) \cdot \vec{E} d^3\vec{r} \\ &= \frac{1}{4\pi} \int_V \underbrace{(c \vec{E} \cdot \nabla \times \vec{B} - \frac{1}{2} \frac{\partial}{\partial t} |\vec{E}|^2)}_{\substack{\Downarrow \\ \nabla \cdot (\vec{B} \times \vec{E}) - \vec{B} \cdot (\nabla \times \vec{E}) \\ \hookrightarrow -\frac{1}{2} \frac{\partial}{\partial t} |\vec{B}|^2}} d^3\vec{r} \end{aligned}$$

$$\Rightarrow \frac{dw}{dt} = - \int \left( \frac{\partial}{\partial t} u + \nabla \cdot \vec{S} \right) d^3 \vec{r}$$

$$\left\{ \begin{array}{ll} u = \frac{1}{8\pi} (|\vec{E}|^2 + |\vec{B}|^2) & \text{Energy density} \\ \vec{S} = \frac{c}{4\pi} (\vec{E} \times \vec{B}) & \text{Poynting vector} \end{array} \right.$$

if no work between field and source,

$$\frac{du}{dt} = 0 \Rightarrow \frac{\partial u}{\partial t} + \nabla \cdot \vec{S} = 0$$

3) momentum Conservation

$$\frac{d\vec{p}}{dt} = \vec{F} = \int (\rho \vec{E} + \frac{1}{c} \vec{j} \times \vec{B}) \cdot d^3\vec{r}$$

$$\begin{aligned} \frac{d\vec{p}}{dt} &= \int (\rho \vec{E} + \frac{1}{c} \vec{J} \times \vec{B}) d^3\vec{r} \\ &= \int \frac{1}{4\pi} (\nabla \cdot \vec{E}) \vec{E} d^3\vec{r} + \int \frac{1}{4\pi} (\nabla \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t}) \times \vec{B} d^3\vec{r} \end{aligned}$$



$$\frac{d\vec{p}}{dt} = - \left\{ \frac{1}{4\pi} \int [\vec{E} \times (\nabla \times \vec{E}) + \vec{E} (\nabla \cdot \vec{E})] d^3\vec{r} \right. \\ \left. + \frac{1}{4\pi} \int (\nabla \times \vec{B}) \times \vec{B} d^3\vec{r} - \frac{1}{4\pi c} \int \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) d^3\vec{r} \right\}$$

定义

$$= \int \left[ \frac{1}{4\pi} \vec{E} (\nabla \cdot \vec{E}) - \frac{1}{4\pi} \vec{E} \times (\nabla \times \vec{E}) + \frac{1}{4\pi} (\nabla \times \vec{B}) \times \vec{B} - \frac{1}{4\pi c} \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) \right] d^3\vec{r}$$

Notice that

$$\vec{E} (\nabla \cdot \vec{E}) - \vec{E} \times (\nabla \times \vec{E}) = \vec{E} (\nabla \cdot \vec{E}) - \frac{1}{2} \nabla (\vec{E} \cdot \vec{E}) + (\vec{E} \cdot \nabla) \vec{E}$$

$$= \partial_i (\vec{E}_i \vec{E}_j + \vec{B}_i \vec{B}_j - \frac{1}{2} (\vec{E}^2 + \vec{B}^2) \delta_{ij}) \hat{e}_i$$

or

Momentum Conservation

$$\frac{d\vec{p}^{(src)}}{dt} = - \int \left[ \frac{\partial}{\partial t} \vec{g} + \partial_i T^{ij} \hat{e}_j \right] d^3\vec{r}$$

$$\vec{g} \equiv \frac{1}{4\pi c} (\vec{E} \times \vec{B}) = \frac{\vec{S}}{c^2}$$

$$T^{ij} = -\frac{1}{4\pi} \left( \vec{E}_i \vec{E}_j + \vec{B}_i \vec{B}_j - \frac{1}{2} (\vec{E}^2 + \vec{B}^2) \delta_{ij} \right)$$

If no momentum transfer:

$$\frac{\partial \vec{g}}{\partial t} + \partial_i T^{ij} \hat{e}_j = 0$$

$$\Rightarrow \partial_\mu T^{\mu j} = 0 \quad , \quad \text{with } T^{0j} = g^j$$

# Symmetries of Maxwell Eqs

## 1) Discrete symmetries

a) inversion  $\vec{r} \rightarrow -\vec{r} \Rightarrow \begin{matrix} \vec{E} \rightarrow -\vec{E} \\ \vec{B} \rightarrow \vec{B} \end{matrix} \quad (\text{pseudo vector})$

rank  $(n, m)$  vector

$$T^{(n, m)} \rightarrow (-1)^{n+m} T^{(n, m)}$$

b) Time-reversal  $t \rightarrow -t \Rightarrow \begin{matrix} \vec{E} \rightarrow \vec{E} \\ \vec{B} \rightarrow -\vec{B} \end{matrix}$

## \* Gauge Symmetry

$$\begin{cases} \vec{B} = \nabla \times \vec{A} \\ \vec{E} + \frac{1}{c} \frac{\partial \vec{A}}{\partial t} = -\nabla \phi \Rightarrow \vec{E} = -\nabla \phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \end{cases}$$

$(\phi, \vec{A})$  is not unique

$$\begin{cases} \phi \rightarrow \phi - \frac{1}{c} \frac{\partial \lambda}{\partial t} \\ \vec{A} \rightarrow \vec{A} + \nabla \lambda \end{cases}$$

Gauge condition

### 1) Lorentz Gauge

$$\nabla \cdot \vec{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0$$

### 2) Coulomb Gauge.

$$\nabla \cdot \vec{A} = 0$$

wave

$$\psi \xrightarrow{\vec{B}} \psi(\vec{r}) e^{i \frac{q}{\hbar} \int \vec{A} \cdot d\vec{r}}$$

$$\text{AB effect.} \Rightarrow \oint \vec{A} \cdot d\vec{l} = \frac{q}{\hbar} \Phi.$$

Rewrite :

$$\left\{ \begin{array}{l} \nabla^2 \phi - \frac{1}{c} \frac{\partial^2}{\partial t^2} \phi = -4\pi \rho \\ \nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{A} = -\frac{4\pi}{c} \vec{J} \end{array} \right.$$

(under Lorentz gauge)

### Chapter III. Electrostatics

$$\nabla^2 \phi = -4\pi \rho_f$$

$$\text{Poisson Eqn} \Rightarrow \text{Solution} \quad \phi = \int \frac{\rho_f(\vec{r}') d^3\vec{r}'}{|\vec{r} - \vec{r}'|}$$

$$\begin{cases} \nabla^2 \phi = -4\pi \rho \\ \phi|_S \\ \frac{\partial \phi}{\partial n}|_S \end{cases} \quad \text{boundary}$$

### Unique Theorem

Let  $\phi_1, \phi_2$  both

$$\nabla^2 \phi_1 = -4\pi \rho$$

and the same boundary.

$$\nabla^2 \phi_2 = -4\pi \rho$$

$$\text{introduce } \psi = \phi_1 - \phi_2 \Rightarrow \nabla^2 \psi = 0$$

$$\begin{aligned} \int \psi \nabla^2 \phi d^3\vec{r} &= \int \nabla \cdot (\psi \nabla \phi) - \nabla \psi \cdot \nabla \phi d^3\vec{r} \\ &= \int \psi \nabla \phi \cdot d\vec{s} - \int |\nabla \psi|^2 d^3\vec{r} = 0. \end{aligned}$$

$$\Rightarrow |\nabla \psi| = 0.$$

$$1) \text{ Dirichlet boundary} \quad \phi|_S = f(\vec{r})$$

$$\Rightarrow \psi|_S = 0 \Rightarrow \psi = 0$$

$$2) \text{ Neumann boundary} \quad \frac{\partial \phi}{\partial n}|_S = g(\vec{r})$$

$$\Rightarrow \frac{\partial \psi}{\partial n}|_S = 0 \Rightarrow \psi = \text{const.}$$

\* Green theorem

Let  $\psi$  be arbitrary function

we have that

$$\begin{aligned} & \int (\phi \nabla^2 \psi - \psi \nabla^2 \phi) dV \\ &= \int \nabla (\phi \nabla \psi - \psi \nabla \phi) \cdot d\vec{S} \\ &= \oint_{\partial V} (\phi \nabla \psi - \psi \nabla \phi) \cdot d\vec{S} \end{aligned}$$

we take that  $\nabla^2 \psi = -4\pi \delta^{(3)}(\vec{r} - \vec{r}')$

$$\Rightarrow -4\pi \int \phi \delta^{(3)}(\vec{r} - \vec{r}') dV - \int \underbrace{\psi \nabla^2 \phi}_{-4\pi \rho(\vec{r})} dV$$

$$= -4\pi \phi(\vec{r}') - 4\pi \int \rho(\vec{r}) dV$$

$$\therefore \phi(\vec{r}') = \frac{1}{4\pi} \int \rho(\vec{r}) \psi dV - \frac{1}{4\pi} \oint (\phi \nabla \psi - \psi \nabla \phi) \cdot d\vec{S}$$

$\vec{r} \leftrightarrow \vec{r}'$

$$\phi(\vec{r}) = \int \rho(\vec{r}') \psi(\vec{r}, \vec{r}') d^3\vec{r}' - \frac{1}{4\pi} \oint_{\partial V} [\phi \nabla \psi - \psi \nabla \phi] \cdot d\vec{S}'$$

If we choose  $\psi(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|}$

$$\Rightarrow \phi(\vec{r}) = \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3\vec{r}' - \frac{1}{4\pi} \oint_{\partial V} \left[ \phi(\vec{r}') \overset{\substack{\uparrow \\ \text{Dirichlet}}}{\nabla' \frac{1}{|\vec{r} - \vec{r}'|}} - \frac{1}{|\vec{r} - \vec{r}'|} \overset{\substack{\uparrow \\ \text{Neumann}}}{\nabla' \phi} \right] \cdot d\vec{S}'$$

Green function

$$\psi(\vec{r}, \vec{r}') \longrightarrow G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} + F(\vec{r}, \vec{r}')$$

$$\hookrightarrow \nabla^2 F = 0$$

Thus,  $\nabla^2 G(\vec{r}, \vec{r}') = -4\pi \delta^3(\vec{r}, \vec{r}')$

1) Dirichlet boundary.

$$G \rightarrow G_D(\vec{r}, \vec{r}') \begin{cases} \nabla'^2 G_D(\vec{r}, \vec{r}') = -4\pi \delta^3(\vec{r}, \vec{r}') \\ G_D(\vec{r}, \vec{r}') \Big|_{\partial V} = 0. \end{cases}$$

we have

$$\phi(\vec{r}) = \int \rho(\vec{r}') G_D(\vec{r}, \vec{r}') d^3 \vec{r}' - \frac{1}{4\pi} \int_{\partial V} \phi(\vec{r}') \nabla' G_D(\vec{r}, \vec{r}') \cdot d\vec{s}'$$

2) Neumann boundary.

$$G \rightarrow G_N(\vec{r}, \vec{r}') \begin{cases} \nabla'^2 G_N(\vec{r}, \vec{r}') = -4\pi \delta^3(\vec{r}, \vec{r}') \\ \frac{\partial G_N(\vec{r}, \vec{r}')}{\partial n} \Big|_{\partial V} = \dots \end{cases}$$

Notice that  $\int \nabla^2 G_N dV = \int_{\partial V} \nabla' G_N \cdot d\vec{s}' = \int_{\partial V} \frac{\partial G_N}{\partial n} ds'$   
 $\rightarrow = -4\pi.$

we can take  $\frac{\partial G_N}{\partial n} \Big|_s = -\frac{4\pi}{A}.$

$$\phi(\vec{r}) = \int \rho(\vec{r}') G_N(\vec{r}, \vec{r}') d^3 \vec{r}' + \frac{1}{4\pi} \int_{\partial V} G_N \nabla' \phi(\vec{r}') \cdot d\vec{s}' - \frac{1}{4\pi} \int_{\partial V} \phi(\vec{r}') \frac{\partial G_N}{\partial n'} ds'$$

||  
 $+ \frac{1}{4\pi} \frac{4\pi}{A} \int \phi(\vec{r}') ds' = \langle \phi \rangle_s.$

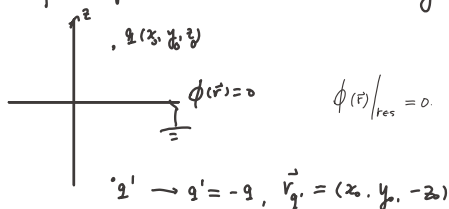
symmetric property  $G_D(\vec{r}, \vec{r}') = G_D(\vec{r}', \vec{r}).$

# Method of image charge

basic idea: for the eq  $\begin{cases} \nabla^2 \phi(\vec{r}) = 4\pi \rho(\vec{r}) \\ \text{boundary} \end{cases}$

Replace the effect of boundary with certain image charge(s).

1) Infinite planar conductor which is grounded



Then 
$$\phi(\vec{r}) = \frac{q}{|\vec{r} - \vec{r}_2|} + \frac{q'}{|\vec{r} - \vec{r}'_2|}$$

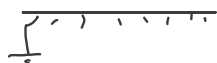
\*  $G_D(\vec{r}, \vec{r}')$  for infinite planar geometry

$$\begin{cases} \nabla'^2 G_D(\vec{r}, \vec{r}') = -4\pi \delta^{(3)}(\vec{r} - \vec{r}') \\ G_D(\vec{r}, \vec{r}')|_{\text{res}} = 0 \end{cases}$$

$$G_D(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} - \frac{1}{|\vec{r} - \vec{r}''|}$$

$$= \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} - \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z+z')^2}}$$

$\Rightarrow \dots \rho(\vec{r}), \phi(\vec{r}) = ?$



The solution 
$$\phi(\vec{r}) = \int \rho(\vec{r}') G_D(\vec{r}, \vec{r}') dV' - \frac{i}{4\pi} \int \phi(\vec{r}') \frac{\partial G_D}{\partial n'} dz' dy'$$

$$\nabla' G_D \cdot d\vec{s}' = ? \quad d\vec{s}' \text{ " " from inside to outside.}$$

$$\therefore \frac{\partial G_D}{\partial n} \Big|_0 = - \frac{\partial G_D}{\partial z} \Big|_{z=0}$$

$$\text{Thurs. } \frac{\partial G_D}{\partial n'} = \dots \quad \checkmark$$

The generic solution

$$\phi(\vec{r}) = \int \frac{\rho(\vec{r}')}{\dots} d^3\vec{r}' - \int \frac{\rho(\vec{r}')}{\sqrt{\dots}} d^3\vec{r}'$$

$$- \frac{1}{4\pi} \int \underbrace{\phi(\vec{r}')}_{\text{image}} \cdot \frac{(1-2z)}{(\dots)^{3/2}} dx' dy'$$

$$= \phi_1 + \underbrace{\phi_2}_{\text{boundary}} + \phi_3$$

$$\phi_3 = \dots = \frac{1}{2\pi} \int \phi(\vec{r}') \frac{\partial}{\partial z} \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + z^2}} dx' dy' \underbrace{\delta(z')}_{\left( \frac{\partial}{\partial z} \delta(z) \right) f(z)} = - \int \left( \frac{\partial f}{\partial z} \right) \delta(z) dz.$$

$$= \frac{1}{2\pi} \int \left[ \phi(\vec{r}') \frac{\partial}{\partial z} \delta(z) \right] \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + z^2}} dx' dy' dz'$$

$$\left\{ \begin{array}{l} \phi_3 = \int \frac{\rho_b(\vec{r}')}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} d^3\vec{r}' \\ \rho_b(\vec{r}') = - \frac{\phi(x', y')}{2\pi} \frac{d}{dz} \delta(z') \end{array} \right. \quad \text{Effective boundary charge}$$

$$Q_b = \int \rho_b dv' = 0$$



\* N Green function  $G_n(\vec{r}, \vec{r}')$

$$\begin{cases} \nabla^2 G_n(\vec{r}, \vec{r}') = -4\pi \delta(\vec{r} - \vec{r}') \\ \frac{\partial G_n}{\partial n'} = -\frac{4\pi}{A} \rightarrow 0. \end{cases}$$

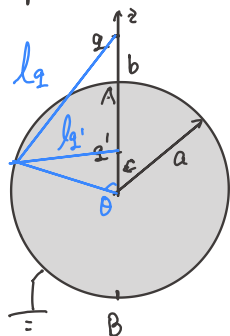
$$G_n = \frac{1}{\sqrt{\quad}} + \frac{1}{\sqrt{\quad}}$$

$$\Rightarrow \phi = \phi_1 + \phi_2 + \phi_3 + \langle \phi \rangle_s$$

$$\phi_3 = \frac{1}{2\pi} \int \frac{E_z(x', y')}{\sqrt{(x-x')^2 + (y-y')^2 + z^2}} dx' dy' = \int \frac{\sigma_s(x', y')}{\sqrt{\quad}} dx' dy'$$

$\sigma_s = \frac{E_z}{2\pi}$

\* Spherical conductor grounded.



$$\phi(\vec{r}) = ? \quad r > a.$$

$$q' = ? \quad c = ?$$

$$A, B \quad \phi = 0$$

$$\Rightarrow c = \frac{a^2}{b}, \quad q' = -\frac{a}{b} q$$

$$\begin{cases} l_2^2 = a^2 + b^2 - 2ab \cos \theta \\ l_{q1}^2 = a^2 + b^2 + 2ab \cos \theta. \end{cases}$$

\* Generalize:

$$\phi = \frac{Q}{a}. \quad \Rightarrow \phi(\vec{r}) = \frac{q}{|\vec{r} - \vec{r}_1|} + \frac{q'}{|\vec{r} - \vec{r}_2|} + \underline{\frac{Q}{r}}$$

Notice

$$q' = -\frac{a}{r_2} q$$

$$\vec{r}_{q'} = \frac{a^2}{r_2^2} \vec{r}_2$$

$$\phi(r) = \frac{q}{|\vec{r} - \vec{r}_2|} - \frac{(a/r_2) q}{|\vec{r} - \frac{a^2}{r_2^2} \vec{r}_2|}$$

the Dirichlet Green function : ( $q=1$ )

$$G_D(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} - \frac{(a/r')}{|\vec{r} - \frac{a^2}{r'^2} \vec{r}'|}$$

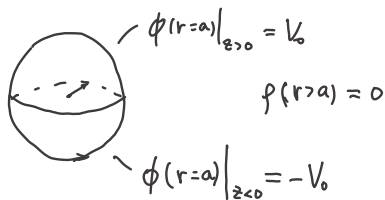
$$\phi(\vec{r}) = \int \rho(\vec{r}') G_D(\vec{r}, \vec{r}') d^3\vec{r}' - \frac{1}{4\pi} \int_{\partial V} \phi(\vec{r}') \nabla' G_D(\vec{r}, \vec{r}') \cdot d\vec{s}'$$

If we take  $\vec{r} = r(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

$$\vec{r}' = r'(\sin\theta'\cos\varphi', \sin\theta'\sin\varphi', \cos\theta') \Rightarrow \vec{r} \cdot \vec{r}' = rr'\cos\gamma$$

$$\text{where } \cos\gamma = \sin\theta\sin\theta'\cos(\varphi-\varphi') + \cos\theta\cos\theta'$$

eg.



$$\phi(\vec{r}) = -\frac{1}{4\pi} \int_{\partial V} \phi(\vec{r}') \nabla' G_D(\vec{r}, \vec{r}') \cdot d\vec{s}'$$

$$= -\frac{1}{4\pi} \int_{\partial V} \phi(\vec{r}') \left[ -\frac{\partial G_D}{\partial r'} \right] dS'$$

$$\left. \frac{\partial G_D}{\partial r'} \right|_{r'=a} = \left[ \frac{-r' + r\cos\gamma}{(r^2 + r'^2 - 2r'r\cos\theta)^{3/2}} + \frac{r^2 \frac{r'}{a^2} - r\cos\gamma}{(r^2 + r'^2/a^2 - 2r'r\cos\theta)^{3/2}} \right]_{r'=a}$$

$$\therefore \frac{\partial \phi}{\partial r} \Big|_{r=a} = - \frac{a - r^3/a}{(r^2 + a^2 - 2ar \cos \theta)^{3/2}}$$

$$\phi(\vec{r}) = \frac{1}{4\pi} \int \phi(\vec{r}') \frac{r'^3/a - a}{(r^2 + a^2 - 2ar' \cos \theta')^{3/2}}$$

$$= \frac{r^2 - a^2}{4\pi a} \int_V \phi(\vec{r}') \frac{a^2 \sin \theta' d\theta' d\varphi'}{(r^2 + a^2 - 2ar' \cos \theta')^{3/2}}$$

$$= \frac{a(r^2 - a^2)}{4\pi} \int d\varphi' \int \frac{d\theta' \sin \theta'}{\dots}$$

Notice  $\theta' = \gamma$   
when  $\vec{r} = r \hat{e}_z$

$$\therefore \phi(\vec{r}) = \frac{aV_0(r^2 - a^2)}{4\pi} \left[ \int_0^{\pi/2} \frac{d\theta' \sin \theta'}{(r^2 + a^2 - \dots)^{3/2}} - \int_{\pi/2}^{\pi} \dots \right]$$

$$\left\{ \text{if only on } \vec{r} = r \hat{e}_z, \phi = V \left( 1 - \frac{z^2 - a^2}{2\sqrt{a^2 + z^2}} \right) \right\}$$

Separation of variables

$$\nabla^2 \phi = 0$$

$$\phi(\vec{r}) = X(x) Y(y) Z(z)$$

$$\nabla^2 \phi = X'' Y Z + Y'' Z X + Z'' X Y = 0$$

$$\Rightarrow \frac{1}{X} X'' + \frac{1}{Y} Y'' + \frac{1}{Z} Z'' = 0 \quad (\gamma^2 = \alpha^2 + \beta^2)$$

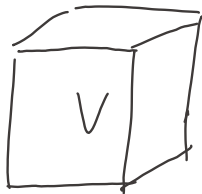
$$\begin{matrix} \parallel & \parallel & \parallel \\ -\alpha^2 & -\beta^2 & \gamma^2 \end{matrix}$$

$$\rightarrow X \sim e^{\pm i\alpha x} \quad Y \sim e^{\pm i\beta y} \quad Z \sim e^{\pm \gamma z}$$

$$\Rightarrow \phi(\vec{r}) = \int d\alpha d\beta e^{i\alpha x} e^{i\beta y} [A(\alpha, \beta) e^{\gamma z} + B(\alpha, \beta) e^{-\gamma z}]$$

Rectangle box boundary

$$\phi|_{\partial V} = 0 \quad \text{except} \quad \phi|_{z=c} = V_0(x, y)$$



$$\Rightarrow A = -B. \quad X_n, Y_n \rightarrow \sin \frac{n\pi x}{a}, \sin \frac{n\pi y}{b}.$$

$$Z \Rightarrow \gamma = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$$

$$\begin{cases} V|_{z=c} = V_0(x, y) \\ V|_{z=0} = 0 \end{cases}$$

$$\Rightarrow \phi = \sum_{n,m} \sin(\dots) \sin(\dots) \sinh(\dots) A_{nm}$$

For a system with variable  $[0, b]$ ,  $f(x) = \sum a_n u_n(x)$  generic function

If  $u_n(x)$  are orthogonal bases

$$\int \bar{u}_n(x') f(x) dx = \sum a_n \delta_{nn'} = a_n$$

$$\sum \bar{u}_n(x') u_n(x) = \delta(x - x')$$

$$f(x) = \sum_n \left[ \int f(x') \bar{u}_n dx' \right] u_n(x)$$

Generalization.

$$\begin{cases} \nabla^2 \phi = -4\pi \rho \\ \phi|_{\partial V} = 0 \end{cases}$$

Hard to obtain the solution

Note that  $\frac{d^2}{dx^2} \cos \frac{n\pi}{a} x = - \left( \frac{n\pi}{a} \right)^2 \cos \left( \frac{n\pi}{a} x \right)$ , and  $y, z$ .

$$\text{so } \nabla^2 \sin(\dots) \sin(\dots) \sin(\dots) = - \left[ (\dots)^2 + (\dots)^2 + (\dots)^2 \right] \sin \sin \sin.$$

$$g(\vec{r}, \vec{r}') = \sum_{nkl} \left[ \sin\left(\frac{n\pi}{a} k\right) \sin\left(\frac{n\pi}{a} k'\right) \cdot s s' s s' \right] \cdot \frac{1}{(\dots)^2 + (\dots)^2 + (\dots)^2}.$$

$$\nabla^2 g \sim -\delta^{(3)}(\vec{r} - \vec{r}').$$

$$\epsilon_D(\vec{r}, \vec{r}') = 4\pi g(\vec{r}, \vec{r}') / (abc/8).$$

Neumann boundary  $\frac{\partial \phi}{\partial n} = -\frac{q\pi}{A}$

Curvilinear coordinates

$$(x, y, z) \rightarrow (u_1, u_2, u_3)$$

$$dl_i = h_i du_i, \dots (h_i)$$

$$\nabla f = \frac{1}{h_1} \frac{df}{du_1} \hat{e}_1 + \dots$$

$$\nabla \cdot \vec{g} = \frac{1}{h_1 h_2 h_3} \left[ \frac{d}{du_1} (h_2 h_3 g_1) + \text{cyc} \right].$$

$$\nabla \times \vec{g} = \frac{1}{h_2 h_3} \left[ \frac{\partial}{\partial u_2} (h_3 g_3) - \frac{\partial}{\partial u_3} (h_2 g_2) \right] \hat{e}_1 + \text{cyc}$$

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial u_1} \left( \frac{h_2 h_3}{h_1} \frac{\partial}{\partial u_1} \right) + \dots + \text{cyc} \right] f(\vec{r})$$

$$1) (r, \theta, \varphi) \quad 1, r, r \sin \theta \quad \frac{1}{r^2} \left( \frac{\partial}{\partial r} (r^2 \frac{\partial}{\partial r}) \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta})$$

$$2) (p, \theta, z) \quad 1, p, 1 \quad + \frac{1}{p^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}$$

$$\frac{1}{p^2} \frac{\partial}{\partial p} (p \frac{\partial}{\partial p}) + \frac{1}{p^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

spheric:  $\phi(\vec{r}) = \frac{1}{r} R(r) Y(\theta, \varphi)$

when the system has azimuthal symmetry  $\phi(\vec{r}) = \frac{1}{r} R(r) \Theta(\theta)$

the Legendre eq.

$$\sin \theta \frac{d}{d\theta} \left( \sin \theta \frac{d}{d\theta} \Theta \right) + \lambda \sin^2 \theta \Theta = 0$$

if we take  $x = \cos \theta$ ,

$$(1-x^2) \frac{d}{dx} \left[ (1-x^2) \frac{d}{dx} \Theta \right] + \lambda (1-x^2) \Theta = 0$$

$$\Rightarrow \frac{d}{dx} \left[ (1-x^2) \frac{d}{dx} \Theta \right] + \lambda \Theta = 0$$

$$\hookrightarrow \lambda = l(l+1) \text{ with } l \in \mathbb{Z}$$

$$\Theta(x) \rightarrow P_l(x) \quad \text{Legendre polynomials}$$

$$P_l(x) = \sum_{k=0}^{\lfloor l/2 \rfloor} \frac{(-1)^k (2l-2k)!}{2^l (l-k)! (l-2k)!} x^{l-2k}$$

$$P_l(-x) = (-1)^l P_l(x) \quad P_l(1) = 1$$

$$P_0(x) = 1 \quad P_1(x) = x \quad P_2(x) = \frac{1}{2}(3x^2 - 1)$$

Rodrigues formula

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2 - 1)^\ell$$

Generating formula

$$\frac{1}{(1+t^2+2t\cos\theta)^{1/2}} \stackrel{t \leq 1}{=} \sum_{\ell} t^\ell P_\ell(\cos\theta)$$

then  $R(r) \sim r^l, r^{l+1}$

$$\Rightarrow \frac{1}{r} R(r) \sim r^l, r^{-l-1}$$

$$\phi(r, \theta) = \sum_{\ell=0}^{\infty} (A_\ell r^\ell + B_\ell r^{-\ell-1}) P_\ell(\cos\theta)$$

Note that

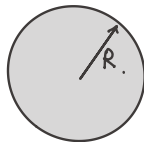
$$\int_{-1}^{+1} P_\ell P_{\ell'} dx = \frac{2}{2\ell+1} \delta_{\ell\ell'}$$

If  $r < R \rightarrow \phi|_{r=0}$  is finite  $\Rightarrow B_\ell = 0$ .

If  $r > R \rightarrow \phi|_{r \rightarrow \infty}$  is finite  $\Rightarrow A_\ell = 0$ .

$$\text{at } r = R \Rightarrow \phi|_{R^+} = \phi|_{R^-}$$

$$D_n \Rightarrow \epsilon_2 \frac{\partial \phi}{\partial n} \Big|_{R^+} = \epsilon_1 \frac{\partial \phi}{\partial n} \Big|_{R^-}$$



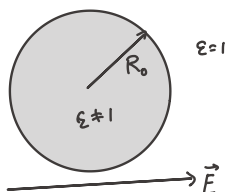
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Consider Boundary

$$\phi|_{R^+} = V(R\cos\theta) = \sum B_\ell \frac{1}{r^{\ell+1}} P_\ell(\cos\theta)$$

$$\Rightarrow B_\ell = \frac{2\ell+1}{2} R^{\ell+1} \int V(R\cos\theta) P_\ell(\cos\theta) d(\cos\theta)$$

Dirichlet Boundary . sphere



$$\phi = \left( C_l r^l + \frac{D_l}{r^{l+1}} \right) P_l(\cos \theta).$$

$$1) \quad r=0 \quad \phi=0 \Rightarrow D_l=0.$$

$$2) \quad r \rightarrow \infty. \quad \phi(r, \theta) \rightarrow -Er \cos \theta.$$

$$\Rightarrow A_1 = -E, \quad A_l = 0 \quad (l \neq 1)$$

$$\Rightarrow \phi = \begin{cases} -Er \cos \theta + \sum \frac{B_l}{r^{l+1}} P_l(\cos \theta) & , \quad r > R \\ C_l r^l P_l(\cos \theta) & , \quad r < R. \end{cases}$$

at  $r=R$ .

$$\begin{cases} -ER \cos \theta + \sum \frac{B_l}{R^{l+1}} P_l(\cos \theta) = \sum C_l R^l P_l(\cos \theta) \\ -\epsilon E \cos \theta - \epsilon \sum \frac{(l+1) B_l}{R^{l+2}} P_l(\cos \theta) = \sum l C_l P_l(\cos \theta) R^{l-1} \end{cases}$$

$$\Rightarrow B_1 = \frac{\epsilon-1}{\epsilon+2} ER^3$$

$$C_1 = -\frac{3}{\epsilon+2} E$$

$$\Rightarrow \begin{cases} \phi_1 = -Er \cos \theta + \frac{\epsilon-1}{\epsilon+2} ER^3 \cos \theta / r^2. \\ \phi_2 = -\frac{3E}{\epsilon+2} r \cos \theta. \end{cases}$$

$$\vec{P} = \chi(\omega) \vec{E} = \frac{\epsilon-1}{4\pi} \cdot \frac{3E}{\epsilon+2} \hat{e}_z$$

notice:  $\phi_{\text{dipole}} = \frac{\vec{p} \cdot \vec{r}}{r^3}$



for metal,  $\epsilon \rightarrow \infty$ ,

$$\phi(r \leq R) = 0$$

$$\phi(r > R, \theta) = -Er \cos \theta + \frac{ER^3}{r^2} \cos \theta.$$

polarization surface

$$4\pi \sigma_{\text{surf}} = E_r|_{R^+} - E_r|_{R^-} \\ = \frac{3}{4\pi} E \cos \theta.$$

without symmetry  $\phi = \frac{1}{r} R(r) Y(\theta, \varphi) \quad - \nabla^2 Y_{\ell, m} = \ell(\ell+1) Y_{\ell, m}$

$$Y \sim \textcircled{H} \quad \Phi \sim P_\ell^m(\cos \theta) e^{im\varphi}.$$

$\Downarrow$

$$\frac{d}{dx} \left[ (1-x^2) \frac{d}{dx} P_\ell^m \right] + \left( \lambda - \frac{m^2}{1-x^2} \right) P_\ell^m = 0$$

$\quad \quad \quad \downarrow \quad \quad \quad \bar{L}_\ell(\ell+1),$

$$P_\ell^m(x) \sim \frac{\partial^m}{\partial x^m} P_\ell(x)$$

$$Y_{\ell m}(\theta, \varphi) = \sqrt{\frac{2\ell+1}{4\pi}} \sqrt{\frac{(\ell-m)!}{(\ell+m)!}} \frac{P_\ell^m(\cos \theta)}{\quad \quad \quad \downarrow} e^{im\varphi}.$$

$$P_\ell^m(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_\ell(x). \quad (m \geq 0).$$

$$P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x)$$

for  $m \geq 0 \quad P_\ell^m \sim \frac{d^{\ell+m}}{dx^{\ell+m}} (x^2-1)^\ell$

$$\int Y_{\ell m} Y_{\ell' m'} d\Omega = \delta_{\ell\ell'} \delta_{mm'}$$

$$\bar{Y}_{\ell m}(\theta, \varphi) = (-1)^m Y_{\ell m}(\theta, \varphi),$$

In QM,  $-D^2 \sim \hat{L}^2$ .

finally we get 
$$\phi = \sum_{m,l} \left( A_{ml} r^l + \frac{B_{ml}}{r^{l+1}} \right) Y_{ml}(\theta, \varphi)$$

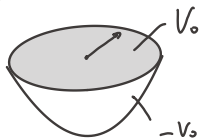
\* off-axis expansion

1) with azimuthal symmetry

Consider  $\vec{r} = z\hat{e}_z$ ,  $\cos\theta = 1$ .

$$\begin{aligned} \phi(r, \theta=0) &= \phi(z) = \sum (A_l \dots + \frac{B_l}{z^{l+1}}) P_{l,1}(1) = 1 \\ &= \sum (A_l z^l + \frac{B_l}{z^{l+1}}). \end{aligned}$$

example.



$$\phi = \begin{cases} V_0 & , \theta = \frac{\pi}{2} \\ -V_0 & , \frac{\pi}{2} < \theta < \pi, (r) \end{cases}$$

$$\phi(r, \theta) = ?$$

easily get: 
$$\phi(z) = V_0 \left( 1 - \frac{z^2 - a^2}{z\sqrt{z^2 + a^2}} \right).$$

$\Rightarrow$

$$\phi(z) = V_0 \left[ 1 - \left( 1 - \frac{a^2}{z^2} \right) \left[ \sum_{n=1}^{\infty} \frac{(-1)^n (2n-1)!!}{2^n n!} \left( \frac{a}{z} \right)^{2n} + 1 \right] \right]$$

$$= V_0 \left[ \frac{a^2}{z^2} - \sum_{n=1}^{\infty} \frac{(-1)^n (2n-1)!!}{2^n n!} \left( \frac{a}{z} \right)^{2n} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} \dots}{\dots} \right]$$

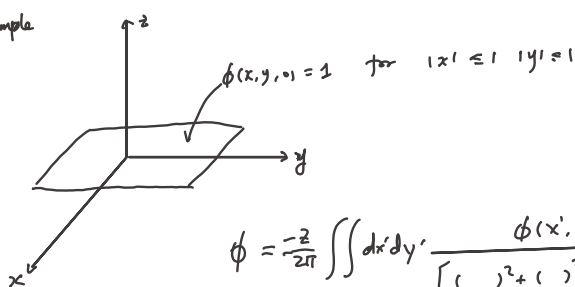
$$= V_0 \left[ \frac{a^2}{z^2} + \sum_{n=2}^{\infty} \frac{(-1)^n (2n-3)!!}{2^{n-1} (n-1)!} \left[ -\frac{(-1)(2n-1)}{2-n} + 1 \right] \left( \frac{a}{z} \right)^{2n} \right]$$

notice  $n=0!$

so

$$\phi(r, \theta) = \frac{1}{2} V_0 \frac{a^2}{r^2} P_2(\cos\theta) + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} (4n-1)(2n-3)!!}{2^{n-1} (2n)(n-1)!} \left( \frac{a}{z} \right)^{2n} P_{2n}(\cos\theta)$$

Example



use  $(r, \theta, \varphi)$ :

$$\phi = \sum (A_{lm} r^l + B_{lm} \frac{1}{r^{l+1}}) Y_{lm}(\theta, \varphi)$$

$$\theta \rightarrow 0. \quad \phi(z) = \dots$$

notice  $\phi(r, \theta, \varphi + \frac{\pi}{2}) = \phi(r, \theta, \varphi)$ .  $\varphi$  - symmetry

$$Y_{lm}(\theta, \varphi + \frac{\pi}{2}) = Y_{lm}(\theta, \varphi) \Rightarrow e^{im(\varphi + \pi/2)} = e^{im\varphi} \Rightarrow m = 4k$$

$$\therefore \phi(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=4n}^{\infty} (A_{lm} r^l + B_{lm} \frac{1}{r^{l+1}}) Y_{lm}(\theta, \varphi)$$

The Green function in the spherical coordinates

1) Methods of image charge

2) Expansion.

$$\text{Recall that } \frac{1}{\sqrt{1+t^2-2t\cos\theta}} = \sum_{l=0}^{\infty} t^l P_l(\cos\theta)$$

$$\Rightarrow \frac{1}{|\vec{r} - \vec{r}'|} = \sum_{l=0}^{\infty} \frac{r'^l}{r^{l+1}} P_l(\cos\theta) \quad (r' > r)$$

$$\text{or, def } \theta(r-r') = \begin{cases} 1 & r > r' \\ 0 & r < r' \end{cases}$$

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_l \left( \frac{r^l}{r'^{l+1}} \theta(r'-r) + \frac{r'^l}{r^{l+1}} \theta(r-r') \right) P_l(\cos \gamma)$$

$$\hookrightarrow \vec{r} \cdot \vec{r}' = rr' \cos \gamma$$

$$\Rightarrow \frac{1}{|\vec{r}-\vec{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \left[ \frac{r^l}{r'^{l+1}} \theta(r'-r) + \frac{r'^l}{r^{l+1}} \theta(r-r') \right] \bar{Y}_{lm}(\theta', \varphi') Y_{lm}(\theta, \varphi)$$

$$\text{now } G_D(\vec{r}, \vec{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{a/r'}{|\mathbf{r}-\frac{a^2}{r'^2} \mathbf{r}'|} = \dots \dots \frac{r^l}{r'^{l+1}} - \frac{a^{2l+1}}{r'^{2l+1} r^{l+1}}$$

Bilinear construction

$$\text{consider the eq. } \hat{L} = P \frac{d^2}{dx^2} + \overset{\text{dP/dx}}{\underbrace{P'}} \frac{d}{dx} + q(x) \quad x_1 \leq x \leq x_2$$

suppose solutions  $u(x) v(x)$ .

$$\hat{L} u = 0, \quad \hat{L} v = 0.$$

$u(x) \rightarrow$  the boundary at  $x=x_1$

$v(x) \rightarrow$  the boundary at  $x=x_2$ .

so we get a Green func:

$$g(x, x') = \frac{u(x_1)v(x')\theta(x'-x_1) + v(x_2)u(x')\theta(x-x_2)}{p(x)w(u, v; x')}$$

$$w(u, v, x') = u(x') \frac{dv}{dx'} - v(x') \frac{du}{dx'}$$

$$\hat{L} g = \delta(x-x').$$

notice: recall  $R(r)$ .

$$\underbrace{r^2}_{\substack{\uparrow \\ P}} \frac{d^2 \tilde{R}}{dr^2} + 2r \frac{d\tilde{R}}{dr} - \lambda \underbrace{\tilde{R}}_Q = 0.$$

$$(\tilde{R} = \frac{1}{r} R(r))$$

$$\Rightarrow u_\ell = r^\ell \quad v_\ell = r^{-\ell-1}$$

$$W(u, v, r') = -(2\ell+1) \frac{1}{r'^2} \Rightarrow p(r') W = -(2\ell+1),$$

$$\therefore g_\ell(r, r') = \frac{r^\ell r'^{-\ell-1} \theta(r'-r) + r^{-\ell} r'^{\ell+1} \theta(r-r')}{-(2\ell+1)}$$

$$\psi \sim \sum_{\ell} g_\ell \bar{Y} Y_{\frac{4\pi}{2\ell+1}}$$

Note that

$$\int \bar{Y} Y \sin\theta d\theta d\phi = \delta_{\ell\ell'} \delta_{mm'}$$

$$\Rightarrow \sum_{\ell m} \bar{Y} Y = \frac{1}{\sin\theta} \delta(\theta-\theta') \delta(\phi-\phi')$$

Recall

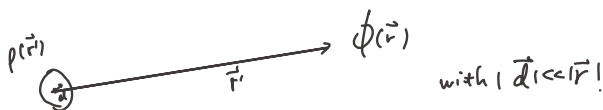
$$\nabla^2 = \underbrace{\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) - \frac{\ell(\ell+1)}{r^2}}_{D_r^2} + \underbrace{\frac{1}{r^2} D_{\theta\phi}^2 + \frac{\ell(\ell+1)}{r^2}}_{\frac{1}{r^2} \tilde{D}_{\theta\phi}^2}$$

$$\epsilon_{\vec{r}}(\vec{r}, \vec{r}') = \sum_{lm} \frac{4\pi}{2l+1} \left[ \left( \frac{r'^l}{r^{l+1}} - \frac{a^{2l+1}}{r^{l+1} r'^{l+1}} \right) \mathcal{Y}(r-r') + \left( \frac{r^l}{r'^{l+1}} - \frac{a^{2l+1}}{r'^{l+1} r^{l+1}} \right) \mathcal{Y}(r'-r) \right]$$

$$\cdot \bar{Y}_{lm}(\theta', \varphi') Y_{lm}(\theta, \varphi)$$

$$\frac{1}{r^{l+1}} \left( r'^l - \frac{a^{2l+1}}{r'^{l+1}} \right) = u(r) v(r') \quad \hat{L} = r^2 \frac{d^2}{dr^2} + 2r \frac{d}{dr} - l(l+1)$$

Multiple Expansion.



$$f(\vec{x} + \vec{a}) = f(\vec{x}) + a_i \partial_i f(\vec{x}) + \frac{a_i a_j}{2!} \partial_i \partial_j f(\vec{x}) + \dots$$

$$\Rightarrow \frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{r} - x'_i \partial_i \frac{1}{r} + \frac{x'_i x'_j}{2} \partial_i \partial_j \frac{1}{r} - \frac{\dots}{3!} \partial \dots \frac{1}{r}$$

$$\Rightarrow \phi = \int \rho(r') \left[ \frac{1}{r} - x'_i \partial_i \frac{1}{r} + \frac{x'_i x'_j}{2} \partial_i \partial_j \frac{1}{r} - \frac{\dots}{3!} \partial \dots \frac{1}{r} \right] d^3 r'$$

$$Q = \int \rho(\vec{r}') d^3 \vec{r}', \quad \phi^{(0)} = \frac{Q}{r}$$

$$\phi^{(1)} = - \int \rho(\vec{r}') x'_i \partial_i \frac{1}{r} d^3 \vec{r}'$$

$$\phi^{(2)} = \frac{1}{2!} \int \rho \underbrace{x'_i x'_j \partial_i \partial_j \frac{1}{r}}_{(x'_i x'_j - \alpha \delta_{ij} r'^2) \partial_i \partial_j \frac{1}{r^3}} d^3 \vec{r}'$$

$\rightarrow (x'_i x'_j - \alpha \delta_{ij} r'^2) \partial_i \partial_j \frac{1}{r^3}$ . Require  $\text{Tr}(\dots) = 0$ .

$$\Rightarrow \alpha = 1/3$$

$$\Rightarrow \phi^{(2)} = \frac{1}{3} \frac{1}{2!} (\partial_i \partial_j \frac{1}{r}) \int \rho(\vec{r}') (3x'_i x'_j - \delta_{ij} r'^2) d^3 \vec{r}'$$

$$\text{def } Q_{ij} = \int \rho(\vec{r}') (\beta x_i' x_j' - \delta_{ij} r'^2) d^3 \vec{r}'.$$

$$\phi^{(2)} = \frac{1}{3} \frac{1}{2!} (\partial_i \partial_j \frac{1}{r}) Q_{ij}.$$

$$\phi^{(3)} = -\frac{1}{3!} \int \text{---} d^3 \vec{r}'$$

$$\text{Tr} (x_i' x_j' x_k' - \beta (\delta_{ij} x_k' + \delta_{jk} x_i' + \delta_{ki} x_j') r'^2) = 0.$$

$$\Rightarrow \beta = \frac{1}{5}$$

$\Rightarrow$

$$\phi^{(3)} = -\frac{1}{5} \frac{1}{3!} (\partial_i \partial_j \partial_k \frac{1}{r}) Q_{ijk}$$

$$Q_{ijk} = \int \rho(\vec{r}') [5 x_i x_j x_k - (\delta_{ij} x_k' + \delta_{jk} x_i' + \delta_{ki} x_j') r'^2] d^3 \vec{r}'$$

$$\therefore \phi(\vec{r}) = \frac{Q}{r} - p_i \partial_i \frac{1}{r} + \frac{1}{3} \frac{1}{2!} Q_{ij} \partial_i \partial_j \frac{1}{r^2} - \frac{1}{5} \frac{1}{3!} Q_{ijk} \partial_i \partial_j \partial_k \frac{1}{r} + \dots$$

$p_i \rightarrow 3$  components

$Q_{ij} \rightarrow$  symmetric  $9-3-1=5$  components

$Q_{ijk} \rightarrow 7$  independent.

The  $l$ -th multipole

$$Q_{i_1 \dots i_l}^{(l)} = \int \rho(\vec{r}') [\gamma x_{i_1}' \dots x_{i_l}' - \text{Tr}(x_{i_1} \dots x_{i_l})] d^3 \vec{r}'$$

Independent components  $\rightarrow 2l+1$

\* the electric dipole

$$\rho = \int \rho(\vec{r}') d^3\vec{r}'$$

$$\phi^{(1)} = + p_i \frac{\partial}{\partial r^i} = \frac{\vec{p} \cdot \vec{r}}{r^3}$$

$$\Rightarrow \vec{E} = -\nabla \phi = \left( \partial_i p_i \frac{x_j}{r^5} - \frac{p_j}{r^3} \right) \vec{e}_j = \frac{3(\vec{p} \cdot \vec{r}) \vec{r}}{r^5} - \frac{\vec{p}}{r^3}$$

Similarly for  $Q_{ij}$ .

$$\phi^{(2)} = \frac{1}{3} \frac{1}{2!} \left( \frac{\partial_i \partial_j}{r^5} - \frac{\delta_{ij}}{r^3} \right) Q_{ij}$$

$$= \frac{1}{2} \frac{\vec{r} \cdot \vec{Q} \cdot \vec{r}}{r^3}$$

$$\phi^{(3)} = \frac{1}{2} Q_{ijk} n_i n_j n_k / r^4$$

energy

$$\mathcal{E} = \int \rho(\vec{r}') \phi_{ex}(\vec{r}') d^3\vec{r}'$$

$$= \int \rho(\vec{r}') \left[ \phi_{ex}^{(0)} + x'_i \partial_i \phi_{ex}^{(0)} + \dots \right] d^3\vec{r}'$$

$$= \mathcal{E}^{(0)} + \mathcal{E}^{(1)} + \dots$$

$$\mathcal{E}^{(0)} = Q \phi_{ex}^{(0)}$$

$$\mathcal{E}^{(1)} = \int \rho(\vec{r}') x'_i \partial_i \phi_{ex}^{(0)} d^3\vec{r}' = -\vec{p} \cdot \vec{E}_{ex}$$

$$\mathcal{E}^{(2)} = -\frac{1}{6} (\partial_i E_{j,ex}) Q_{ij}$$



dipole - dipole interaction

$$\epsilon_{12} = \frac{\vec{p}_1 \cdot \vec{p}_2 - 3 (\vec{p}_1 \cdot \vec{n}_{12}) (\vec{p}_2 \cdot \vec{n}_{12})}{r_{12}^3}$$



$$\epsilon_{12} = \frac{p_1 p_2}{r_{12}^3} > 0$$

2)



$$\epsilon_{12} = \frac{-2p_1 p_2}{r_{12}^3} < 0$$

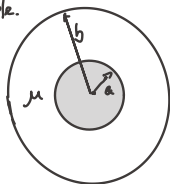
\* Solving boundary problem with magnetic scalar model.

$$\vec{B} = \mu \vec{H}, \quad \nabla \cdot \vec{B} = 0$$

scalar potential  $\vec{H} = -\nabla \Phi_m \Rightarrow \nabla^2 \Phi_m = 0$

$$\Rightarrow \Phi_m = \sum_{l,m} (A_{lm} r^l + B_{lm} r^{-l-1}) Y_{lm}(\theta, \varphi).$$

Example.



[Just as electric]

\* Superconductivity.  $\mu \rightarrow 0. \Rightarrow \text{inside } B = 0.$

→ The magnetic field due to local current distribution

⇒ magnetic multipole expansion

$$\vec{A} = \frac{1}{c} \int \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3\vec{r}'$$

$$= \frac{1}{c} \int \vec{J} \cdot \frac{1}{r} d^3\vec{r}' - \frac{1}{c} \int x'_i \vec{J} \partial_i \frac{1}{r} d^3\vec{r}'$$

$$+ \frac{1}{2c} \int x'_i x'_j \partial_i \partial_j \frac{1}{r} d^3\vec{r}' + \dots$$

$$\vec{A}^{(0)} = \frac{1}{c} \int \vec{J}(\vec{r}') d^3\vec{r}' \quad \text{monopole.} \rightarrow 0.$$

$$\vec{A}^{(1)} = -\frac{1}{c} \left( \partial_i \frac{1}{r} \right) \int x'_i \vec{J}(\vec{r}') d^3\vec{r}'$$

Note that

$$\partial_i (x'_j x'_k J_i) = \delta_{ij} x'_k J_i + \delta_{ik} x'_j J_i + x'_j x'_k \partial_i J_i$$

$$\therefore \vec{A}^{(1)} = -\frac{1}{2c} \partial_i \frac{1}{r} \int (x_i' j_j - x_j' j_i) \hat{e}_i d^3 \vec{r}'$$

$$= \frac{1}{2c} \frac{x_i}{r^3} \epsilon_{ijk} \int (\vec{r}' \times \vec{j})_k \hat{e}_i d^3 \vec{r}'$$

$$= \frac{1}{2c} \left[ \underbrace{(\vec{r}' \times \vec{j})_k d^3 \vec{r}'}_{\vec{m}} \times \frac{\vec{r}}{r^3} \right]$$

$$\hookrightarrow \vec{m} = \frac{1}{2c} \int \vec{r}' \times \vec{j} d^3 \vec{r}' \quad , \quad \vec{A}^{(1)} = \frac{\vec{m} \times \vec{r}}{r^3}$$

$$\Rightarrow \vec{B}^{(1)} = \frac{3(\vec{m} \cdot \vec{r}) \vec{r}}{r^5} - \frac{\vec{m}}{r^3}$$

— Midterm: Nov. 08th

\* If we have point dipole moment

we shouldn't take

$$\partial_j \frac{x_i}{r^3} = -\partial^2 \frac{1}{r} = 0 \text{ for } r > 0$$

instead we take

$$\partial_j \frac{x_i}{r^3} = 4\pi \delta^{(3)}(\vec{r})$$

define

$$M_{ij} = \partial_j \frac{x_i}{r^3}$$

$$\Rightarrow \partial_j \frac{x_i}{r^3} = \frac{1}{3} \delta_{ij} \text{Tr}(M) + \frac{3x_i x_j}{r^5} - \frac{\delta_{ij}}{r^3}$$

$$\text{Tr}(M_{ij}) = 4\pi \delta^{(3)}(\vec{r})$$

finally

$$\vec{B} = \frac{3(\vec{m} \cdot \vec{n})\vec{n} - \vec{m}}{r^3} + 4\pi \vec{m} \delta^{(3)}(\vec{r}) - \frac{4\pi}{3} \vec{m} \delta^{(3)}(\vec{r})$$

$$= \frac{3(\vec{m} \cdot \vec{n})\vec{n} - \vec{m}}{r^3} + \frac{8\pi}{3} \vec{m} \delta^{(3)}(\vec{r})$$

⚡ The magnetic dipole in external magnetic field

$$\vec{F} = \frac{1}{c} \int \vec{J}(\vec{r}') \times \vec{B}_{ex}(\vec{r}') d^3 \vec{r}'$$

$$= \frac{1}{c} \int \underbrace{\vec{J}(\vec{r}') \times \left( \vec{B}_{ex}^{(0)} + \chi_j' \partial_j \vec{B}_{ex}^{(0)} \right)}_{\substack{|| \\ 0.}} d^3 \vec{r}'$$

$$F_i = \frac{1}{c} \epsilon_{ijk} \partial_j B_k \int \chi_e' J_i(\vec{r}') d^3 \vec{r}'$$

$$= \epsilon_{ijk} \epsilon_{ljm} \partial_i B_k m_m$$

$$= (\partial_i B_m) m_m - (\partial_i B_i) m_m$$

$$= -\partial_i U$$

$$U = -\vec{m} \cdot \vec{B}$$

Consider two magnetic dipole



$$U = \frac{\vec{m}_1 \cdot \vec{m}_2 - 3(\vec{m}_1 \cdot \vec{n})(\vec{m}_2 \cdot \vec{n})}{r_{12}^3}$$

# Relativistic electrodynamics

## Chapter 5 Maxwell theory with special relativity

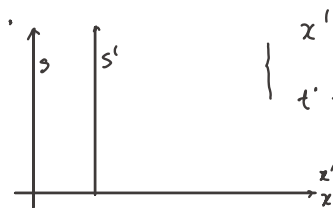
$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \rightarrow \text{Const. in vacuum} \rightarrow \begin{cases} \text{without gravity} \\ \text{without media} \end{cases}$$

- 1) the speed of light in vacuum is const.
- 2) the physical laws are the same for all inertial observers

From Maxwell Eqns, 
$$\begin{cases} \nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{E} = 0 \\ \nabla^2 \vec{B} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{B} = 0 \end{cases}$$

$$(\vec{E}, \vec{B}) \longrightarrow (\vec{E}', \vec{B}')$$

### \* Lorentz Transformation



$$\begin{cases} x' = \gamma(x - vt) \\ t' = \gamma(t - \frac{v}{c^2}x) \end{cases}$$

$$x^{\mu'} = \Lambda^{\mu'}_{\nu} x^{\nu}$$

$$\Lambda^0_0 = \gamma \quad \Lambda^0_i = \gamma \frac{v_i}{c^2}$$

$$\Lambda^i_0 = -\gamma \frac{v_i}{c^2} \quad \Lambda^i_j = 1 + (\gamma - 1) \frac{v_i v_j}{v^2}$$

### Metric tensor

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

→ Lorentz tensor

$$\Lambda^0_0 = \gamma, \quad \Lambda^0_i = \Lambda^i_0 = -\gamma v_i/c$$

$$\Lambda^i_j = \delta_{ij} + \frac{(\gamma-1)}{v^2} v_i v_j$$

all physics transform as tensors, given by Lorentz transf:

$$T^{\mu\nu}(x^\mu) = 0$$

$$S \rightarrow S': T'^{\mu\nu}(x'^\mu) = 0$$

Maxwell Eqn:  $\nabla \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0$

$$\Downarrow$$

$$\nabla' \times \vec{E}' + \frac{1}{c} \frac{\partial \vec{B}'}{\partial t'} = 0$$

$$\longrightarrow E_i = \Lambda^j_i E_j ? \quad \times$$

$$\Rightarrow (E, B)$$

Lorentz Tensors.

$$(m, n) \text{ Type Tensor } T^{\mu_1 \dots \mu_m}_{\nu_1 \dots \nu_n} = \Lambda^{\mu_1}_{\rho_1} \dots \Lambda^{\mu_m}_{\rho_m} \Lambda_{\nu_1}^{\sigma_1} \dots \Lambda_{\nu_n}^{\sigma_n} T^{\rho_1 \dots \rho_m}_{\sigma_1 \dots \sigma_n}$$

with

$$\Lambda_\mu^\nu = \eta_{\mu\rho} \eta^{\rho\sigma} \Lambda^\sigma_\mu$$

1) 4D current  $J^\mu$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0 \Rightarrow \partial_\mu J^\mu = 0$$

with  $J^\mu = (c\rho, \vec{j})$

2) 4D gauge

$$\square = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} = \partial_\mu \partial^\mu$$

$$A^\mu = (\phi, \vec{A})$$

$$\Rightarrow \begin{cases} \square A^\mu = -\frac{4\pi}{c} J^\mu \\ \partial_\mu A^\mu = 0 \end{cases} \longrightarrow \text{4D vector}$$

Can always choose  $A^\mu$  to make  $\nabla \cdot \vec{A} = 0$  true for any frame

3) 4D field tensor  $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$

$$A_\mu = \eta_{\mu\nu} A^\nu$$

rank-2 4D tensor.  $F_{\mu\nu} = -F_{\nu\mu}$

$$\begin{cases} F_{ij} = \epsilon_{ijk} B_k \\ -F_{0i} = F_{i0} = E_i \end{cases} \Rightarrow F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_3 & -B_2 \\ E_y & -B_3 & 0 & B_1 \\ E_z & B_2 & -B_1 & 0 \end{pmatrix}$$

$$\Rightarrow E'_i = F'_{0i} = \Lambda^0_\mu \Lambda^i_\nu F^{\mu\nu}$$

$$= \Lambda^0_0 \Lambda^i_j F^{0j} + \Lambda^0_j \Lambda^i_0 F^{j0} + \Lambda^0_j \Lambda^i_k F^{jk}$$

$$= \gamma \left( \delta_{ij} + \frac{\gamma-1}{v^2} v_i v_j \right) E_j + (-\gamma \frac{v_i}{c}) (-\gamma \frac{v_j}{c}) (-E_j)$$

$$+ (-\gamma \frac{v_i}{c}) \left( \delta_{ik} + \frac{\gamma-1}{v^2} v_i v_k \right) \left[ \epsilon_{jkl} B_k \right]$$

$$\vec{E}' = \gamma \vec{E} + \left( \frac{\gamma-1}{v^2} - \frac{\gamma}{c^2} \right) \vec{v} (\vec{v} \cdot \vec{E}) + \frac{1}{c} \gamma (\vec{v} \times \vec{B})$$

$$\gamma = \sqrt{1 - v^2/c^2} \Rightarrow$$

$$\vec{E}' = \gamma(\vec{E} + \frac{1}{c}(\vec{v} \times \vec{B})) - \frac{\gamma-1}{v^2}(\vec{v} \cdot \vec{E})\vec{v}$$

Similarly

$$\vec{B}' = \gamma(\vec{B} - \frac{1}{c}\vec{v} \times \vec{E}) - \frac{\gamma-1}{v^2}(\vec{v} \cdot \vec{B})\vec{v}$$

\* natural units  $c = 1$   $[L] = [T]$ .

\* Maxwell Eqn  $\nabla_\mu F^{\mu\nu} = -4\pi J^\nu$

$$\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0. \rightarrow \text{Bianchi Identity}$$

for any antisymmetry rank-2 Tensor

4D velocity  $U^\mu = \frac{dx^\mu}{d\tau}$

$$\Rightarrow U^0 = \gamma_u \quad U^i = \gamma_u u^i$$

$$U^{\mu'} = \Lambda^{\mu'}_{\mu} U^\mu$$

$$\Downarrow \quad \Downarrow$$

$$\underline{\gamma_{u'} u'} \quad \underline{\gamma_u u}$$

$$\gamma_{u'} u'^i = \gamma_v \gamma_u v^i - \gamma_u u^i + \frac{\gamma-1}{v^2} v_i^2 \gamma_u u^i$$



4D force  $f^\mu$

$$f^\mu = m \frac{dU^\mu}{d\tau} = \frac{dP^\mu}{d\tau}$$
$$= m \gamma_u \left( u^\mu \frac{d\gamma_u}{d\tau} + \gamma_u \frac{du^\mu}{d\tau} \right)$$

---

In the presence of  $\vec{E}$ ,  $\vec{B}$  fields,

$$f^\mu = e F^{\mu\nu} u_\nu = \frac{dP^\mu}{d\tau}$$

$$\Rightarrow f^0 = e \gamma_u (\vec{E} \cdot \vec{u})$$

$$f^i = e F^{i0} u_0 + e F^{ij} u_j$$

$$= \gamma_u e E^i + e \epsilon_{ijk} \gamma_u u_j B_k$$

$$= \gamma_u e (E^i + (\vec{v} \times \vec{B})^i)$$

$$\Rightarrow \frac{dP^0}{dt} = e (\vec{u} \cdot \vec{E})$$

$$\frac{dP^i}{dt} = e (E^i + (\vec{v} \times \vec{B})^i)$$

Action Principle in  $\vec{E}, \vec{B}$  field

$$S_{\text{tot}} = S_{\text{charge}} + S_{\text{field}} + S_{\text{tot}}$$

Lorentz Invariant

Free particle

$$S = - \int_{\tau_1}^{\tau_2} m d\tau$$

$$= - m \int (-\eta_{\mu\nu} dx^\mu dx^\nu)^{\frac{1}{2}}$$

$$= - m \int (1 - u^2)^{\frac{1}{2}} dt$$

The variation principle

$$\begin{aligned}
 \delta S_{\text{charge}} &= -m \int \delta (1-u^2)^{\frac{1}{2}} dt \\
 &= m \int \frac{u_i}{(1-u^2)^{\frac{3}{2}}} \frac{d \delta x^i}{dt} dt \\
 &= m \int \left( \frac{d}{dt} \left( \frac{u^i \delta x^i}{(1-u^2)^{\frac{3}{2}}} \right) - \frac{d}{dt} \frac{u^i}{(1-u^2)^{\frac{3}{2}}} \delta x^i \right) dt
 \end{aligned}$$

$$\delta x^i \Big|_{t_1}^{t_2} = 0 \Rightarrow$$

$$\delta S_{\text{charge}} = -m \int \frac{d}{dt} \left( \frac{u_i}{\sqrt{1-u^2}} \right) \delta x^i dt.$$

$$\text{free charge: } \delta S = 0 \Rightarrow \frac{d p^i}{dt} = 0.$$

Now in the presence of  $\vec{E}, \vec{B}$  fields

$$\begin{aligned}
 S_{\text{int}} &= e \int A_\mu dx^\mu \\
 &= e \int A_\mu U^\mu d\tau
 \end{aligned}$$

$$\Rightarrow S = -m \int d\tau + e \int A^\mu U_\mu d\tau$$

$$\begin{aligned}
 \delta S &= -m \int \frac{d}{dt} (\gamma U^i) \delta x^i dt - e \int \frac{\partial \phi}{\partial x^i} \delta x^i dt + e \int \frac{\partial A_i}{\partial x^i} u^i \delta x_i dt \\
 &\quad + e \int A_i \left( \frac{d}{dt} \delta x^i \right) dt
 \end{aligned}$$

$$\delta S = 0 \Rightarrow$$

$$\begin{aligned}
 \frac{d p^i}{dt} &= -e \frac{\partial \phi}{\partial x^i} + e \frac{\partial A_i}{\partial x^i} u_i - e \frac{d A_i}{dt} u_i \\
 &= e E_i + e (\vec{v} \times \vec{B})_i
 \end{aligned}$$

$$\text{OR, } \frac{d p^\mu}{d\tau} = e F^{\mu\nu} U_\nu$$

The motion of charges in  $\vec{E}, \vec{B}$  fields

1) only  $\vec{E}$  field

$$\begin{cases} \frac{dP^0}{dt} = q(\vec{u} \cdot \vec{E}) \\ \frac{d\vec{P}}{dt} = q\vec{E} \end{cases} \quad \text{let } \vec{E} = (E, 0, 0), \vec{u}|_{t=0} = (0, u_y, 0)$$

$$\Rightarrow P_y(t) = P_{y_0}, \quad P_x(t) = qEt$$

$$\begin{aligned} \therefore y(t) &= \int \frac{P_y}{\sqrt{\epsilon_0^2 + (eEt)^2}} dt & \epsilon_0^2 &= m^2 + p_y^2 \\ &= p_y \operatorname{arcsinh}\left(\frac{eEt}{\epsilon_0}\right) \end{aligned}$$

$$x(t) = \frac{1}{eE} \sqrt{\epsilon_0^2 + (eEt)^2}$$

For field,

$\rightarrow S_2$  无致.

$$I_1 = F_{\mu\nu} F^{\mu\nu}$$

$$I_2 = \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

$$S_{\text{field}} = -\frac{1}{16\pi} \int F_{\mu\nu} F^{\mu\nu} d^4x$$

$$\delta S_f = -\frac{1}{16\pi} \int (\delta F_{\mu\nu}) F^{\mu\nu} + F_{\mu\nu} (\delta F^{\mu\nu}) d^4x$$

$$F_{\mu\nu} \delta F^{\mu\nu} = \eta^{\mu\rho} \eta^{\nu\sigma} \delta F_{\rho\sigma} \cdot F_{\mu\nu}$$

$$= \delta F_{\mu\nu} F^{\mu\nu}$$

$$\begin{aligned}
 \therefore \delta S &= -\frac{1}{8\pi} \int F^{\mu\nu} \delta(\partial_\mu A_\nu - \partial_\nu A_\mu) d^4x \\
 &= \frac{1}{8\pi} \int F^{\mu\nu} \delta \partial_\mu A_\nu + F^{\nu\mu} \delta \partial_\nu A_\mu d^4x \\
 &= -\frac{1}{4\pi} \int \underbrace{F^{\mu\nu} \delta \partial_\mu A_\nu}_{\parallel} d^4x \\
 &\quad \quad \quad \delta(F^{\mu\nu} \partial_\mu A_\nu) \stackrel{0}{=} -(\partial_\mu F^{\mu\nu}) \delta A_\nu \\
 &= \frac{1}{4\pi} \int \partial_\mu F^{\mu\nu} \delta A_\nu dx \quad \Rightarrow \partial_\mu F^{\mu\nu} = 0
 \end{aligned}$$

$$\partial_\mu F^{\mu\nu} = 0 \text{ ————— Eqn. of motion without source}$$

If there  $J^\mu$ ,

$$S = S_{\text{source}} + S_{\text{int}}$$

$$= -\frac{1}{16\pi} \int F_{\mu\nu} F^{\mu\nu} d^4x + \int J_\mu A^\mu d^4x$$

$$\delta S = 0 \Rightarrow$$

$$\frac{1}{4\pi} \partial_\mu F^{\mu\nu} + J^\nu = 0.$$

Energy-momentum Tensor

Lagrangian in 4D

$\mathcal{L}(\phi, \partial_\mu \phi)$  which is a function of scalar field  $\phi$ ,  $\partial_\mu \phi$

$$\frac{\partial \mathcal{L}}{\partial t} = 0 \rightarrow \text{energy conservation}$$

$$S = \int \mathcal{L} d^4x$$

$$\delta S = \int \left( \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi} \delta \partial_\nu \phi \right) d^4x$$

$$\downarrow$$

$$\cancel{\partial_\nu \left( \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi} \delta \phi \right)} - \left( \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi} \right) \delta \phi.$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi} = 0 \quad \text{Euler-Lagrange}$$

then

$$T_\rho{}^\nu \equiv - \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi} \partial_\rho \phi + \delta_\rho{}^\nu \mathcal{L}, \quad \partial_\nu T^{\mu\nu} = 0.$$

symmetric  $T^{\mu\nu}$ :  $\psi^{\mu\nu} = -\psi^{\nu\mu}$

$$T^{\mu\nu} \longrightarrow T^{\mu\nu} + \partial_\sigma \psi^{\mu\sigma\nu} = \tilde{T}^{\mu\nu}. \quad \partial_\mu \tilde{T}^{\mu\nu} = 0.$$

then require  $\tilde{T}^{\mu\nu} = \tilde{T}^{\nu\mu}$

$$\Rightarrow T^{\mu\nu} - T^{\nu\mu} = \partial_\sigma \psi^{\mu\sigma\nu} - \partial_\sigma \psi^{\nu\sigma\mu}$$

so in the future we assume for  $T$ ,  $T^{\mu\nu} = T^{\nu\mu}$

consider  $\partial_\nu T^{\mu\nu} = 0$

$$\partial_0 T^{\mu 0} + \partial_i T^{\mu i} = 0$$

for  $\mu=0$ :  $\partial_0 T^{00} + \partial_i T^{0i} = 0 \rightarrow \text{Energy Cons.}$

$\mu=j$ :  $\partial_0 T^{j0} + \partial_i T^{ji} = 0 \rightarrow \text{Momentum Cons.}$

The E.M. field:  $\phi \rightarrow A_\mu$

$$\mathcal{L}(A_\mu, \partial_\nu A_\mu)$$

Then  $T_{\rho}^{\nu} = -\frac{\partial \mathcal{L}}{\partial \partial_{\nu} A_{\rho}} \partial_{\rho} A_{\mu} + \delta_{\rho}^{\nu} \mathcal{L}$  .  $\mathcal{L} = -\frac{1}{16\pi} F_{\rho\mu} F^{\rho\mu}$

$$\frac{\partial \mathcal{L}}{\partial \partial_{\nu} A_{\mu}} = -\frac{1}{4\pi} F^{\mu\nu}$$

$$\therefore T_{\rho}^{\nu} = \frac{1}{4\pi} F^{\mu\nu} \partial_{\rho} A_{\mu} + \delta_{\rho}^{\nu} \left( -\frac{1}{16\pi} F^{\mu\sigma} F_{\mu\sigma} \right)$$

$$\Rightarrow T^{\mu\nu} = \eta^{\mu\rho} T_{\rho}^{\nu}$$

$$(* \eta^{\mu\rho} \delta_{\mu}^{\nu} = \eta^{\rho\nu})$$

$$= \frac{1}{4\pi} F^{\nu\sigma} \partial^{\mu} A_{\sigma} - \frac{1}{16\pi} \eta^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \neq T^{\nu\mu}$$

choose  $\psi^{\mu\nu\sigma} = -\frac{1}{4\pi} A^{\mu} F^{\nu\sigma}$  .  $\partial_{\sigma} \psi^{\mu\nu\sigma} = -\frac{1}{4\pi} A^{\mu} \partial_{\sigma} F^{\nu\sigma}$

$$\begin{aligned} \Rightarrow T^{\mu\nu} &= \frac{1}{4\pi} F^{\nu\sigma} \partial^{\mu} A_{\sigma} - \frac{1}{4\pi} (\partial_{\sigma} A^{\mu}) F^{\nu\sigma} - \frac{1}{16\pi} \eta^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \\ &= \frac{1}{4\pi} F^{\nu\sigma} (\partial^{\mu} A_{\sigma} - \partial_{\sigma} A^{\mu}) - \frac{1}{16\pi} \eta^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \end{aligned}$$

Concrete terms:

$$T^{00} = \frac{1}{4\pi} F^{0i} (\partial^0 A_i - \partial_i A^0) + \frac{1}{16\pi} (2|B|^2 - 2|E|^2)$$

$$= \frac{1}{8\pi} (|E|^2 + |B|^2) = W$$

energy density

$$T^{0i} = \frac{1}{4\pi} F^{i\sigma} (\partial^0 A_{\sigma} - \partial_{\sigma} A^0)$$

$$= \frac{1}{4\pi} (\vec{E} \times \vec{B})_i = S_i$$

Poynting vector

$$\Rightarrow \frac{\partial W}{\partial t} + \partial_i S^i = 0$$

for  $(i, j)$ .

$$\begin{aligned} T^{ij} &= -\frac{1}{4\pi} E^i E^j - \frac{1}{4\pi} B^i B^j + \frac{1}{8\pi} \delta_{ij} (|B|^2 + |E|^2) \\ &\equiv \sigma^{ij} \end{aligned}$$

Maxwell stress tensor

$$T = \begin{pmatrix} W & S^i \\ S^i & T^{ij} \end{pmatrix}$$

$$\partial_t W + \partial_i S^i = 0$$

$$\partial_t S^i + \partial_j \sigma^{ij} = 0$$

4D Momentum

$$P^\mu = \int T^{\mu\nu} d\Sigma_\nu$$

$$M^{\mu\nu} = \int (x^\mu dp^\nu - x^\nu dp^\mu).$$

$$= \int (x^\mu T^{\nu\sigma} d\Sigma_\sigma - x^\nu T^{\mu\sigma} d\Sigma_\sigma)$$

Also need symmetric:  $f^{\mu\nu\sigma} = x^\mu T^{\nu\sigma} - x^\nu T^{\mu\sigma}$

## Ch. 5 Propagation of E, B fields - Electronic Wave

$$\nabla \cdot \vec{E} = 0 \quad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0 \quad \nabla \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} = 0$$

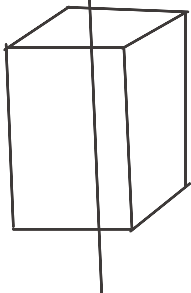
$$\vec{E} = \vec{E}^{(0)} e^{i\vec{k} \cdot \vec{r} - i\omega t}, \quad \vec{B} = \vec{B}^{(0)} e^{i\vec{k} \cdot \vec{r} - i\omega t}$$

$$\nabla \times \vec{E} = i\vec{k} \times \vec{E}, \quad \frac{\partial \vec{B}}{\partial t} = -i\omega \vec{B}$$

$$\therefore i\vec{k} \times \vec{E} - \frac{1}{c} i\omega \vec{B} = 0, \quad \frac{\omega}{c} = k$$

$$\therefore \vec{B} = \frac{\vec{k}}{k} \times \vec{E}$$

Wave guides



1) the geometry of the cross section

2) the material of wave guide

dissipation small enough  $\rightarrow$  ideal translational symmetry along  $\hat{z}$

Take

$$\begin{cases} \vec{E} = \vec{E}(x, y) e^{ikz - i\omega t} \\ \vec{B} = \vec{B}(x, y) e^{ikz - i\omega t} \end{cases}$$

$k$  - the wave vector along  $z$ -axis

consider Rectangle:

$$\nabla \cdot \vec{E} = 0 \quad \nabla \cdot \vec{B} = 0$$

$$(c=1) \quad \nabla \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0 \quad \nabla \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} = 0 \quad (\text{no free source})$$



get  $E_z \Rightarrow$

$$\nabla \cdot \tilde{\mathbf{E}} + ikE_z = 0$$

$$(\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_1 + E_z \hat{\mathbf{z}})$$

$$\vec{\mathbf{r}}_m = (0, 0, 1)$$

$$\nabla_1 \cdot \mathbf{E}_1 = -ikE_z,$$

similarly

$$\nabla_1 \cdot \mathbf{B}_1 = -ikB_z$$

from  $\nabla \times \tilde{\mathbf{E}}, \tilde{\mathbf{B}}$ :

$$\begin{cases} \nabla \times \tilde{\mathbf{E}} = i\omega \tilde{\mathbf{B}} \\ \nabla \times \tilde{\mathbf{B}} = i\omega \tilde{\mathbf{E}} \end{cases}$$

use  $(\vec{m} \cdot)$  and  $(\vec{m} \times)$ .

$$\Rightarrow \nabla_1 E_z - ik\tilde{E}_1 = i\omega \vec{m} \times \tilde{\mathbf{B}}_1$$

$\Rightarrow$

$$\begin{cases} ik\tilde{E}_x - i\omega\tilde{B}_y = \partial_x \tilde{E}_z \\ ik\tilde{E}_y + i\omega\tilde{B}_x = \partial_y \tilde{E}_z \end{cases}$$

B similarly

$$\Rightarrow \tilde{E}_x = \frac{i}{\omega^2 - k^2} (k \partial_x \tilde{E}_z + \omega \partial_y \tilde{B}_z)$$

$$\tilde{E}_y = \frac{i}{\omega^2 - k^2} (k \partial_y \tilde{E}_z - \omega \partial_x \tilde{B}_z)$$

$$\tilde{B}_x = \frac{i}{\omega^2 - k^2} (k \partial_x \tilde{B}_z - \omega \partial_y \tilde{E}_z)$$

$$\tilde{B}_y = \frac{i}{\omega^2 - k^2} (k \partial_y \tilde{B}_z + \omega \partial_x \tilde{E}_z)$$

then.  $\nabla^2 E - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = 0$

$$\Rightarrow \nabla_1^2 \tilde{E}_z + (\omega^2 - k^2) \tilde{E}_z = 0$$

Boundary condition

Local surface of the wave guide.  $\vec{n} \times \tilde{\mathbf{E}}|_s = 0 \Rightarrow E_{1s} = E_z|_s = 0,$

so for  $\tilde{\mathbf{B}} \Rightarrow \frac{\partial}{\partial n} \tilde{\mathbf{B}}|_s = 0$

\* TEM Mode  $\vec{E}_z = \vec{B}_z = 0.$

$$\begin{cases} \nabla \cdot \vec{E} = 0 \\ \nabla \times \vec{E} = i\omega \vec{B} \end{cases} \Rightarrow \begin{cases} \nabla_1 \cdot \vec{E}_1 = 0 \\ \nabla_1 \times \vec{E}_1 = 0 \end{cases}$$

take  $E_1 = -\nabla_1 \phi \Rightarrow \nabla_1^2 \phi = 0 \xrightarrow{\text{Boundary}} \phi = 0$  Trivial

\* TM Mode.  $B_z = 0 \quad E_z \neq 0.$

$$\begin{cases} \nabla_1^2 E_z + \underbrace{(\omega^2 - k^2)}_{\Omega^2} E_z = 0 \\ E_z|_s = 0 \end{cases}$$

$$\nabla_1^2 E_z + \Omega^2 E_z = 0.$$

$$\Rightarrow \vec{E} = A_{nm} \sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{m\pi}{b} y\right)$$

$$\begin{cases} E_z(r, t) = \sum_{nm} A_{nm} \sin(\dots) \sin(\dots) e^{ikz - i\omega t} \\ \omega^2 = k^2 + \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 \end{cases}$$

$$\therefore v_{pk} = \frac{\omega}{k} > 1.$$

$$v_g = \frac{d\omega}{dk} = \frac{1}{v_{pk}} < 1.$$

Retarded Potential

$\Rightarrow$  Lecture note

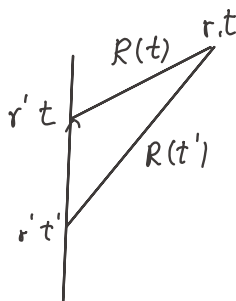
The electric field

$$\vec{E} = \frac{e(1-v^2)(\vec{R} - \vec{v}R)}{(R - \vec{v} \cdot \vec{R})^3} + \frac{e\vec{R} \times [(\vec{R} - \vec{v}R) \times \vec{a}]}{(R - \vec{v} \cdot \vec{R})^3}$$

① Const  $\vec{v}$ ,  $\vec{a} = 0$ .

$$\vec{E}(\vec{R}, t) = \frac{e(1-v^2)(\vec{R} - \vec{v}R)}{(R - \vec{v} \cdot \vec{R})^3}$$

$$\vec{R} = \vec{R}(t') = \vec{r} - \vec{r}'(t')$$



$$\vec{R}(t') = \vec{r} - \vec{r}'(t')$$

$$\vec{R} - \vec{v}R = \vec{R} - \vec{v}(t - t')$$

(note:  $c=1$ )

$$= \vec{r} - \vec{r}'(t') - \vec{v} \Delta t$$

$$= \vec{r} - \vec{r}'(t) \quad \leftarrow$$

$$\Delta t = t - t', \quad \vec{r}'(t) = \vec{r}'(t') + \vec{v} \Delta t$$

$$\Rightarrow \vec{R}(t) = \vec{r} - \vec{r}'(t)$$

$$\text{and } R(t') - \vec{v} \cdot \vec{R}(t') = (1-v^2)R(t') - \vec{v} \cdot \vec{R}(t)$$

Note that,

$$\vec{R}(t') = \vec{v}R(t') + \vec{R}(t)$$

$$\Rightarrow ( )^2: \quad R(t') = \frac{\vec{v} \cdot \vec{R}(t) + \sqrt{(\vec{v} \cdot \vec{R}(t))^2 + (1-v^2)R^2(t)}}{(1-v^2)}$$

$$\text{Define: } \vec{v} \cdot \vec{R}(t) = vR \cos \theta$$

$$\begin{aligned} R - \vec{v} \cdot \vec{R} &= \left( vR \cos \theta + \sqrt{v^2 R^2 \cos^2 \theta + (1-v^2)R^2} \right) - vR \cos \theta \\ &= \sqrt{v^2 R^2 \cos^2 \theta + (1-v^2)R^2} \end{aligned}$$

$$\Rightarrow \vec{E}(\vec{r}, t) = \frac{e\vec{R}}{R^3} \frac{1-v^2}{(1-v^2 \sin^2 \theta)^{3/2}}$$

# \* Radiation

Pointing vector  $\vec{S} = \frac{1}{4\pi} \vec{E} \times \vec{B}$  if  $S \sim R^{-2} \Rightarrow \checkmark$

$$= \frac{1}{4\pi} \vec{E} \times \left( \frac{\vec{R}}{R} \times \vec{E} \right)$$

$$= \frac{1}{4\pi} (\vec{n} |\vec{E}|^2 - (\vec{n} \cdot \vec{E}) \vec{E})$$

||  
0

$$\vec{E} = \frac{e\vec{R} \times [(\vec{R} - \vec{v}R) \times \vec{a}]}{(\vec{R} - \vec{v} \cdot \vec{R})^3}$$

$$\therefore \vec{S} = \frac{|\vec{E}|^2}{4\pi} \vec{n}$$

$$= \frac{e^2}{4\pi R^2} \left( \frac{\vec{n} \times [(\vec{n} - \vec{v}) \times \vec{a}]}{(1 - v \cos \theta)^3} \right)^2 \vec{n}$$

The power radiated per solid angle.

$$\frac{dp}{d\Omega} = \vec{n} \cdot \vec{S} R^2$$

$$= \frac{e^2}{4\pi} \left( \frac{\vec{n} \times [(\vec{n} - \vec{v}) \times \vec{a}]}{(1 - v \cos \theta)^3} \right)^2$$

$$\vec{n} \cdot \vec{a} \rightarrow a \cos \theta$$

$$= \frac{e^2 a^2}{4\pi} \sin^2 \theta$$

Total Power

$$P = \int \frac{dp}{d\Omega} d\Omega = \frac{e^2}{4\pi} a^2 \int \sin^2 \theta d\Omega = \frac{2}{3} e^2 a^2 \rightarrow \cdot \frac{1}{c^3}$$

The generic case

$$\frac{dp^\mu}{dx^\nu} \rightarrow \text{Scalar.}$$

$$P(t) = \frac{2}{3} e^2 a^2 = \frac{2}{3} \frac{e^2}{m^2} \left( \frac{dp}{dt} \right)^2 \longrightarrow \frac{2}{3} \frac{e^2}{m^2} \frac{dp^\mu}{d\tau} \frac{dp_\mu}{d\tau}$$

$$\frac{dp^\mu}{dt} = \gamma m \left( \frac{d\mathbf{v}}{dt}, \frac{d\gamma\mathbf{v}}{dt} \right)$$

$$= \gamma m \left( \gamma^3 \mathbf{v} \cdot \mathbf{a}, \gamma^3 \mathbf{v} (\mathbf{v} \cdot \mathbf{a}) + \gamma \mathbf{a} \right)$$

$$\begin{aligned} \therefore P(t) &= \frac{2e^2}{3m^2} m^2 \gamma^2 \left( -\gamma^6 (\mathbf{v} \cdot \mathbf{a})^2 + \gamma^2 a^2 + \gamma^6 v^2 (\mathbf{v} \cdot \mathbf{a})^2 + 2\gamma^4 (\mathbf{v} \cdot \mathbf{a})^2 \right) \\ &= \frac{2e^2 \gamma^2}{3} \left[ -\gamma^6 (\mathbf{v} \cdot \mathbf{a})^2 (1-v^2) + \gamma^2 [a^2 + 2\gamma^2 (\mathbf{v} \cdot \mathbf{a})^2] \right] \\ &= \frac{2e^2}{3} \gamma^6 [(1-v^2) a^2 + (\mathbf{v} \cdot \mathbf{a})^2] \\ &= \frac{2e^2}{3} \gamma^6 (a^2 - (\mathbf{v} \times \mathbf{a})^2) \end{aligned}$$

$$\mathbf{a} \parallel \mathbf{v} \quad P(t) = \frac{2e^2}{3} \gamma^6 a^2 = \frac{2e^2}{3m} \left( \frac{dp}{dt} \right)^2$$

$$\dagger \text{ Energy: } E^2 = m^2 + p^2$$

$$E dE = p dp \Rightarrow \frac{dE}{dx} = \frac{p}{E} \frac{dp}{dx} = v \frac{dp}{dx}$$

$$\frac{\text{radiation energy}}{\text{Energy supplied}} = \frac{P(t)}{\frac{dE}{dt} + P(t)} \quad P \ll \frac{dE}{dt}$$

$$\simeq \frac{P(t)}{dE/dt} \longrightarrow v dE/dx$$

$$= \frac{2e^2}{3m} \frac{1}{v^2} \left( \frac{dE}{dx} \right)^2 \frac{1}{v \frac{dE}{dx}} = \frac{2e^2}{3mv} \left( \frac{dE}{dx} \right)$$

for  $v \rightarrow c = 1$ :

$$\frac{2e^2}{3m} \frac{dE}{dx} \sim 10^{-12} / 10^{-13} \sim 10 \text{ MeV per meter}$$

2) Circular accelerator

$$\left\{ \begin{array}{l} |\vec{v}| \rightarrow v \sim \text{const} \\ a = \frac{v^2}{R} \end{array} \right.$$

$$P(t) = \frac{e^2}{3} \gamma^6 (a^2 - (\vec{v} \times \vec{a})^2) = \frac{e^2}{3} \gamma^4 a^2$$

$$\frac{d\vec{p}}{dt} \approx m\gamma \vec{a}$$

$$P(t) = \frac{e^2 \gamma^2}{3m} \left( \frac{d\vec{p}}{dt} \right)^2$$

For  $R \sim 100$  meter,  $m\gamma \sim 10$  GeV,  $v \rightarrow c = 1$ ,  $T = \frac{2\pi R}{c} \sim 1 \mu s$ .

$$P(t) \cdot \frac{2\pi R}{c} \sim 10 \text{ MeV} \quad \sim 1\% \text{ of Energy}$$

\* Angular distribution of power

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi} \left( \frac{|\vec{n} \times [(\vec{n} \times \vec{v}) \times \vec{a}]|}{(1 - v \cos\theta)^3} \right)^2$$

Recall  $t, t'$ :

$$t = t' + |\vec{r} - \vec{r}'|$$

$$\Delta P = \int_{t_1}^{t_2} \frac{dP}{d\Omega} dt = \int_{t'_1}^{t'_2} \frac{dP}{d\Omega} \frac{dt}{dt'} dt'$$

$$\frac{dt}{dt'} = 1 + \frac{d|\vec{r} - \vec{r}'|}{dt'} = 1 - \frac{\frac{d\vec{r}}{dt'} \cdot (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|}$$

$$= 1 - \vec{v} \cdot \vec{n}$$

$$\therefore \Delta P = \int_{t'_1}^{t'_2} \frac{dP(t)}{d\Omega} (1 - v \cos\theta) dt' \equiv \int_{t'_1}^{t'_2} \frac{dP(t')}{d\Omega} dt'$$

$$\therefore \frac{dP(t')}{d\Omega} = \frac{dP(t)}{d\Omega} (1 - v \cos\theta)$$

$$\Rightarrow \frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi} \frac{|\vec{n} \times [(\vec{n} - \vec{v}) \times \vec{a}]|^2}{(1 - v \cos\theta)^5}$$

not  
~ Larmor formula

\* Linear  $\vec{v} \parallel \vec{a}$

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi} \frac{a^2 \sin^2 \theta}{(1 - v \cos \theta)^5}$$

$$\frac{d}{d\theta} \left( \frac{dP}{d\Omega} \right) = 0 \Rightarrow \frac{2 \sin \theta \cos \theta}{(1 - v \cos \theta)^5} - \frac{5 \sin^2 \theta \sin \theta v}{(1 - v \cos \theta)^6} = 0.$$

$$\left\{ \begin{array}{l} \textcircled{1} \sin \theta = 0 \end{array} \right.$$

$$\left\{ \begin{array}{l} \textcircled{2} \cos \theta = \frac{\sqrt{1 + 15v^2} - 1}{3v} \end{array} \right.$$

total Power

$$\int \frac{dP(r)}{d\Omega} d\Omega = \frac{2}{3} e^2 a^2 \gamma^6$$

\* Circular,  $\vec{v} \perp \vec{a}$

$$\left\{ \begin{array}{l} \vec{v} \rightarrow (0, 0, v) \\ \vec{a} \rightarrow (0, a, 0) \end{array} \right. \quad \vec{n} \rightarrow (s\theta \cos \varphi, s\theta \sin \varphi, c\theta).$$

$$\vec{n} \cdot [(\vec{n} \times \vec{v}) \times \vec{a}] = (\vec{n} - \vec{v}) a \sin \theta \cos \varphi - (1 - v \cos \theta) \vec{a}$$

$$\Rightarrow \frac{dP}{d\Omega} = \frac{e^2 a^2}{4\pi} \frac{1}{(1 - v \cos \theta)^3} \left[ 1 - \frac{\sin^2 \theta \cos^2 \varphi}{\gamma^2 (1 - v \cos \theta)^2} \right] \rightarrow \theta_{\max} = 0.$$

$$\int \frac{dP}{d\Omega} = \frac{2}{3} e^2 \gamma^4 a^2$$

Frequency distribution.

$$\tilde{G}(t) = \frac{1}{\sqrt{4\pi}} |R\tilde{E}|, \quad \frac{dP}{d\Omega} = |\tilde{G}(t)|^2$$

$$\hat{g}(\omega) = \frac{1}{\sqrt{2\pi}} \int \tilde{G} e^{i\omega t} dt$$

$$|g(\omega)|^2 + |g(-\omega)|^2 = 2|g(\omega)|^2$$

↓

$$\Rightarrow \frac{dW}{d\Omega} = \int_{-\infty}^{+\infty} \hat{g}(\omega) \hat{g}^*(\omega) d\omega = \int_{-\infty}^{+\infty} d\omega \frac{d^2 I(\omega, \vec{n})}{d\omega d\Omega}$$

(Parseval)

$$\therefore \hat{g}(\omega) = \frac{e}{2\sqrt{2}\pi} \int_{-\infty}^{+\infty} e^{i\omega t} \left[ \frac{\vec{n} \times (\vec{n} - \vec{v}) \times \vec{a}}{(1 - v \cos\theta)^{3/2}} \right] dt$$

ret.

$$= \frac{e}{2\sqrt{2}\pi} \int_{-\infty}^{+\infty} e^{i\omega t} \left[ \frac{\vec{n} \times (\vec{n} - \vec{v}) \times \vec{a}}{(1 - v \cos\theta)^{3/2}} \right] dt'$$

↗ t → t'

t' + R/c

R ~ r - n · r\_0(t')

$$\frac{d}{dt'} \left( \frac{\vec{n} \times (\vec{n} \times \vec{v})}{1 - \vec{n} \cdot \vec{v}} \right)$$

$$\longrightarrow \frac{1}{c^3} \frac{e^2 \omega^2}{4\pi} \left| \int \vec{n} \times (\vec{n} \times \vec{v}) e^{i\omega(t' - \frac{\vec{n} \cdot \vec{r}_0(t')}{c})} dt' \right|^2$$

Cerenkov radiation

$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2 \omega^2}{4\pi c^3} \sqrt{\epsilon} \left| \int \vec{n} \times (\vec{n} \times \vec{v}) e^{i\omega(t' - \sqrt{\epsilon} \vec{n} \cdot \vec{r}_0(t')/c)} dt' \right|^2$$

for  $\vec{v}$  and  $\vec{a} = 0$

$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2 \omega^2 \sqrt{\epsilon}}{c^3} v^2 \sin^2 \theta \left| \delta\left(\omega(1 - \sqrt{\epsilon} \frac{v}{c} \cos\theta)\right) \right|^2$$

$$= \frac{e^2 \sqrt{\epsilon}}{c^3} v^2 \sin^2 \theta \left| \delta(1 - \sqrt{\epsilon} \frac{v}{c} \cos\theta) \right|^2$$



Thompson scattering

non-rel

$$\vec{n} \cdot \vec{a} = \cos(\theta)$$

$$\frac{dP}{d\Omega} = \frac{e^2 a^2}{4\pi} \sin^2(\theta) \rightarrow \frac{e^2}{4\pi} (a^2 - (\vec{n} \cdot \vec{a})^2)$$

classic

$$\vec{E} = E_0 \vec{e} e^{i(\vec{k} \cdot \vec{r} - \omega t)} = m \vec{a}$$

↓

$$(\cos\psi, \sin\psi, 0)$$

$$\vec{n} = (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta)$$

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{e^4}{8\pi m^2} |E_0|^2 (1 - \sin^2\theta \cos^2(\varphi - \psi))$$

↓

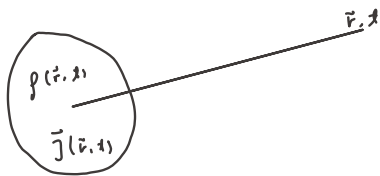
$$\left\langle \left\langle \frac{dP}{d\Omega} \right\rangle \right\rangle_{\varphi} = \frac{e^4}{16\pi m^2} |E_0|^2 (1 + \cos^2\theta)$$

$$\text{Cross section } \frac{d\sigma}{d\Omega} = \frac{dP/d\Omega}{I} = \frac{e^4}{2m^2} (1 + \cos^2\theta)$$

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{2}{3} \cdot 4\pi R_0^2$$

$\hookrightarrow R_0 = \frac{e^2}{mc^2}$

# Ch. 8. Radiating Systems



$$\phi(\vec{r}, t) = \int \frac{\rho(\vec{r}', t - |\vec{r} - \vec{r}'|)}{|\vec{r} - \vec{r}'|} d^3\vec{r}'$$

$$\vec{A}(\vec{r}, t) = \frac{1}{c} \int \frac{\vec{J}(\vec{r}', t - |\vec{r} - \vec{r}'|)}{|\vec{r} - \vec{r}'|} d^3\vec{r}'$$

Note  $\vec{E} = -\frac{1}{c} \nabla \times \vec{B}$ .  $\rightarrow$  Only to study  $\vec{A}$

$$\rho \sim e^{-i\omega t}, \quad \vec{J} \sim e^{-i\omega t}$$

$$\vec{A}(\vec{r}, t) = e^{-i\omega t} \int \frac{\vec{J}(\vec{r}') e^{ik|\vec{r} - \vec{r}'|}}{|\vec{r} - \vec{r}'|} d^3\vec{r}' = \vec{A}(\vec{r}) e^{-i\omega t}$$

$$\begin{aligned} & \frac{e^{ikr}}{r} - x'_i \partial_i \frac{e^{ikr}}{r} + \frac{1}{2} x'_i x'_j \partial_i \partial_j \frac{e^{ikr}}{r} \\ &= \frac{e^{ikr}}{r} \left( 1 + \frac{x'_i x'_i}{r^2} - \dots \right) \end{aligned}$$

$$\therefore \vec{A} = \frac{e^{ikr}}{r} \underbrace{\left( \int \vec{J} d^3\vec{r}' \right)}_{\substack{\text{static case, } = 0}} + \frac{e^{ikr}}{r^2} \left( \frac{x'_i}{r} - ik x'_i \right) \int J x'_i d^3\vec{r}'$$

Apply the formula  $\partial_i (x'_j J_j) = J_j + x'_j \nabla \cdot \vec{J}$ .

$$= J_j - x'_j \frac{\partial \rho}{\partial t} \Rightarrow \int \partial'_i (x'_j J_j) d^3\vec{r}' = 0.$$

$$\int J_i d^3\vec{r}' = -i\omega \int x'_j J_j d^3\vec{r}'$$

$$= -i\omega p_j \quad \leftarrow \text{Electric Dipole}$$

$$A'' = -\frac{e^{ikr}}{r} \cdot i\omega p$$

$$\therefore \vec{A} = -\frac{ik}{r} \vec{p} e^{ikr}$$

$$\vec{B} = \nabla \times \vec{A}$$

$$\Rightarrow B_i = -\epsilon_{ijk} \partial_j \left( \frac{ik}{r} p_k e^{ikr} \right)$$

$$\begin{aligned} \Rightarrow \vec{B} &= \left( \frac{k^2}{r} + \frac{ik}{r^2} \right) (\vec{n} \times \vec{p}) e^{ikr} \\ &= k^2 (\vec{n} \times \vec{p}) e^{ikr} \left( \frac{1}{kr} + \frac{i}{k^2 r^2} \right) \end{aligned}$$

$$\vec{E} = -\frac{1}{ik} \nabla \times \vec{B}$$

$$\begin{aligned} &= ik^2 \epsilon_{ijk} \partial_i \left( \epsilon_{klm} n_l p_m e^{ikr} \left( \frac{1}{kr} + \frac{i}{k^2 r^2} \right) \right) \\ &= -k^3 \left[ \vec{n} \times (\vec{n} \times \vec{p}) \left( \frac{1}{kr} e^{ikr} \right) \right. \\ &\quad \left. + (3(\vec{p} \cdot \vec{n}) \vec{n} - \vec{p}) \left( \frac{1}{r^3} - \frac{ik}{r^2} \right) e^{ikr} \right] \end{aligned}$$

① Near zone  $kr \ll 1$

$$\vec{B} \sim k^2 (\vec{n} \times \vec{p}) \left( \frac{1}{kr} + \frac{i}{k^2 r^2} \right) \ll \frac{1}{r^3} \quad kr \ll 1 \quad \sim r \ll \frac{1}{k}$$

$$\vec{E} \sim E_{\text{static}} \sim \frac{1}{r^3}$$

② Far zone  $kr \gg 1$  (radiation zone)

$$\vec{B} \sim k^2 (\vec{n} \times \vec{p}) \frac{1}{r} e^{ikr}$$

$$\vec{E} \sim -k^2 (\vec{n} \times (\vec{n} \times \vec{p}) \frac{1}{r}) e^{ikr} \propto \frac{1}{r}$$

$$\left\langle \frac{dP}{d\Omega} \right\rangle_T = \frac{1}{8\pi} |\vec{E} \times \vec{B}|^2$$

$$P = \int d\Omega = \frac{4}{3} k^4 p^2$$

Electric quadrupole

$$\vec{A}^{(4)} = \frac{ik}{2} e^{ikr} \left( \frac{ik}{r} - \frac{1}{r^2} \right) \int \vec{r}' (\vec{n} \cdot \vec{r}') \rho(\vec{r}') d^3 \vec{r}'$$

$kr \gg 1$ , the leading order

$$\vec{A}^{(4)} = \frac{ik}{2} e^{ikr} \frac{ik}{r} \int \vec{r}' (\vec{n} \cdot \vec{r}') \rho(\vec{r}') d^3 \vec{r}'$$

$$\vec{B}^{(4)} = \nabla \times \vec{A}^{(4)} = \frac{1}{2} (ik)^3 \frac{1}{r} \vec{n} \times \int \vec{r}' (\vec{n} \cdot \vec{r}') \rho(\vec{r}') d^3 \vec{r}' \quad (\nabla \mapsto ik \vec{n})$$

$$\vec{E} = \frac{i}{k} \nabla \times \vec{B} = -\vec{n} \times \vec{B}$$

$$\langle \vec{S} \rangle = \frac{1}{8\pi} \vec{E} \times \vec{B}$$

$$= -\frac{1}{8\pi} (\vec{n} \times \vec{B}) \times \vec{B}$$

$$= \frac{1}{8\pi} (\vec{n} |\vec{B}|^2 - (\vec{n} \cdot \vec{B}) \vec{B}) = \frac{1}{8\pi} \vec{n} |\vec{B}|^2$$

$$\frac{dP}{d\Omega} = (\vec{n} \cdot \vec{S}) r^2 = \frac{1}{8\pi} |\vec{B}|^2 r^2$$

$$Q_{ij} = \int (3x_i' x_j' - r'^2 \delta_{ij}) \rho(\vec{r}') d^3 \vec{r}' \quad Q_i = Q_{ij} n_j$$

$$\mapsto \frac{1}{3} (\vec{n} \times \vec{Q})_i = \frac{1}{3} \varepsilon_{ijk} n_j Q_k$$

$$= \frac{1}{3} \varepsilon_{ijk} \int (3n_j x_k' n_k x_i' - n_j n_k r'^2) \rho(\vec{r}') d^3 \vec{r}'$$

$$\Rightarrow \vec{B}^{(4)} = \frac{(ik)^3}{2r} \cdot \frac{1}{3} (\vec{n} \times \vec{Q})$$

$$\frac{dP}{d\Omega} = \frac{k^6}{288\pi} |\vec{n} \times \vec{Q}|^2$$

$$\therefore \frac{dP}{d\Omega} = \frac{k^6}{288\pi} [Q_i Q_i^* - n_i Q_j \eta_j Q_j^*]$$

$$= \frac{k^6}{288\pi} (Q_j \eta_j Q_{i_k}^* n_k - Q_j Q_{k_k}^* n_i \eta_j n_k n_k)$$

Let  $\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$$\int n_i \eta_j d\Omega \quad \text{and} \quad \int n_i \eta_j n_k n_k d\Omega.$$

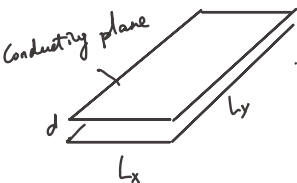
$$\int \eta_j n_k d\Omega = \int \eta_j n_k \sin\theta d\theta d\phi = \begin{cases} 0 & j \neq k. \\ \int n_k^2 d\Omega = \frac{4}{3}\pi. \end{cases}$$

$$\Rightarrow P = \frac{k^6}{288\pi} \left[ \frac{4\pi}{3} Q_j Q_j^* - \frac{4\pi}{15} (Q_j Q_{kk}^* + Q_j Q_j^* + Q_{ij} Q_{ji}^*) \right]$$

$$= \frac{k^6}{288\pi} \frac{12\pi}{15} Q_j Q_j^* = \frac{k^6}{360} Q_j Q_j^* \propto k^6$$

And  $\frac{dP}{d\Omega} = \frac{k^6 Q_j^2}{512\pi} \sin^2 2\theta$ , with  $Q_{ij} = \begin{pmatrix} -\frac{1}{2}Q_3 & & \\ & -\frac{1}{2}Q_3 & \\ & & Q_3 \end{pmatrix}$

Casimir force



$$L_x L_y \gg d.$$

electromagnetic vacuum.

$$E_\omega = \frac{1}{2} \hbar \omega$$

momentum for the eigen-mode

$$k_i = \frac{\pi}{L_i} n_i$$

$$L_{x,y} \rightarrow \infty.$$

$$k_z = \frac{\pi}{d} n_z \text{ discrete.}$$

$$\omega = k c = c \left[ \left( \frac{2}{L_x} n_x \right)^2 + \left( \frac{2}{L_y} n_y \right)^2 + \left( \frac{2\pi}{d} n_z \right)^2 \right]^{1/2}$$

For each  $k$ -mode, 2 states

$$\begin{aligned} \therefore E &= \sum_n \frac{1}{2} \hbar \omega \cdot 2 = \sum \hbar \pi c \left( \frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{d^2} \right)^{1/2} \\ &= \int dx dy \sum_z \left( x^2 + y^2 + \frac{n_z^2}{d^2} \right)^{1/2} \hbar \pi c L_x L_y \\ &= 2\pi^2 \hbar c L_x L_y \sum_z \int_0^\infty r dr \left( r^2 + \left( \frac{n_z}{d} \right)^2 \right)^{1/2} \end{aligned}$$

If no conducting planes, the total energy is

$$\begin{aligned} E_{\text{total, free}} &= \hbar c \pi L_x L_y d \int_0^\infty dx dy dz (x^2 + y^2 + z^2)^{1/2} \\ &= 2\pi^2 \hbar c L_x L_y d \int_0^\infty (r^2 + z^2)^{1/2} r dr \end{aligned} \quad \begin{array}{l} r \sim \frac{1}{L} \\ z \sim \frac{1}{d} \end{array}$$

$$E = \frac{2^2 \hbar c L_x L_y}{d^3} \sum' \int d\xi (\xi^2 + \eta^2)^{1/2}$$

$$\therefore \Delta E = \frac{\pi^2 \hbar c L_x L_y}{d^3} \left[ \sum' \int d\xi (\xi^2 + \eta^2)^{1/2} - \int d\eta \int d\xi (\xi^2 + \eta^2)^{1/2} \right]$$

$$= \pi^2 \hbar c L_x L_y / d^3 \cdot \left[ F(\eta) - \int_0^\infty d\eta F(\eta) \right]$$

$$= \pi^2 \hbar c L_x L_y / d^3 \left( \frac{1}{2} F(0) + \sum_{n=1}^\infty F(n) - \int_0^\infty d\eta F(\eta) \right)$$

Euler-Maclaurin

$$\sum_{n=1}^\infty F(n) - \int_0^\infty d\eta F(\eta) = -\frac{1}{2} F(0) - \frac{1}{2!} B_2 F'(0) - \frac{1}{4!} B_4 F'''(0) - \dots$$

$$B_n : \text{Bernolli numbers} \quad \frac{y}{e^y - 1} = \sum B_n y^n$$

$$\text{If } \eta \rightarrow \infty, \quad F'(\eta) = -\alpha^2 \frac{\partial f}{\partial \eta} \Big|_{\alpha=\eta^2} = -2\eta^2 f\left(\frac{\pi}{2}\eta\right) \quad F''(0) = 0$$

$$F''(0) = -4$$

$$\therefore \Delta E = -\frac{\pi^2 \hbar c L_x L_y}{4 d^3} \left( \frac{1}{4!} B_4 F''(0) \right) = -\pi^2 \hbar c L_x L_y / 24 d^3 \quad F = -\pi^2 \hbar c L_x L_y / 24 d^3 \frac{\omega^2}{d^4}$$

Topological field theory  $\rightarrow$  effective field

$$S_{\text{eff}} = \frac{1}{2} c_1 \frac{e^2}{h} \int dx dy dz \varepsilon^{\alpha\beta\gamma} A_\alpha \partial_\beta A_\gamma$$

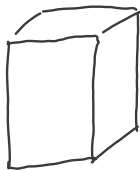
[ Chern number 1st (# of Landau being occupied) ]

$$J^\alpha = \delta S / \delta A_\alpha = c_1 e^2 / h \varepsilon^{\alpha\beta\gamma} \partial_\beta A_\gamma$$

$$\Rightarrow J^1 = c_1 e^2 / h (-\partial_x A_y - \partial_y \phi) = \sigma_{xy} \bar{E}_y = (c_1 \frac{e^2}{h}) E_y$$

$$J^3 = c_1 e^2 / h B_z$$

Magneto-electric effect.



$$\varepsilon_k = \frac{\sqrt{\hbar^2 V_F^2 (k_x^2 + k_y^2) + \Delta^2}}{\sqrt{\hbar^2 c^2 + m^2 c^4}}$$

$$\Delta \varepsilon \approx 2 \Delta.$$

$$\sigma_H^i = \frac{1}{2} \sin(\alpha) \cdot \frac{e^2}{h}$$

$$J_H = \frac{1}{2} \frac{e^2}{h} E_z \hat{e}_y^i$$

$$(I = J L_B)$$

$$\vec{M} = -\frac{1}{2} \frac{e^2}{hc} E_z \hat{e}_z$$

$\uparrow$   
 $\propto \Rightarrow$   
 $n = B - 4\pi M + 4\pi(n + \frac{1}{2})E$

$$p = -(n + \frac{1}{2}) \propto B$$

$$\mathcal{D} = E + 4\pi p - 4\pi(n + \frac{1}{2})E$$

$$S_{\text{tot}} = \int d^4x \left( F^{\mu\nu} F_{\mu\nu} \frac{1}{4\pi} + \frac{1}{2} F^{\mu\nu} p_{\mu\nu} + J_\mu A^\mu \right) + \frac{1}{4\pi} \int d^4x \, p_3 \varepsilon^{\mu\nu\sigma} F_{\mu\nu} F_\sigma$$

$$\delta S = 0 :$$

$$\frac{1}{4\pi} \partial_\mu F^{\mu\nu} + \partial_\mu p^{\mu\nu} + \frac{1}{4\pi} \partial_\mu (p^3 \varepsilon^{\mu\nu\sigma} F_{\sigma\rho}) = -J^\nu$$

$$\text{For } 2n\text{-dim: } S_{\text{top}} = c_2 \frac{e^2}{h} \frac{1}{8} \int d^5x \varepsilon^{abcde} A_a \partial_b A_c \partial_d A_e$$

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