

第九章 电磁波

§1. 一维波

波方程

Intro. 一维波的波方程可写为 $f(x,t) = g(x-vt)$ (向右以 v 传播)

Def. 经典波方程: $\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$, f 为位移, v 为波速, 称为经典波方程

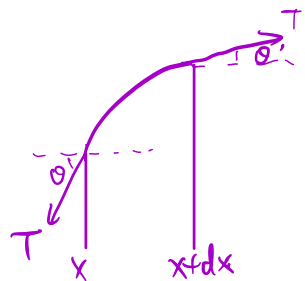
Tip. $f(x,t) = g(x-vt)$ or $f(x,t) = h(x+vt)$ 均满足该方程

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial \left(\frac{\partial g}{\partial u} \frac{\partial u}{\partial x} \right)}{\partial u} \frac{\partial u}{\partial x} = \frac{\partial^2 g}{\partial u^2}, \quad \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} = \frac{1}{v^2} \frac{\partial \left(\frac{\partial g}{\partial u} \frac{\partial u}{\partial t} \right)}{\partial u} \frac{\partial u}{\partial t} = \frac{1}{v^2} (-v)^2 \frac{\partial^2 g}{\partial u^2} = \frac{\partial^2 g}{\partial u^2}, \text{ 满足方程}$$

另一同理, 根据它为一阶偏微分方程, 其通解 $f(x,t)$ 可表示为

$$f(x,t) = \lambda_1 g(x-vt) + \lambda_2 h(x+vt)$$

Ex. 弦波方程



$$T(\sin \theta' - \sin \theta) = dl \times \lambda \times a \quad (a \approx \frac{\partial^2 f}{\partial t^2}, \text{ 由于 } \theta \rightarrow 0, \sin \theta \approx \tan \theta, dl \approx dx)$$

$$\Rightarrow T \left[\left(\frac{\partial f}{\partial x} \right)_{x+dx} - \left(\frac{\partial f}{\partial x} \right)_x \right] = \lambda \frac{\partial^2 f}{\partial t^2} dx \Rightarrow T \frac{\partial^2 f}{\partial x^2} = \lambda \frac{\partial^2 f}{\partial t^2}$$

$$\text{即 } \frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}, \quad \text{或 } \frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}, \quad v = \sqrt{\frac{T}{\lambda}}$$

边界条件: 透射反射

Scene: 有两根半无限长弦连在一起, 弦上张力均为 T , 但线密度不同, 为 λ_1 与 λ_2 , 向右以 v_1 传播

设节点为 $x=0$, 则波入射波的复开式为

$$\tilde{f}_I(z,t) = \tilde{A}_I e^{i(k_1 z - \omega t)} \quad (z < 0, \quad k_1 = \frac{2\pi}{\lambda_1}) \quad \text{并假设其延伸至 } -\infty.$$

若否, 总可展为如此的延伸至 $-\infty$ 函数的叠加, 即 $\tilde{f}(z,t) = \int_{-\infty}^{+\infty} \tilde{A}(k) e^{i(kz - \omega t)} dk$

设产生反射波 $\tilde{f}_R(z,t)$ 与透射波 $\tilde{f}_T(z,t)$, 形式分别为

$$\tilde{f}_R(z,t) = \tilde{A}_R e^{i(k_1 z - \omega t)}, \quad \tilde{f}_T(z,t) = \tilde{A}_T e^{i(k_2 z - \omega t)} \quad (\omega \text{ 由入射波决定})$$

$$\text{则有 } \tilde{f}(z,t) = \begin{cases} \tilde{f}_I(z,t) + \tilde{f}_R(z,t) & (z < 0) \\ \tilde{f}_T(z,t) & (z > 0) \end{cases}$$

$$\text{且需满足: } 1^\circ \operatorname{Re}(\tilde{f}_T(0,t)) = \operatorname{Re}(\tilde{f}_I(0,t) + \tilde{f}_R(0,t)) \quad (z=0 \text{ 处 绳子连续})$$

$$2^\circ \left(\frac{\partial}{\partial z} \operatorname{Re}(\tilde{f}_T(z,t)) \right)_{z=0} = \frac{\partial}{\partial z} \left(\operatorname{Re}(\tilde{f}_I(z,t) + \tilde{f}_R(z,t)) \right)_{z=0} \quad (\text{在 } z=0 \text{ 处 绳子光滑})$$

$$1^\circ \Leftrightarrow \tilde{f}_T(0,t) = \tilde{f}_I(0,t) + \tilde{f}_R(0,t) \Leftrightarrow \tilde{A}_T = \tilde{A}_I + \tilde{A}_R$$

$$2^\circ \Leftrightarrow \frac{\partial}{\partial z}(\tilde{f}_T(z,t)) = \frac{\partial}{\partial z}(\tilde{f}_R(z,t) + \tilde{f}_I(z,t)) \Leftrightarrow k_2 \tilde{A}_T = k_1(\tilde{A}_I - \tilde{A}_R)$$

$$\text{由 (1) 得 } \tilde{A}_R = \frac{k_1 - k_2}{k_1 + k_2} \tilde{A}_I, \quad \tilde{A}_T = \frac{2k_1}{k_1 + k_2} \tilde{A}_I$$

$$\text{or } \tilde{A}_R = \frac{v_2 - v_1}{v_1 + v_2} \tilde{A}_I, \quad \tilde{A}_T = \frac{2v_2}{v_1 + v_2} \tilde{A}_I$$

代入波数相同子 δ 得

$$A_R e^{i\delta_R} = \frac{v_2 - v_1}{v_1 + v_2} A_I e^{i\delta_I}, \quad A_T e^{i\delta_T} = \frac{2v_2}{v_1 + v_2} A_I e^{i\delta_I}$$

$$1^\circ \text{ 若 } v_2 > v_1, \text{ 即 密} \rightarrow \text{疏}, \text{ 有 } \delta_R = \delta_I = \delta_T, \quad A_R = \frac{v_2 - v_1}{v_1 + v_2} A_I, \quad A_T = \frac{2v_2}{v_1 + v_2} A_I, \quad (A_T > A_R)$$

$$2^\circ \text{ 若 } v_2 < v_1, \text{ 即 疏} \rightarrow \text{密}, \text{ 有 } \delta_R + \pi = \delta_I = \delta_T, \text{ 有半波损失 } A_R = \frac{v_1 - v_2}{v_1 + v_2} A_I, \quad A_T = \frac{2v_2}{v_1 + v_2} A_I \quad (A_T < A_R)$$

且若 $v_2 \rightarrow 0$ (极密, 例如固 \rightarrow 固) 则有 $A_T = 0$, 全反射

Tip. 当 $z=0$ 处为位置为 m 的结点, 则

$$\begin{aligned} \tilde{f}_I(0,t) + \tilde{f}_R(0,t) &= \tilde{f}_T(0,t) & \tilde{f}_I(z,t) &= \tilde{A}_I e^{i(k_1 z - \omega t)}, \quad \tilde{f}_R(z,t) = \tilde{A}_R e^{i(-k_1 z - \omega t)}, \quad \tilde{f}_T(z,t) = \tilde{A}_T e^{i(k_2 z - \omega t)} \\ \left[\frac{\partial}{\partial z} (\tilde{f}_T - \tilde{f}_R - \tilde{f}_I) \right]_{z=0} &= -\frac{m}{T} \left(\frac{\partial^2 \tilde{f}}{\partial t^2} \right)_{z=0} \\ \begin{cases} \tilde{A}_I + \tilde{A}_R = \tilde{A}_T \\ k_2 \tilde{A}_T + k_1 \tilde{A}_R - k_1 \tilde{A}_I = -\frac{m}{iT} \omega^2 \tilde{A}_T \end{cases} &\Rightarrow \begin{cases} \tilde{A}_R = \frac{k_1 - k_2 - i\frac{m}{T}\omega^2}{k_1 + k_2 + i\frac{m}{T}\omega^2} \tilde{A}_I \\ \tilde{A}_T = \frac{2k_1}{k_1 + k_2 + i\frac{m}{T}\omega^2} \tilde{A}_I \end{cases} \end{aligned}$$

Especially. 若 $v_2 \rightarrow +\infty$ 即 $k_2 \rightarrow 0$ (如右端绳无限长)

$$\tilde{A}_R \approx \frac{k_1 - \frac{m}{T}\omega^2}{k_1 + i\frac{m}{T}\omega^2} \tilde{A}_I, \quad \tilde{A}_T \approx \frac{2k_1}{k_1 + i\frac{m}{T}\omega^2} \tilde{A}_I$$

$$\text{or } \tilde{A}_R = \frac{k_1 + i\frac{m}{T}\omega^2}{k_1 - i\frac{m}{T}\omega^2} \tilde{A}_I, \quad \tilde{A}_T = \frac{2k_1}{k_1 - i\frac{m}{T}\omega^2} \tilde{A}_I$$

$$\text{令 } \beta = \frac{m}{T k_1} \omega^2, \quad \omega^2 = v_1^2 k_1^2 \Rightarrow \beta = \frac{m}{T k_1} \frac{T}{\mu_1} k_1^2 = m \frac{k_1}{\mu_1} \quad (\mu_1 \text{ 为线密度})$$

$$\text{则 } \tilde{A}_R = \frac{1+i\beta}{1-i\beta} \tilde{A}_I, \quad \tilde{A}_T = \frac{2}{1-i\beta} \tilde{A}_I$$

$$A_R e^{i\delta_R} = e^{2i\phi} A_I e^{i\delta_I}, \quad A_T e^{i\delta_T} = \frac{2}{\sqrt{1+\beta^2}} e^{i\phi} A_I e^{i\delta_I}$$

$$\Rightarrow A_R = A_I, \quad \tilde{A}_R = \tilde{A}_I e^{2i\phi}, \quad A_T = \frac{2}{\sqrt{1+\beta^2}} A_I, \quad \tilde{A}_T = \frac{2}{\sqrt{1+\beta^2}} e^{i\phi} \tilde{A}_I$$

偏振.

Def. 偏振: 横波的振动方向在传播方向的现象

Tip. 横波振动方向与传播方向垂直, 有二独立偏振态

$$\tilde{f}_v(z,t) = \tilde{A}_1 e^{i(kz - \omega t)} \hat{x}, \quad \tilde{f}_h(z,t) = \tilde{A}_2 e^{i(kz - \omega t)} \hat{y}$$

若 \tilde{A}_1 与 \tilde{A}_2 同相 但最后 $\tilde{f}_v + \tilde{f}_h$ 为单偏光,

若频率 ω ，全波电磁波

§2. 真空中的电磁波

E与B的波动方程

Intro. 无源区的电磁场中

$$\begin{cases} \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \nabla \cdot \vec{E} = 0 \end{cases} \quad \begin{cases} \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \end{cases}$$

$$\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \nabla \times (-\nabla \times \vec{E}) = -\nabla \times (\nabla \times \vec{E}) = (\nabla \cdot \nabla) \vec{E} - \nabla (\nabla \cdot \vec{E}) = (\nabla \cdot \nabla) \vec{E} = \nabla^2 \vec{E}$$

$$\text{即 } \nabla^2 \vec{E} = \frac{1}{(\sqrt{\mu_0 \epsilon_0})^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$(\text{写成三个方向}) \quad \frac{\partial^2 E_i}{\partial x^2} + \frac{\partial^2 E_i}{\partial y^2} + \frac{\partial^2 E_i}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 E_i}{\partial t^2} \quad (i=x, y, z)$$

$$\text{同理有 } \nabla^2 \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}$$

Thm. 无源区下的电磁波方程: 电场与磁场分别满足三维矢量波方程

$$\begin{cases} \nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} \\ \nabla^2 \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} \end{cases} \quad \text{三维波方程普通形式: } \nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$$

单色平面波

Intro. 由于前理止, 先研究正弦波, 单一频率时, 光(电磁波)的色也单一, 则也称为单色光

又进一步, 若波沿z方向传播, 且与x, y无关, 但称平面波, 则平面正弦单色波可写为

$$\vec{E}(z, t) = \vec{E}_0 e^{i(kz - \omega t)}, \quad \vec{B}(z, t) = \vec{B}_0 e^{i(kz - \omega t)}$$

(由于 $\text{Re}(\tilde{E})$ 满足 Maxwell, \tilde{E} 也满足)

$$\nabla \cdot \tilde{E} = 0 \Rightarrow i k \tilde{E}_z e^{i(kz - \omega t)} = 0 \quad \text{对 } \forall t, z \text{ 成立} \quad \therefore (\tilde{E}_0)_z = 0, \text{ 同理 } (\tilde{B}_0)_z = 0 \quad [\text{横波, 振动方向与传播方向垂直}]$$

$$\text{又 } \nabla \times \tilde{E} = -\frac{\partial \tilde{B}}{\partial t}, \text{ 有 } (ik)(\tilde{E})_y = (-i\omega)(\tilde{B})_x, \quad (ik)(\tilde{E})_x = +i\omega(\tilde{B})_y$$

$$\text{or } k\tilde{E}_y = -\omega\tilde{B}_x, \quad k\tilde{E}_x = \omega\tilde{B}_y \quad \Rightarrow \boxed{\tilde{B} = \frac{k}{\omega} (\hat{z} \times \tilde{E})}$$

$$\text{即 } \boxed{\tilde{E} \text{ 与 } \tilde{B} \text{ 同相且互相垂直}}, \text{ 且 } B_0 = \frac{k}{\omega} E_0 = \frac{k}{kc} E_0 = \frac{1}{c} E_0 \quad \rightarrow \quad \tilde{B} = \frac{1}{c} (\hat{k} \times \tilde{E})$$

Thm. 当无电荷, 无流时, 在真空中沿 \hat{k} 方向传播的单一平面电磁波的电场与磁场为横波, $\hat{n} \cdot \hat{k} = 0$

$$\begin{aligned} \tilde{E}(\vec{r}, t) &= E_0 e^{i(\vec{k} \cdot \vec{r} - \omega t + \delta)} \hat{n}, \quad \tilde{B}(\vec{r}, t) = \frac{E_0}{c} e^{i(\vec{k} \cdot \vec{r} - \omega t + \delta)} (\hat{k} \times \hat{n}) \\ &= \tilde{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \hat{n} \quad = \frac{\tilde{E}_0}{c} e^{i(\vec{k} \cdot \vec{r} - \omega t)} (\hat{k} \times \hat{n}) \end{aligned}$$

$$\text{实际为 } \vec{E}(\vec{r}, t) = E_0 \cos(\vec{k} \cdot \vec{r} - \omega t + \delta) \hat{n} \quad \vec{B}(\vec{r}, t) = \frac{E_0}{c} \cos(\vec{k} \cdot \vec{r} - \omega t + \delta) (\hat{k} \times \hat{n})$$

电磁波的能量与动量

Thm. 电磁波的能量密度, 动量密度与能流密度:

$$u_{em} = \frac{1}{2} (\epsilon_0 E^2 + \frac{1}{\mu_0} B^2) = \frac{1}{2} (\epsilon_0 E^2 + \frac{1}{c^2 \mu_0} E^2) = \epsilon_0 E^2 = \epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)$$

$$\vec{S} = \frac{1}{\mu_0} (\vec{E} \times \vec{B}) = \frac{1}{c\mu_0} E^2 \hat{z} = \frac{1}{c\mu_0 \epsilon_0} \epsilon_0 E^2 \hat{z} = c \epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta) \hat{z} = c u_{em} \hat{z}$$

$$\vec{p}_{em} = \mu_0 \epsilon_0 \vec{S} = \frac{1}{c} u_{em} \hat{z}$$

Tip. \vec{S} 作为能流密度时, 表示在 (z, t) 时, 单位时间通过单位面积的能量.

由于 $\omega = \frac{c}{\lambda}$, 周期很短, 观测时一般只考虑关于时间的平均. [$\langle \rangle$ 代表关于时间的平均]

$$\langle u_{em} \rangle = \epsilon_0 E_0^2 \frac{2\pi}{\omega} \int_0^{2\pi} \cos^2(kz - \omega t + \delta) dt = \frac{1}{2} \epsilon_0 E_0^2$$

$$\langle \vec{S} \rangle = c \langle u_{em} \rangle \hat{z} = c \left(\frac{1}{2} \epsilon_0 E_0^2 \right) \hat{z}$$

$$\langle \vec{p}_{em} \rangle = \frac{1}{c} \langle u_{em} \rangle \hat{z} = \frac{1}{c} \left(\frac{1}{2} \epsilon_0 E_0^2 \right) \hat{z}$$

Def. 电磁波的强度: 通过单位面积传递的电磁波平均功率, 记为 I

$$\text{即 } I := \langle S \rangle = \frac{1}{2} \epsilon_0 E_0^2 c$$

Def. 电磁波的辐射压: 单位时间由光传递动量引起的单位面积上的平均力

$$E_x, \text{ 例如垂直于 } xy \text{ 平面上, } P = \frac{A |\vec{p}_{em}| c \Delta t}{\Delta t A} = \frac{1}{2} \epsilon_0 E_0^2 = \langle u_{em} \rangle = \frac{I}{c}$$

可考虑成先由 \vec{E} 驱动了撞击物体上的电荷, 产生 \vec{v} 后由于 \vec{B} , 受 $\vec{f} = q \vec{v} \times \vec{B}$ 为沿 z 方向力.

§3. 物质中的电磁波

在线性介质中的传播

Intro. 无 \vec{J}_f, \vec{J}_f 时的 Maxwell 方程组,

$$\begin{array}{ll} \nabla \cdot \vec{D} = 0 & \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} & \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \end{array} \Rightarrow \begin{array}{ll} \nabla \cdot \vec{E} = 0 & \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} & \nabla \times \vec{B} = \mu \epsilon \frac{\partial \vec{E}}{\partial t} \end{array}$$

$$\begin{cases} \vec{D} = \epsilon \vec{E} \\ \vec{B} = \mu \vec{H} \end{cases} \quad \text{不同之处在于} \quad \begin{cases} \nabla^2 \vec{E} = -\mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} \\ \nabla^2 \vec{B} = -\mu \epsilon \frac{\partial^2 \vec{B}}{\partial t^2} \end{cases} \quad \text{中, } v = \frac{c}{n} = \frac{1}{\sqrt{\mu \epsilon}}$$

$v = \frac{c}{n} < c$, 这物理意义代表, 在外加电磁场下,

束缚电荷, 极化电流所产生的电磁场与原场合成为一个频率相同, 振幅变小的场
相当于起延缓作用

Def. 折射率: $n := \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}}$

Tip. 对一般材料, μ 接近 μ_0 , 则 $n \approx \sqrt{\epsilon_r}$

Thm. 线性介质中电磁波的能流密度, 能量密度, 动量密度, 强度 (单位面积)

$$u_{em} = \frac{1}{2} (\vec{B} \cdot \vec{H} + \vec{E} \cdot \vec{D}) = \frac{1}{2} (\mu B^2 + \epsilon E^2)$$

$$\vec{S} = \vec{E} \times \vec{H} = \frac{1}{\mu} (\vec{E} \times \vec{B}) = \frac{1}{\mu v} E^2 \hat{z} = v \epsilon E^2 \hat{z} \quad (v = \frac{1}{\sqrt{\mu\epsilon}})$$

$$\vec{p}_{em} = \mu \epsilon \vec{S} = \vec{D} \times \vec{B} = \frac{1}{v} \epsilon E^2 \hat{z}$$

$$I = \langle S \rangle = \frac{1}{2} v \epsilon E_0^2$$

垂直入射时的反射与折射.

Intro: 考虑电磁场边界条件. (以外为 n_1 区)

$$D_{\text{外}}^\perp - D_{\text{内}}^\perp = \sigma_f$$

$$B_{\text{外}}^\perp = B_{\text{内}}^\perp$$

$$\vec{E}_{\text{外}}^\parallel = \vec{E}_{\text{内}}^\parallel$$

$$\vec{H}_{\text{外}}^\parallel - \vec{H}_{\text{内}}^\parallel = \vec{K}_f \times \hat{n}$$

在无 $\rho_f(\sigma_f)$, $\vec{J}_f(\vec{K}_f)$ 时, 且为线性介质, 可改为

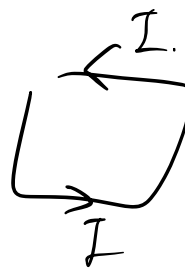
$$\epsilon_{\text{外}} E_{\text{外}}^\perp = \epsilon_{\text{内}} E_{\text{内}}^\perp \quad \dots \textcircled{1} \quad B_{\text{外}}^\perp = B_{\text{内}}^\perp \quad \dots \textcircled{2}$$

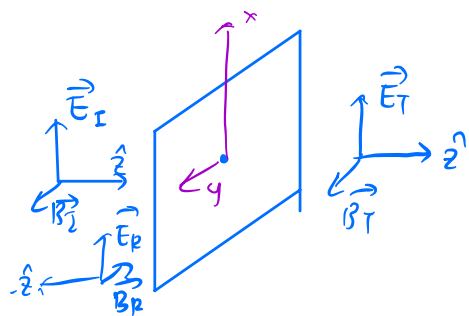
$$\vec{E}_{\text{外}}^\parallel = \vec{E}_{\text{内}}^\parallel \quad \dots \textcircled{3} \quad \frac{1}{\mu_{\text{外}}} B_{\text{外}}^\parallel = \frac{1}{\mu_{\text{内}}} B_{\text{内}}^\parallel \quad \dots \textcircled{4}$$

由此可得这反情况

Scene 1. 垂直入射, 反射, 透射

$$\vec{E}_I = \vec{E}_{0,I} e^{i(kz - \omega t)} \hat{x}$$





$$\begin{cases} \tilde{B}_I = \tilde{B}_{0I} e^{i(k_1 z - \omega t)} & \hat{y} = \frac{k_1}{\omega} \tilde{E}_{0I} e^{i(k_1 z - \omega t)} \hat{y} \\ \tilde{E}_R = \tilde{E}_{0R} e^{i(-k_1 z - \omega t)} & \hat{x} \\ \tilde{B}_R = -\frac{k_1}{\omega} \tilde{E}_{0R} e^{i(-k_1 z - \omega t)} \hat{y} \end{cases}$$

$$\begin{cases} \tilde{E}_T = \tilde{E}_{0T} e^{i(k_2 z - \omega t)} & \hat{x} \\ \tilde{B}_T = \frac{k_2}{\omega} \tilde{E}_{0T} e^{i(k_2 z - \omega t)} \hat{y} \end{cases}$$

由于本身无垂直分量, ①③的垂直分量

$$\textcircled{1}: \frac{1}{\mu_2 v_2} \tilde{E}_{0T} e^{i(k_2 z - \omega t)} = \frac{1}{\mu_1 v_1} \tilde{E}_{0I} e^{i(k_1 z - \omega t)} - \frac{1}{\mu_1 v_1} \tilde{E}_{0R} e^{i(-k_1 z - \omega t)} \quad |_{z=0}$$

$$\textcircled{2} \quad \tilde{E}_{0I} e^{i(k_1 z - \omega t)} + \tilde{E}_{0R} e^{i(-k_1 z - \omega t)} = \tilde{E}_{0T} e^{i(k_2 z - \omega t)} \quad |_{z=0}$$

$$\text{即有} \begin{cases} \frac{\tilde{E}_{0T}}{\mu_2 v_2} = \frac{\tilde{E}_{0I} - \tilde{E}_{0R}}{\mu_1 v_1} \\ \tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T} \end{cases} \Rightarrow \begin{cases} \tilde{E}_{0R} = \frac{\mu_2 v_2 - \mu_1 v_1}{\mu_1 v_1 + \mu_2 v_2} \tilde{E}_{0I} \\ \tilde{E}_{0T} = \frac{2\mu_2 v_2}{\mu_1 v_1 + \mu_2 v_2} \tilde{E}_{0I} \end{cases}$$

$$\text{若记 } \beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1} \quad \tilde{E}_{0R} = \frac{1-\beta}{1+\beta} \tilde{E}_{0I}, \quad \tilde{E}_{0T} = \frac{2}{\beta+1} \tilde{E}_{0I}$$

若 $\mu_1 \approx \mu_2 \approx \mu_0$, 则 $\beta \approx \frac{v_1}{v_2} = \frac{n_2}{n_1}$, $\tilde{E}_{0R}, \tilde{E}_{0T}, \tilde{E}_{0I}$ 关系接近于弦波反射透射关系

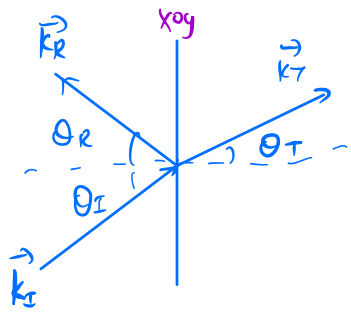
$$\text{振幅: } E_{0R} = \left| \frac{v_1 - v_2}{v_1 + v_2} \right| E_{0I}, \quad E_{0T} = \frac{2v_2}{v_1 + v_2} E_{0I}. \quad (\text{Att: 作}) \mu_1 \approx \mu_2 \approx \mu_0 \text{ 近似}$$

若 $v_2 > v_1$, 无相位, $v_2 < v_1$, E_{0R} 落后 π 相位

$$\text{强度: } I \propto v E^2, \quad \text{则 } R := \frac{I_R}{I_0} = \left(\frac{v_1 - v_2}{v_1 + v_2} \right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2, \quad T := \frac{I_T}{I_0} = \frac{E_{0T}^2}{E_{0I}^2} \left(\frac{2n_2}{n_1 + n_2} \right)^2 = \frac{4n_1 n_2}{(n_1 + n_2)^2}$$

$$I_R + I_T = I_0 \quad (\text{能量守恒}) \quad \begin{matrix} \text{反射系数} \\ \text{透射系数} \end{matrix}$$

Science. 斜入射 (3个波偏振方向与 \vec{E}_I 一致)



$$\begin{cases} \vec{E}_I = \vec{E}_{0I} e^{i(\vec{k}_I \cdot \vec{r} - \omega t)} \\ \vec{B}_I = \frac{1}{v_1} (\hat{k}_I \times \vec{E}_{0I}) e^{i(\vec{k}_I \cdot \vec{r} - \omega t)} \\ \vec{E}_R = \vec{E}_{0R} e^{i(\vec{k}_R \cdot \vec{r} - \omega t)} \\ \vec{B}_R = \frac{1}{v_1} (\hat{k}_R \times \vec{E}_{0R}) e^{i(\vec{k}_R \cdot \vec{r} - \omega t)} \end{cases}$$

$$\begin{cases} \vec{E}_T = \vec{E}_{0T} e^{i(\vec{k}_T \cdot \vec{r} - \omega t)} \\ \vec{B}_T = \frac{1}{v_2} (\hat{k}_T \times \vec{E}_{0T}) e^{i(\vec{k}_T \cdot \vec{r} - \omega t)} \end{cases}$$

Tip. $Ae^{iax} + Be^{ibx} = Ce^{icx}$. 若 $\forall x$ 成立, 且 $A, B, C, a, b, c \neq 0$, 则 $A+B=C$, $a=b=c$

证 取 $x=0$ 有 $A+B=C$

再对 x 求导 $\begin{cases} aA + bB = cC \\ a^2A + b^2B = c^2C \end{cases} \Rightarrow a^2A + b^2B = \frac{(aA+bB)^2}{A+B} \Rightarrow (a-b)^2AB=0 \Rightarrow a=b$ 同理有 $a=b=c$

① $(\vec{E}_I + \vec{E}_R) \perp \vec{E}_I = (\vec{E}_T) \perp \vec{E}_T$ 在 xoy 面上恒成立.

又有 $(\vec{E}_I + \vec{E}_{0R}) \perp \vec{E}_I = (\vec{E}_{0T}) \perp \vec{E}_T$ 且 $\boxed{\vec{k}_I \cdot \vec{r} = \vec{k}_R \cdot \vec{r} = \vec{k}_T \cdot \vec{r}}$

由此可知, $\vec{k}_I, \vec{k}_R, \vec{k}_T$ 在同一直线上, 且 $k_I \sin \theta_I = k_R \sin \theta_R = k_T \sin \theta_T$ ($k_I = k_R, k_T/k_R = v_1/v_2$)

Thm. 1° 第一定律: 入, 反, 透波矢在同一平面内, 该面(入射面)过边界法线

2° 第二定律: 入射角 θ_I = 反射角 θ_R

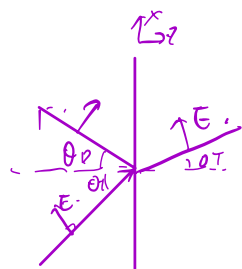
3° 第三定律: 入射角 θ_I 与折射角 θ_T 间

$$\frac{\sin \theta_I}{\sin \theta_T} = \frac{k_T}{k_I} = \frac{v_1}{v_2} = \frac{n_2}{n_1} \quad \text{or } n_1 \sin \theta_I = n_2 \sin \theta_T$$

边界条件:

$$\begin{cases} \vec{E}_1 (\vec{E}_{0I} + \vec{E}_{0R})_{\perp} = \epsilon_2 (\vec{E}_{0T})_{\perp} \\ (\vec{E}_{0I} + \vec{E}_{0R})_{\parallel} = (\vec{E}_{0T})_{\parallel} \\ \frac{1}{\nu_1} (\hat{k}_I \times \vec{E}_{0I} + \hat{k}_R \times \vec{E}_{0R})_{\perp} = \frac{1}{\nu_2} (\hat{k}_T \times \vec{E}_{0T})_{\perp} \\ \frac{1}{\mu_1 \nu_1} (\hat{k}_I \times \vec{E}_{0I} + \hat{k}_R \times \vec{E}_{0R})_{\parallel} = \frac{1}{\mu_2 \nu_2} (\hat{k}_T \times \vec{E}_{0T})_{\parallel} \end{cases} \Rightarrow \begin{cases} \epsilon_1 (\vec{E}_{0I} + \vec{E}_{0R})_z = \epsilon_2 (\vec{E}_{0T})_z \\ (\vec{E}_{0I} + \vec{E}_{0R})_{x,y} = (\vec{E}_{0T})_{x,y} \\ \frac{1}{\nu_1} (\hat{k}_I \times \vec{E}_{0I} + \hat{k}_R \times \vec{E}_{0R})_z = \frac{1}{\nu_2} (\hat{k}_T \times \vec{E}_{0T})_z \\ \frac{1}{\mu_1 \nu_1} (\hat{k}_I \times \vec{E}_{0I} + \hat{k}_R \times \vec{E}_{0R})_{x,y} = \frac{1}{\mu_2 \nu_2} (\hat{k}_T \times \vec{E}_{0T})_{x,y} \end{cases}$$

Especially, 若偏振在 xoy 面内, 则透射反射的偏振方向均在 xoy 面内



$$\begin{cases} \epsilon_1 (\vec{E}_{0I} \sin \theta_I - \vec{E}_{0R} \sin \theta_R) = \epsilon_2 \vec{E}_{0T} \sin \theta_T \\ \vec{E}_{0I} \cos \theta_I + \vec{E}_{0R} \cos \theta_R = \vec{E}_{0T} \cos \theta_T \\ \frac{1}{\mu_1 \nu_1} (\vec{E}_{0I} - \vec{E}_{0R}) = \frac{1}{\mu_2 \nu_2} \vec{E}_{0T} \end{cases}$$

令 $\beta = \frac{\mu_1 \nu_1}{\mu_2 \nu_2}$, $\alpha = \frac{\cos \theta_T}{\cos \theta_I}$

$$\Rightarrow \boxed{\vec{E}_{0R} = \frac{\alpha - \beta}{\alpha + \beta} \vec{E}_{0I}, \vec{E}_{0T} = \frac{2}{\alpha + \beta} \vec{E}_{0I}}$$

$$\beta = \frac{\epsilon_2 \eta_1}{\epsilon_1 \eta_2} = \frac{\epsilon_2 \nu_2}{\epsilon_1 \nu_1} \quad (\text{正常})$$

偏振方向在入射面内的菲涅尔方程

(由一式得)

Especially: β 对于固定介电质, 则 TSR 取决于 α . 又 $\alpha = \frac{\sqrt{1 - \sin^2 \theta_I}}{\cos \theta_I} = \frac{\sqrt{1 - (\frac{n_2}{n_1} \sin \theta_I)^2}}{\cos \theta_I}$ (仅由 θ_I 决定)

在 $\theta_I \rightarrow \frac{\pi}{2}$ (掠入射) 时, $\alpha \rightarrow \infty$, $\vec{E}_{0R} \rightarrow \vec{E}_{0I}$, $\vec{E}_{0T} \rightarrow 0$. (几乎全反射)

在 $\alpha = \beta$ 即 $\sin^2 \theta_B = \frac{1 - \beta^2}{(n_1/n_2)^2 - \beta^2}$

, $R=0$, 全透射, θ_B 称为布儒斯特角

Especially, $\mu_1 \approx \mu_2 \approx \mu_0$ 时, 有 $\beta \approx \frac{n_2^2}{n_1^2}$, $\sin^2 \theta_B \approx \frac{1 - (\frac{n_2}{n_1})^2}{(\frac{n_1}{n_2})^2 - (\frac{n_2}{n_1})^2} = \frac{\beta^2}{1 + \beta^2} \Rightarrow \tan \theta_B \approx \beta$

此时, $\theta_B \approx \arctan \frac{n_2}{n_1}$

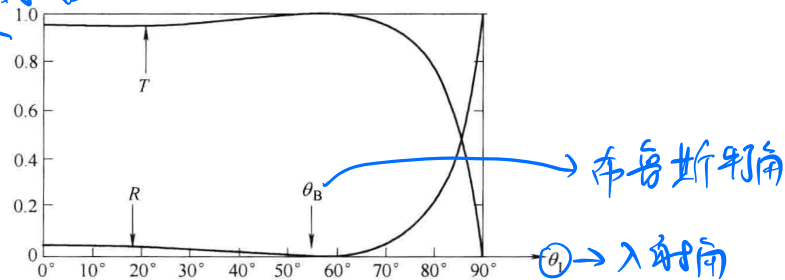
Tip. 考虑光强有, $I_I = \langle S_I \rangle \cos \theta_I = \frac{1}{\mu_1 \nu_1} E_{0I}^2 \times \frac{1}{2} \cos \theta_I = \frac{1}{2} \epsilon_1 \nu_1 E_{0I}^2 \cos \theta_I$

同理, $I_R = \frac{1}{2} \epsilon_1 \nu_1 E_{0R}^2 \cos \theta_R$, $I_T = \frac{1}{2} \epsilon_2 \nu_2 E_{0T}^2 \cos \theta_T$

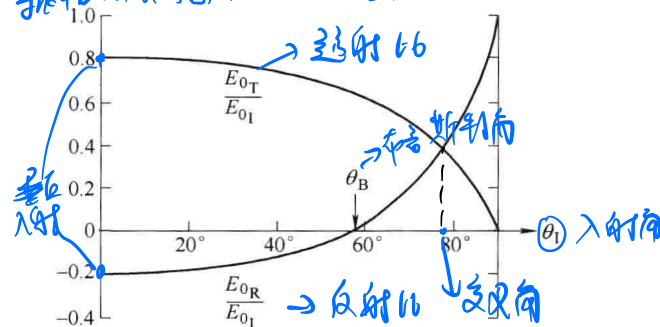
$$R = \frac{I_R}{I_I} = \left(\frac{E_{0R}}{E_{0I}} \right)^2 = \left(\frac{\alpha - \beta}{\alpha + \beta} \right)^2 \quad T = \frac{I_T}{I_I} = \frac{\epsilon_2 V_2}{\epsilon_1 V_1} \left(\frac{E_{0T}}{E_{0I}} \right)^2 \frac{\cos \theta_T}{\cos \theta_I} = \frac{\mu_1 V_1}{\mu_2 V_2} \frac{\cos \theta_T}{\cos \theta_I} \left(\frac{E_{0T}}{E_{0I}} \right)^2 = \alpha \beta \left(\frac{2}{\alpha + \beta} \right)^2$$

$R+T=1$ (能量守恒) 且当 $\theta_I = \theta_B$ 时有 $R=0, T=1$. R 与 T 随 θ_I 变化图像如下

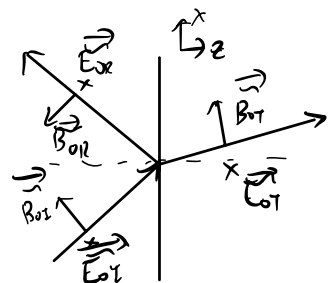
R, T 绝对数值



振幅比例(考虑相位, 无负号)



Ex. 导出偏振方向垂直于入射面的菲涅尔方程, 且证明对 $\forall n_1, n_2$, 不存在 θ_B 且 $\mu_1 = \mu_2, \mu_1 \neq \mu_2$ 时 $E_{0R} \neq 0$. 并验证在垂直入射时化为正确形式



$$\begin{cases} E_{\perp I} = 0 = E_{\perp T} \quad \checkmark \\ \vec{E}_{0I} + \vec{E}_{0R} = \vec{E}_{0T} \end{cases}$$

$$\vec{B}_{0R} \sin \theta_R + \vec{B}_{0T} \sin \theta_T = \vec{B}_{0I} \sin \theta_I$$

$$\frac{1}{\mu_1} (\vec{B}_{0I} \cos \theta_I - \vec{B}_{0R} \cos \theta_R) = \frac{1}{\mu_2} \vec{B}_{0T} \cos \theta_T$$

$$(\vec{B} = \frac{1}{v} \hat{k} \times \vec{E})$$

$$\Rightarrow \begin{cases} \vec{E}_{0I} + \vec{E}_{0R} = \vec{E}_{0T} \\ \frac{1}{v_1} (\vec{E}_{0R} \sin \theta_R + \vec{E}_{0I} \sin \theta_I) = \frac{1}{v_2} \vec{E}_{0T} \sin \theta_T \\ \frac{1}{\mu_1 v_1} (\vec{E}_{0I} \cos \theta_I - \vec{E}_{0R} \cos \theta_R) = \frac{1}{\mu_2 v_2} \vec{E}_{0T} \cos \theta_T \end{cases} \quad \text{全反}$$

$$\Rightarrow \begin{cases} \vec{E}_{0R} = \frac{1 - \alpha \beta}{1 + \alpha \beta} \vec{E}_{0I} \\ \vec{E}_{0T} = \frac{2}{1 + \alpha \beta} \vec{E}_{0I} \end{cases}$$

(当 $\theta_T = \theta_I = 0$ 时, 右边退化为垂直入射)

$$\theta_B \text{ 为 } 1 - \alpha \beta = 0 \text{ 即 } 1 - \frac{\cos \theta_T}{\cos \theta_I} \frac{\mu_1 V_1}{\mu_2 V_2} = 0 \text{ 即 } \sin^2 \theta_B = \frac{\beta^2 - 1}{\beta^2 (n_1/n_2)^2 - 1} = \frac{\mu_1^2 V_1^2 - \mu_2^2 V_2^2}{\mu_1^2 V_2^2 - \mu_2^2 V_2^2} = \frac{\mu_1^2 n_2^2 - \mu_2^2 n_1^2}{\mu_1^2 n_1^2 - \mu_2^2 n_1^2}$$

若 $\mu_1 \neq \mu_2$, 仅在 $V_1 \neq V_2$ 时才有 $\sin \theta_B$ 为有限值.

$$R = \left(\frac{\vec{E}_{0R}}{\vec{E}_{0I}} \right)^2 = \left(\frac{1 - \alpha \beta}{1 + \alpha \beta} \right)^2, \quad T = \frac{\epsilon_2 V_2}{\epsilon_1 V_1} \left(\frac{\vec{E}_{0T}}{\vec{E}_{0I}} \right)^2 \frac{\cos \theta_T}{\cos \theta_I} = \alpha \beta \left(\frac{2}{1 + \alpha \beta} \right)^2, \quad R + T = 1.$$

§ 4. 吸引与色散

导体中的电磁波

Intro. 对于电导率为 σ 的均匀线性介质

$$\begin{cases} \nabla \cdot \vec{E} = \frac{1}{\epsilon} \rho_f \\ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \end{cases} \quad \begin{cases} \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{B} = \mu \sigma \vec{E} + \mu \epsilon \frac{\partial \vec{E}}{\partial t} \end{cases}$$

且有 $\nabla \cdot \vec{J}_f = -\frac{\partial \rho_f}{\partial t}$, $\Rightarrow \frac{\partial \rho_f}{\partial t} = -\nabla \cdot (\sigma \vec{E}) = -\frac{\sigma}{\epsilon} \rho_f$

即 $\rho_f(t) = e^{-\frac{\sigma}{\epsilon} t} \rho_f(0)$ 时间常数 $\frac{\epsilon}{\sigma}$ 反映了 ρ_f 衰减的速度

当 t 足够长后, $\rho_f = 0$, 此时有

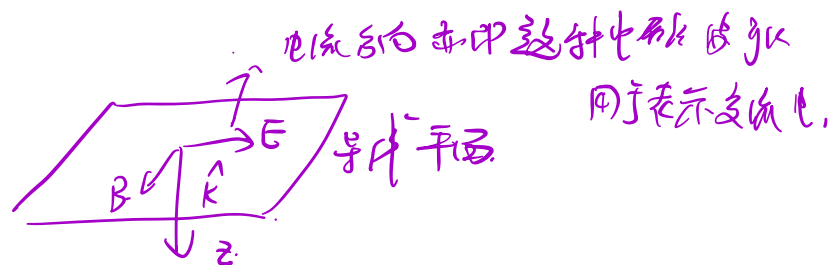
$$\begin{cases} \nabla \cdot \vec{E} = 0 \\ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \end{cases} \quad \begin{cases} \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{B} = \mu \epsilon \frac{\partial \vec{E}}{\partial t} + \mu \sigma \vec{E} \end{cases}$$

Thm: \Rightarrow 波方程为
$$\begin{cases} \nabla^2 \vec{E} = \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu \sigma \frac{\partial \vec{E}}{\partial t} \\ \nabla^2 \vec{B} = \mu \epsilon \frac{\partial^2 \vec{B}}{\partial t^2} + \mu \sigma \frac{\partial \vec{B}}{\partial t} \end{cases}$$

设解为 $\vec{E}(z,t) = \vec{E}_0 e^{i(kz - \omega t)}$

代入得 $k^2 = \mu \epsilon \omega^2 + i \mu \sigma \omega$ 设 $k = a + ib$

$$\Rightarrow \begin{cases} a^2 - b^2 = \mu \epsilon \omega^2 \\ 2ab = \mu \sigma \omega \end{cases} \Rightarrow \begin{cases} a = \omega \sqrt{\frac{\mu \epsilon}{2}} [\sqrt{1 + (\frac{\sigma}{\omega \epsilon})^2} + 1]^{\frac{1}{2}} \\ b = \omega \sqrt{\frac{\mu \epsilon}{2}} [\sqrt{1 + (\frac{\sigma}{\omega \epsilon})^2} - 1]^{\frac{1}{2}} \end{cases} \Rightarrow k = a + bi$$



则 $\vec{E}(z,t) = \vec{E}_0 e^{i(\alpha z - \omega t)} e^{-bz}$ (即随着 z 个, $|E|$ 在衰减)

Def. 趋肤深度: 当 E 的振幅衰减下来 $\frac{1}{e}$ 时所深入的距离, 记为 d

$$d = \frac{1}{b}$$

Tip. 例如上例子, $\frac{1}{b} = \frac{2a}{\mu\sigma\omega} = \sqrt{\frac{2\varepsilon}{\mu\sigma^2}} [\sqrt{1 + (\frac{\sigma}{\varepsilon\omega})^2} + 1]^{\frac{1}{2}}$, 量化了波进入导体的深度.

$\text{Im}(k)$ 决定衰减, $\text{Re}(k)$ 决定了正波的性质. 例如 $\lambda = \frac{2\pi}{\text{Re}(k)}$, $v = \frac{\omega}{\text{Re}(k)}$, $n = \frac{c \text{Re}(k)}{\omega}$.

同样根据 Maxwell 方程有 $\vec{E}_0 \cdot \hat{k} = 0$, $\vec{B}_0 \cdot \hat{k} = 0$

$$\vec{B}_0 = \frac{\vec{k} \times \vec{E}_0}{\omega} \quad (\text{只不过 } k \text{ 值为复数})$$

若记 $k = K e^{i\phi}$, 则 $B_0 e^{i\delta_B} = \frac{K e^{i\phi}}{\omega} E_0 e^{i\delta_E}$, 即由 $\delta_B = \delta_E + \phi$. 不再同相.

$\delta_B - \delta_E = \phi$. B 的相位滞后于 E

$$\frac{B_0}{E_0} = \frac{K}{\omega} = \sqrt{\mu\varepsilon} \sqrt{1 + (\frac{\sigma}{\varepsilon\omega})^2}$$

最后 $\begin{cases} \vec{E}(z,t) = E_0 e^{-bz} \cos(\alpha z - \omega t + \delta_E) \hat{x} \\ \vec{B}(z,t) = B_0 e^{-bz} \cos(\alpha z - \omega t + \delta_E + \phi) \hat{y} \end{cases}$ (例如取 $\hat{k} = \hat{z}$, $\hat{E} = \hat{x}$)
($k = a + bi$, $\phi = \arg k$)

1° 趋肤深度, $d = \sqrt{\frac{2\varepsilon}{\mu\sigma^2}} [1 + \sqrt{1 + (\frac{\sigma}{\varepsilon\omega})^2}]^{\frac{1}{2}}$

在不良导体中, $\sigma \ll \omega\varepsilon$ 时, $d \approx 2\sqrt{\frac{\varepsilon}{\mu\sigma^2}}$ 与频率无关 对不良导体一般成立, 证明.

在良导体中, $\sigma \gg \omega\varepsilon$ 时, 由 $a \ll b$, $\Rightarrow \frac{1}{a} = \frac{1}{\text{Re}(k)} = \frac{\lambda}{2\pi} \approx \frac{1}{b} = d$

$$\text{or } d = \sqrt{\frac{2\varepsilon}{\mu\sigma^2}} \times \sqrt{\frac{\sigma}{\omega\varepsilon}} = \sqrt{\frac{2}{\sigma\mu\omega}} \approx \frac{\lambda}{2\pi}$$

E_x 在典型金属, $\sigma \approx 10^7 (\Omega \cdot m)^{-1}$, 频率 $\omega \approx 10^{15} s^{-1}$ ($\epsilon \approx \epsilon_0, \mu \approx \mu_0$) 中的趋肤深度

$$\frac{\omega \epsilon}{\sigma} \approx \frac{10^{15} \times 10^{-11}}{10^7} = 10^{-3} \ll 1.$$

代入 $d = \sqrt{\frac{2}{\sigma \mu \omega}} \approx 1.26 \times 10^{-8} m \approx 12.6 nm$ (金属不透光)

2° 能量密度 $\langle u \rangle = \frac{d^2}{2\mu\omega^2} E_0^2 e^{-2bz}$

$$I = \langle S \rangle = v \langle u \rangle = \frac{a}{2\mu\omega} E_0^2 e^{-2bz}.$$

导体表面的反射. (金属界面上不是绝缘体有 $J_f, J_f \neq 0$)

Intro:
$$\begin{cases} \epsilon_{\text{外}} E_{\text{外}}^\perp - \epsilon_{\text{内}} E_{\text{内}}^\perp = \sigma_f & E_{\text{外}}^\parallel = E_{\text{内}}^\parallel \\ B_{\text{外}}^\perp = B_{\text{内}}^\perp & \frac{1}{\mu} B_{\text{外}}^\parallel - \frac{1}{\mu_1} B_{\text{内}}^\parallel = K_f \times \hat{n} \end{cases} \quad (\hat{n} \text{ 由 } 1 \rightarrow 2, \text{ 外为 } 2 \text{ 到内为 } 1 \text{ 区}, \perp \text{ 正向为 } \hat{n} \text{ 正向})$$

下考虑偏方向在入射面内的垂直入射情况

且考虑由一个线性绝缘介质 (1) 到欧姆导体 (2)

(给出认为, 欧姆导体中, $\vec{J} = \sigma \vec{E}$, 而 K_f 相当于无限大的 \vec{J}_f , 因而需无限大 \vec{E} , 可看成 $K_f = 0$)

$\begin{matrix} \uparrow \\ x \\ \leftarrow z \end{matrix}$

$$\begin{cases} \vec{E}_I(z,t) = \vec{E}_{0I} e^{i(k_1 z - \omega t)} \hat{x} \\ \vec{B}_I(z,t) = \frac{1}{v_1} \vec{E}_{0I} e^{i(k_1 z - \omega t)} \hat{y} \end{cases} \quad \begin{cases} \vec{E}_R(z,t) = \vec{E}_{0R} e^{i(-k_1 z - \omega t)} \hat{x} \\ \vec{B}_R(z,t) = \frac{1}{v_1} \vec{E}_{0R} e^{i(-k_1 z - \omega t)} (-\hat{y}) \end{cases}$$

$$\begin{cases} \vec{E}_T(z,t) = \vec{E}_{0T} e^{i(\tilde{k}_2 z - \omega t)} \hat{x} \\ \vec{B}_{0T}(z,t) = \frac{\tilde{k}_2}{\omega} \vec{E}_{0T} e^{i(\tilde{k}_2 z - \omega t)} \hat{y} \end{cases} \quad (\text{标上 } \sim \text{ 代表复数})$$

根据边界条件有

$$E_\perp = 0 \Rightarrow \sigma_f = 0, \quad B_\perp = 0$$

$$\vec{E}_{0R} = \frac{1-\tilde{\beta}}{1+\tilde{\beta}} \vec{E}_{0I} \quad (\tilde{\beta} := \frac{\mu_1 v_1}{\mu_2 \omega} \tilde{k}_2)$$

$$\begin{cases} \tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T} \\ \frac{1}{\mu_1 v_1} (\tilde{E}_{0I} - \tilde{E}_{0R}) = \frac{k_2}{\mu_2 \omega} \tilde{E}_{0T} \end{cases} \Rightarrow \begin{cases} \tilde{E}_{0T} = \frac{2}{1+\beta} \tilde{E}_{0I} \end{cases}$$

Especially

1° 理想导体, $\sigma = \infty, k_2 = \infty$ 则 $\tilde{E}_{0R} \approx -\tilde{E}_{0I}, \tilde{E}_{0T} = 0$

全反射, 有 π 相位 (类似由疏到密)

Ex. 计算在空气-铜界面的 R. ($\mu_1 = \mu_2 = \mu_0, \epsilon_1 = \epsilon_0, \sigma = 6 \times 10^7, \omega = 4 \times 10^{15} s^{-1}$)

$$R = \left| \frac{\tilde{E}_{0R}}{\tilde{E}_{0I}} \right|^2 = \left| \frac{1-\tilde{\beta}}{1+\tilde{\beta}} \right|^2, \quad \tilde{k}_2 = a+bi, \quad \begin{cases} a = \omega \sqrt{\frac{\mu_0 \epsilon_0}{2}} \left[\sqrt{1 + \left(\frac{\sigma}{\omega \epsilon_0}\right)^2} + 1 \right]^{\frac{1}{2}} \\ b = \omega \sqrt{\frac{\mu_0 \epsilon_0}{2}} \left[\sqrt{1 + \left(\frac{\sigma}{\omega \epsilon_0}\right)^2} - 1 \right]^{\frac{1}{2}} \end{cases}$$

know, $\sigma \gg \omega \epsilon_0 \therefore a \approx b \approx \sqrt{\frac{\sigma \mu_0 \omega}{2}} \gg 1, \quad \tilde{\beta} = c \sqrt{\frac{\sigma \mu_0}{2 \omega}} (1+i) \approx 29.124 (1+i)$

$R \approx 93.36\%$

介电常数对频率的依赖

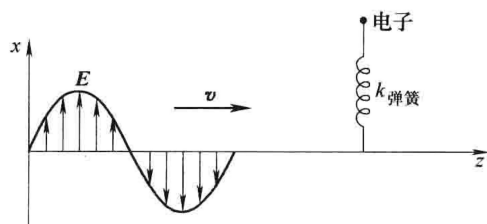
Def. 有色散性介质: 若材料中波速依赖于频率, 则称该介质有色散性, 该现象称为色散.

Tip. 相速: 单个波的波前传播速度 $v = \frac{\omega}{k}$ (不随传播距离, 因而有时大于 c)

群速: 多波合成 (不同频率, 差到不大, 形成拍) 拍即波包, 波包传播速度称为群速 $v_g = \frac{d\omega}{dk}$ (小于 c)

下述通过电子在原子尺度的经典模型解释色散机理.

模型.



1° 电子作为带电量为 $q = -e$ 的小球连接在弹性系数为 $k_{eff} = m \omega_0^2$ 的虚拟弹簧上

(ω_0 为本征频率, $\omega_0 = \sqrt{\frac{k_{eff}}{m}}$)

实际上, 这忽略了约束力的第一近似. $U(x) = U_0 + \frac{1}{2} U''(0) x^2 + \dots$

2° 束缚电子受到阻尼作用, $F_{阻} = -m\gamma \frac{dx}{dt}$ (与速度相反, γ 为阻尼系数)

阻尼产生的原因之一是运动电荷会辐射电磁波, 因而减速, 此为辐射阻尼

3° 驱动力类似由一个正弦形式的电场使极化,

$$F_{SE} = qE = qE_0 \cos(\omega t)$$

综上, 由牛顿定律

$$m \frac{d^2 x}{dt^2} = F_{SE} + F_{阻} + F_{弹} = qE_0 \cos(\omega t) - m\gamma \frac{dx}{dt} - m\omega_0^2 x$$

$$\text{or } m \frac{d^2 x}{dt^2} + m\gamma \frac{dx}{dt} + m\omega_0^2 x = qE_0 \cos(\omega t) \quad (\text{有阻尼受迫振动})$$

(假设原子核静止)

$$\text{将其视为受迫谐振, } \frac{d^2 \tilde{x}}{dt^2} + \gamma \frac{d\tilde{x}}{dt} + \omega_0^2 \tilde{x} = \frac{qE_0}{m} e^{-i\omega t}$$

$$\text{在稳态时, } \tilde{x}(t) = \tilde{x}_0 e^{-i\omega t} \quad \text{代入得}$$

$$-\omega^2 \tilde{x}_0 - i\gamma\omega \tilde{x}_0 + \omega_0^2 \tilde{x}_0 = \frac{qE_0}{m} \Rightarrow \tilde{x}_0 = \frac{q/m}{\omega_0^2 - \omega^2 - i\gamma\omega} E_0$$

因为 \vec{P} 与 E 同相 \Rightarrow 电偶极矩 $\vec{p}(t) = \text{Re}(\tilde{p}(t))$, $\tilde{p}(t) = q\tilde{x}(t) = \frac{q^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega} E_0 e^{-i\omega t}$

由于阻尼因子存在, $\vec{p}(t)$ 并不与 $E(t)$ 同相, 而是延迟了 $\theta = \arctan \frac{\gamma\omega}{\omega_0^2 - \omega^2}$ 的相位

当 $\omega \ll \omega_0$ 时很小, 当 $\omega \gg \omega_0$ 时, 趋于 π

同一分子的不同电子具有不同的 ω_j (共振) 与 γ_j (阻尼). 假设一种材料单位体积有 N 个分子

$$\text{电极化矢量 } \vec{P} = \frac{Nq^2}{m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega} \vec{E}$$

显然 ω_j \vec{P} 不正比于 \vec{E} (由于二者不同相), 但 $\vec{P} \propto \vec{E}$

\Rightarrow 定义为 $\chi(\omega)$ 的介电函数

则 $\vec{P} = \epsilon_0 \tilde{\chi}_e \vec{E}$, \vec{D} 与 \vec{E} 同向为 $\epsilon_r = (1 + \tilde{\chi}_e) = 1 + \frac{Nq^2}{\epsilon_0 m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j \omega}$ 即为 ϵ_r

这里, 分母中虚部 γ_j 略, 但在 $\omega \approx \omega_j$ 时, 虚部项很重要

下给出在均匀介质中的波动方程

$$\nabla^2 \vec{E} = \mu_0 \tilde{\epsilon} \frac{\partial^2 \vec{E}}{\partial t^2}$$

平面波解为 $\vec{E}(z, t) = \vec{E}_0 e^{i(\vec{k}z - \omega t)}$ ($\vec{k} = \sqrt{\mu_0 \tilde{\epsilon}} \omega = a + bi$)

$$\Rightarrow \vec{E}(z, t) = \vec{E}_0 e^{-bz} e^{i(az - \omega t)}$$

↓
阻尼衰减

由于 $I \propto |\vec{E}|^2$, 则 $\alpha := 2b$ 称为吸收系数,

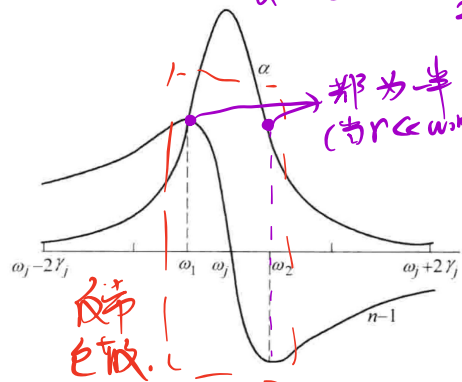
波速仍为 $v = \frac{\omega}{a}$, 折射率 $n = \frac{c}{v} = \frac{c_0}{\omega}$

Especially. 对于气体, 由于 N 很小, 所以 ϵ_r 中第 2 项 = 2 为小量

$$\vec{k} = \sqrt{\mu_0 \tilde{\epsilon}} \omega = \frac{\omega}{c} \sqrt{\epsilon_r} \approx \frac{\omega}{c} \left(1 + \frac{Nq^2}{2m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j \omega} \right)$$

$$n = \frac{c}{v} = \frac{c_0}{\omega} = 1 + \frac{Nq^2}{2m\epsilon_0 c} \sum_j \frac{f_j \gamma_j}{(\omega_j^2 - \omega^2)^2 + (\gamma_j \omega)^2} \quad (\omega = \omega_j \sqrt{1 + \frac{\gamma_j^2}{\omega_j^2}} \text{ 时最大, 对于共振})$$

$$\alpha = 2b = \frac{Nq^2 \omega^2}{2m\epsilon_0 c} \sum_j \frac{f_j \gamma_j}{(\omega_j^2 - \omega^2)^2 + (\gamma_j \omega)^2} \quad (\text{在 } \omega = \omega_j \text{ 时达到最大, 对于共振})$$



此为半透明图, 因为 $\omega = \omega_j$ 附近 α 与 n 的变化.

一般 n 随 ω 上升, 但在 ω_j 附近不同, 该现象称为反常色散.

因为 α 在该区域较大, 因为在此附近 ω 驱动电子, 因而损耗大

Att. 在 $\omega > \omega_j$ 时, $n < 1$ 即 $n < 1$, 此时, 相速度超光速 (注意, 因为波速是 c/n 和 c/n 的比)

若 ω 远小于 ω_j , 则可将阻尼忽略, $n = 1 + \frac{Nq^2}{2m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2}$

对大多数材料, ω_j 有分布, 但对透明材料, 一般忽略 ω_j 在紫外区域

所以在可见光中有 $\omega < \omega_j$, 此时, $n > 1$. 若近似看成 $\omega_j \gg \omega$ 则有

$$\frac{1}{\omega_j^2 - \omega^2} = \frac{1}{\omega_j^2} \left(1 - \frac{\omega^2}{\omega_j^2}\right)^{-1} \approx \frac{1}{\omega_j^2} \left(1 + \frac{\omega^2}{\omega_j^2}\right)$$

$$n = 1 + \frac{Nq^2}{2m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2} + \frac{Nq^2\omega^2}{2m\epsilon_0} \sum_j \frac{f_j}{\omega_j^4} \quad (N \text{ 为分子浓度, } f_j \text{ 为一个分子有几个固有频率为 } \omega_j \text{ 的原子})$$

(f_j, ω_j 由材料决定) 则 $n = 1 + A \left(1 + \frac{B}{\lambda^2}\right)$ 柯西色散公式

适用于一般透明材料 (物质频率近似远大于公式频率或物质波长远小于 λ) (例如在可见光区对空气)

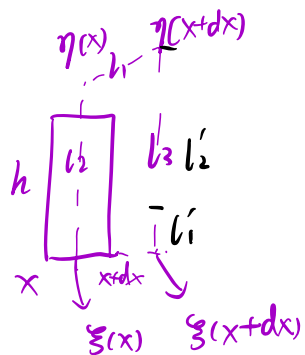
A 称为折射系数, B 称为色散系数

Ex. 对于超光速问题: 群速度 $v_g = \frac{d\omega}{dk} < c$

$$\frac{d\omega}{dk} = \frac{d\left[\frac{\omega}{c} \left(1 + \frac{Nq^2}{2m\epsilon_0} \sum_j \frac{f_j(\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2}\right)\right]}{dk} = \frac{1}{c} \left[1 + \frac{Nq^2}{2m\epsilon_0} \sum_j \frac{f_j(\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2}\right] + \frac{\omega}{c} \frac{Nq^2}{2m\epsilon_0} \sum_j f_j \frac{(\omega_j^2 - \omega^2)(2\omega\omega_j^2 - \omega^3 - \gamma_j^2 \omega^3)}{[(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2]^2}$$

取 $\gamma_j = 0$, $\frac{d\omega}{dk} \uparrow$. 在透明区, $\frac{d\omega}{dk} = \frac{1}{c} \left[1 + \frac{Nq^2}{2m\epsilon_0} \sum_j \frac{f_j(\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2}\right] > \frac{1}{c} \therefore \frac{d\omega}{dk} = v_g < c \quad \left(\frac{d\omega}{dk} = \frac{d\omega}{d(Re(k))}\right)$

Ex. 对浅水波波动方程的推导. η 为水平位移, η 为垂直位移 (浅一般为 $\frac{h}{\lambda} < \frac{1}{2}$)



定线的水柱 \rightarrow 虚线水柱, $\eta \ll h$

$$\text{有 } h dx = [dx + \xi(x+dx) - \xi(x)] [h + \eta(x)]$$

$$\Rightarrow h = \left[1 + \frac{\partial \xi}{\partial x}\right] [h + \eta(x)] \Rightarrow \eta(x) = -\frac{\partial \xi}{\partial x} h$$

再有水平压强提供压力.

l_3 分为 l_2' 与 l_1' , l_2' 上压力与 l_2 平衡, 变变压力来自 $l_1'-l_1$

$$dF = b_x [\eta(x+dx) - \eta(x)] [p_0 + lgh] - b [\eta(x+dx) - \eta(x)] p_0 = lghb [\eta(x+dx) - \eta(x)]$$

$$= lghb \frac{\partial \eta}{\partial x} dx = -lgh^2 b \frac{\partial^2 \xi}{\partial x^2} dx$$

$$dm = b l h dx \quad \Rightarrow \quad dF = dm (-\frac{\partial^2 \xi}{\partial t^2}) \Rightarrow \frac{\partial^2 \xi}{\partial t^2} = gh \frac{\partial^2 \xi}{\partial x^2} \quad \text{或} \quad \frac{\partial^2 \xi}{\partial x^2} = \frac{1}{(\sqrt{gh})^2} \frac{\partial^2 \xi}{\partial t^2}$$

波速 $v = \sqrt{gh}$.

Ex. 对深水波波动的方程推导

§5. 波导.

Def. 波导: 用于定向引导电磁波的结构, 一般指空心金属管.

Tip. 下波导用空心的直埋热导体代表.

Intro. 波导中电磁波满足的方程.

1° 在壳中, $\vec{E} = \vec{0}$, 而壳一开始磁感应为 $\vec{0}$. 根据 $\nabla \times \vec{E} = \vec{0} = -\frac{\partial \vec{B}}{\partial t}$ 得 $\vec{B} = \vec{0}$

2° 由边界条件, 得, $\vec{E}_{\text{内}} = \vec{E}_{\text{外}} = \vec{0}$, $B_{\text{内}} = B_{\text{外}} = 0$

由于对沿轴传播单元感兴趣, 如图建立坐标.



$$\vec{E}(x, y, z, t) = \vec{E}_0(x, y) e^{i(kz - \omega t)} \quad (\text{只考虑不衰减部分})$$

$$\vec{B}(x, y, z, t) = \vec{B}_0(x, y) e^{i(kz - \omega t)} \quad (\text{由于 } \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \vec{B}_0 = \frac{\vec{E}_0 \times \vec{z}}{\omega}, \text{ 即 } \vec{B}_0 \text{ 与 } \vec{E}_0 \text{ 同相垂直})$$

$$\text{代入 } \nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{得} \quad (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}) \vec{E}_0(x, y) e^{i(kz - \omega t)} = -\frac{\omega^2}{c^2} \vec{E}_0$$

$$\Rightarrow (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c^2} - k^2) \vec{E}_0 = 0 \quad \text{或} \quad (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c^2} - k^2) E_z = 0$$

同理有 $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c^2} - k^2) B_z = 0$

分类: 1° $E_z = 0$, 称为 TE 波 (横电磁波)

2° $B_z = 0$, 称为 TM 波 (横电磁波)

3° $E_z = B_z = 0$ 称为 TEM 波, 但在真空中波导中, 该波不存在 (亦即被限制的波一般非TEM波)

证: $\nabla \cdot \vec{E} = 0 \Rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0$

$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, $\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t} = 0$

即 $B_z = E_z = 0 \Rightarrow \vec{E}$ 在 D 内无闭合线

所以, 在某 γ 与 z 轴垂直平面上 \vec{E} 无闭合线, 因而其势函数相同。有闭合等势线

根据边界条件知, 波导边界等势, 而根据唯一性, 势函数 = const

因而 $E = 0$, 即不存在电场, 同理, 若 $E_z = 0 \Rightarrow B = 0$

所以, TEM 波不存在, 这正与 TEM = 横电磁波

At. 平面波是典型 TEM 波, 它的平面场里不存在闭合等势线

平面场沿 z 轴方向为实的区域内, 势与流出极才均存在且同和

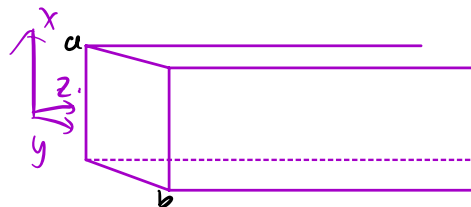
否则只存在一个而不一定同和

矩形波导中的 TE 波

Intro. 波动方程的本解 (B_z)

$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c^2} - k^2) B_z(x, y) = 0$

利用分离变量法, 令 $B_z(x, y) = X(x)Y(y)$



$B_z(x, y, z, t) = B_z(x, y) \cos(kz - \omega t + \phi_0)$

$$\Rightarrow \frac{\partial^2 X(x)}{\partial x^2} Y + \frac{\partial^2 Y(y)}{\partial y^2} X + (\frac{\omega^2}{c^2} - k^2) X Y = 0 \quad \text{同解 } X, Y.$$

$$\Rightarrow \left(\frac{1}{X} \frac{\partial^2 X}{\partial x^2} \right) + \left(\frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} \right) + \frac{\omega^2}{c^2} - k^2 = 0$$

$$\therefore \frac{1}{X} \frac{\partial^2 X}{\partial x^2} = -k_x^2, \quad \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} = -k_y^2, \quad \frac{\omega^2}{c^2} - k^2 - k_x^2 - k_y^2 = 0$$

$$\text{有 } X(x) = A \sin k_x x + B \cos k_x x \quad Y(y) \text{ 同}.$$

$$\text{又有 } x=0, x=a \text{ 时, } B_x = 0$$

Extra. 推导波导中沿轴向传播的波的分量式关系

$$\vec{E}(x, y, z, t) = \vec{E}_0(x, y) e^{i(kz - \omega t)}$$

$$\vec{B}(x, y, z, t) = \vec{B}_0(x, y) e^{i(kz - \omega t)}$$

$$\begin{cases} \nabla \cdot \vec{E} = 0 & \dots \textcircled{1} \\ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} & \dots \textcircled{2} \end{cases} \quad \begin{cases} \nabla \cdot \vec{B} = 0 & \dots \textcircled{3} \\ \nabla \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} & \dots \textcircled{4} \end{cases}$$

先代入 $\textcircled{1} \textcircled{4}$ 的. (略去 \sim 上标与下标)

$$\begin{cases} \partial_y E_z - ik E_y = i\omega B_x \\ ik E_x - \partial_x E_z = i\omega B_y \\ \partial_x E_y - \partial_y E_x = i\omega B_z \end{cases} \quad \begin{cases} \partial_y B_z - ik B_y = -\frac{i\omega}{c^2} E_x \\ ik B_x - \partial_x B_z = -\frac{i\omega}{c^2} E_y \\ \partial_x B_y - \partial_y B_x = -\frac{i\omega}{c^2} E_z \end{cases}$$

$$\begin{cases} B_x = i \frac{\frac{\omega}{c^2} \partial_y E_z - k \partial_x B_z}{k^2 - (\omega/c)^2} \\ B_y = -i \frac{\frac{\omega}{c^2} \partial_x E_z + k \partial_y B_z}{k^2 - (\omega/c)^2} \\ E_x = -i \frac{(\omega \partial_y B_z + k \partial_x E_z)}{k^2 - (\omega/c)^2} \end{cases} + \nabla \cdot \vec{B} = 0 \quad \text{得 } B_z \text{ 的波动方程}$$

$$\left\{ \begin{aligned} E_y &= i \frac{(\omega \alpha B_z - k \partial_y E_z)}{k^2 - (\omega/c)^2} \end{aligned} \right\} + \nabla \cdot \mathbf{E} = 0 \quad \text{得 } E_z \text{ 满足波动方程}$$

接上, $B_x = 0$ 在 $x=0$ 与 $x=a$ 处. $E_z = 0$ 则有 $\partial_x(B_z) = \partial_x(\alpha(x) \gamma(y)) = (\partial_x \alpha(x)) \gamma(y) = 0$

$$\text{由于 } \gamma(y) \text{ 并非恒为 } 0, \text{ 故 } \left. \frac{d\alpha(x)}{dx} \right|_{x=0, x=a} = 0. \Leftrightarrow (A k_x \cos k_x x - B k_x \sin k_x x)|_{x=0, x=a} = 0 \Rightarrow \begin{cases} A=0 \\ k_x = \frac{m\pi}{a} \quad (m \in \mathbb{N}) \end{cases}$$

$$\text{同理, } k_y = \frac{n\pi}{b} \Rightarrow B_z(x, y) = B_0 \cos\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right)$$

记 $B_{z_0}(x, y)$ 的这个解为 TE_{mn} 模式

Tip. 一般第一多极与长边联系, 故一般有 $a \geq b$

Att. 不会有 TE_{00} 模式.

$$m=n=0. \text{ 根据 } \frac{\omega^2}{c^2} - k^2 - k_x^2 - k_y^2 = 0 \Rightarrow \frac{\omega}{c} = k \quad (\text{倘若 } \frac{\omega}{c} \neq k \Rightarrow \frac{\omega}{c} = k, \text{ 矛盾, 因而 } \frac{\omega}{c} = k)$$

则后面推理均不成立, 也不满足 $B_z(x, y) = B_0 \cos(0) \cos(0) = B_0$

$$\text{从 } \frac{\omega}{c} = k \text{ 出发有 } ikE_x = 0 = i\omega B_y \Rightarrow \partial_y B_z = ikB_y - \frac{i\omega}{c} E_x = ik(B_y B_y) = 0$$

同理有 $\partial_x B_z = 0$. 又 $B_z = B_z(x, y)$, 则有 $B_z = \text{const}$, 即取恒定值表面闭合曲线

$$\int_S (\nabla \times \vec{E}) \cdot d\vec{a} = \int_S -\frac{\partial B_z}{\partial t} da = \oint_C \vec{E} \cdot d\vec{l} \quad \text{由于是等势线 } (\vec{E}'' = \vec{0})$$

$$\text{则有 } \int_S -\frac{\partial B_z}{\partial t} da = \frac{\partial}{\partial t} (\iint_S B_z da) \quad \text{若对 } \forall \text{ 面 } S \text{ 均成立, 则有 } B_z = B_z(x, y) = 0.$$

因而 $B_z = E_z = 0$ 为 TEM 模式, 但 TEM 模式在波导中是不存在.

$$\text{接上. } k = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \quad \text{若 } \frac{\omega}{c} < \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \pi, \text{ 则 } k \notin \mathbb{R}$$

$$\text{只有衰减点, 但此时 } \tilde{B}_z = \tilde{B}_{z0} e^{i(kz - \omega t)} = \tilde{B}_{z0} e^{-bz} e^{i(az - \omega t)}$$

$$B_z = B_{z0} e^{-bz} \cos(az - \omega t + \phi), \text{ 会随 } z \text{ 衰减. 不能成为在波导中传递的电磁波.}$$

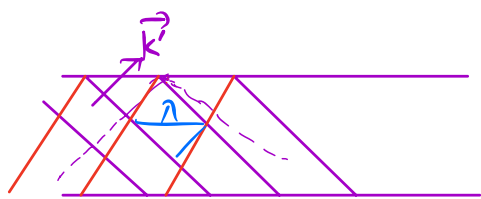
则 $\omega_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$ 是 TE_{mn} 模式的截止频率

而由于 $a > b$, 则可能的最低截止频率为 $\omega_{10} = \frac{c\pi}{a}$

利用截止频率 ω_{mn} , 将 k 写为 $k = \frac{1}{c} \sqrt{\omega^2 - \omega_{mn}^2}$ $v = \frac{\omega}{k} = \frac{c}{\sqrt{1 - (\omega_{mn}/\omega)^2}} > c$

从而 $v_g = \frac{d\omega}{dk} = \frac{1}{dk/d\omega} = c \sqrt{1 - (\omega_{mn}/\omega)^2} < c$

Tip. 另一种看待方法: 把真实的波看成一系列 k, k' 波矢入射的波在波导中经历反射后的波叠加而成



则在 x 方向与 y 方向上的 \vec{k}' 分别为 $\vec{k}' = k_x \hat{x} + k_y \hat{y} + k_z \hat{z}$

(显然, 这两列波叠加后, 从 x 上看为驻波, y 上相同, z 上为行波)

因为合成效果在 z 方向为行波, 波数为 k , 则 $(\vec{k}')_z = k$.

而根据两列波矢为 k 的波叠加后, 振幅因子随位置变化为 $\cos(kx)$, 相位也与

$B_z = B_0 \cos(k_x x) \cos(k_y y)$ 在 x 与 y 上的行为一致, 因而 $\vec{k}' = \frac{m\pi}{a} \hat{x} + \frac{n\pi}{b} \hat{y} + k \hat{z}$

实际的波速上, 每波波速为 c . 即 $\lambda' = cT$, 而 $\lambda = \frac{\lambda'}{\cos\theta}$ ($\theta = \angle \vec{k}', \hat{z}$)

所以看起来 $v = \frac{v'}{\cos\theta} = \frac{c}{\cos\theta}$.

ω 对于不同方向的波相同, 所以 $\omega = |\vec{k}'| v' = |\vec{k}'| c = \sqrt{k_x^2 + k_y^2 + k^2} c$ 从而 $k = \frac{1}{c} \sqrt{\omega^2 - \omega_{mn}^2}$

则有 $\omega = kv = k \frac{c}{\cos\theta} \Rightarrow \cos\theta = \sqrt{1 - (\omega_{mn}/\omega)^2}$ (或 $\theta = \frac{\vec{k}' \cdot \hat{z}}{|\vec{k}'|} = \frac{k}{\sqrt{k_x^2 + k_y^2 + k^2}} = \frac{\sqrt{\omega^2 - \omega_{mn}^2}}{\omega} = \sqrt{1 - (\omega_{mn}/\omega)^2}$)

Ex. 证明在 TE_{mn} 模式中, 能量以群速度传播.

证. 已知 $\langle \vec{S} \rangle$ 在波导截面上的积分 $\int_S \langle \vec{S} \rangle \cdot d\vec{a} = \left(\int_S \langle u \rangle da \right) v_g$

Thm. 对于同频同波长的波, (例如 $f = A_0 \cos(\vec{k} \cdot \vec{r} - \omega t + \delta_A)$, $g = B_0 \cos(\vec{k} \cdot \vec{r} - \omega t + \delta_B)$)

有 $\langle fg \rangle = \frac{1}{2} \text{Re}(\tilde{f} \tilde{g}^*)$ (\tilde{f} 与 \tilde{g} 是 f, g 的复数表示)

$$\text{由此定理有 } \langle u \rangle = \langle \frac{1}{2} \epsilon_0 E^2 + \frac{1}{2\mu_0} B^2 \rangle = \frac{1}{4} \epsilon_0 \text{Re}(\vec{E} \vec{E}^*) + \frac{1}{4\mu_0} \text{Re}(\vec{B} \vec{B}^*)$$

$$\langle \vec{S} \rangle = \langle \frac{1}{\mu_0} \vec{E} \times \vec{B} \rangle = \frac{1}{2\mu_0} \text{Re}(\vec{E} \times \vec{B}^*)$$

$$\vec{E} = \vec{E}_0 e^{i(kz - \omega t)} \quad \vec{B} = \vec{B}_0 e^{i(kz - \omega t)}, \quad \vec{B}^* = \vec{B}_0^* e^{-i(kz - \omega t)}$$

$$\begin{cases} \tilde{B}_{0x} = i \frac{k}{k^2 - (\omega/c)^2} \frac{m\pi}{a} B_0 \sin(\frac{m\pi}{a}x) \cos(\frac{n\pi}{b}y) \\ \tilde{B}_{0y} = i \frac{k}{k^2 - (\omega/c)^2} \frac{n\pi}{b} B_0 \cos(\frac{m\pi}{a}x) \sin(\frac{n\pi}{b}y) \\ \tilde{B}_{0z} = B_0 \cos(\frac{m\pi}{a}x) \cos(\frac{n\pi}{b}y) \end{cases} \quad \begin{cases} \tilde{E}_{0x} = i \frac{\omega}{k^2 - (\omega/c)^2} \frac{n\pi}{b} B_0 \cos(\frac{m\pi}{a}x) \sin(\frac{n\pi}{b}y) \\ \tilde{E}_{0y} = -i \frac{\omega}{k^2 - (\omega/c)^2} \frac{m\pi}{a} B_0 \sin(\frac{m\pi}{a}x) \cos(\frac{n\pi}{b}y) \\ \tilde{E}_{0z} = 0 \end{cases}$$

$$\text{代入 } \langle \vec{S} \rangle = \frac{1}{2\mu_0} \text{Re}(\vec{E} \times \vec{B}^*) = \frac{1}{2\mu_0} \text{Re}[(\tilde{E}_y \tilde{B}_z^* - \tilde{E}_x \tilde{B}_z^*, \tilde{E}_x \tilde{B}_y^* - \tilde{E}_y \tilde{B}_x^*)]$$

$$= \frac{1}{2\mu_0} (0, 0, \frac{k\omega B_0^2}{[k^2 - (\omega/c)^2]^2} [(\frac{n\pi}{b})^2 \cos^2(\frac{m\pi}{a}x) \sin^2(\frac{n\pi}{b}y) + (\frac{m\pi}{a})^2 \sin^2(\frac{m\pi}{a}x) \cos^2(\frac{n\pi}{b}y)])$$

$$\int_S \langle \vec{S} \rangle \cdot d\vec{a} = \int_0^a dx \int_0^b dy \langle S_z \rangle = \frac{k\omega B_0^2 \pi^2 ab}{8\mu_0 [k^2 - (\omega/c)^2]^2} [(\frac{n}{b})^2 + (\frac{m}{a})^2]$$

$$\langle u \rangle = \frac{1}{4} \epsilon_0 (\vec{E} \cdot \vec{E}^*) + \frac{1}{4\mu_0} (\vec{B} \cdot \vec{B}^*)$$

$$= (\frac{1}{4} \epsilon_0 \frac{\omega^2}{[k^2 - (\omega/c)^2]^2} B_0^2 + \frac{1}{4\mu_0} \frac{k^2}{[k^2 - (\omega/c)^2]} B_0^2) [(\frac{n\pi}{b})^2 \cos^2(\frac{m\pi}{a}x) \sin^2(\frac{n\pi}{b}y) + (\frac{m\pi}{a})^2 \sin^2(\frac{m\pi}{a}x) \cos^2(\frac{n\pi}{b}y)] + \frac{1}{4\mu_0} B_0^2 \cos^2 \cos^2$$

$$\text{同样, } \int_S \langle u \rangle da = \int_0^a dx \int_0^b dy \langle u \rangle = \frac{\pi^2 ab B_0^2}{16 [k^2 - (\omega/c)^2]^2} (\epsilon_0 \omega^2 + \frac{k^2}{\mu_0}) [(\frac{n}{b})^2 + (\frac{m}{a})^2] + \frac{B_0^2 ab}{16\mu_0}$$

$$\text{又有 } k^2 - (\frac{\omega}{c})^2 = -(\frac{\omega_{mn}}{c\pi})^2, \quad (\frac{\omega_{mn}}{c\pi})^2 = [(\frac{n}{b})^2 + (\frac{m}{a})^2]$$

$$\int_S \langle \vec{S} \rangle \cdot d\vec{a} = \frac{k\omega B_0^2 abc^2}{8\mu_0 \omega_{mn}^2}, \quad \int_S \langle u \rangle da = \frac{ab B_0^2 c^2}{16\mu_0 \omega_{mn}^2} (\frac{\omega^2}{c^2} + k^2) + \frac{B_0^2 ab}{16\mu_0} = \frac{B_0^2 abc^2}{16\mu_0 \omega_{mn}^2} \times 2 (\frac{\omega}{c})^2$$

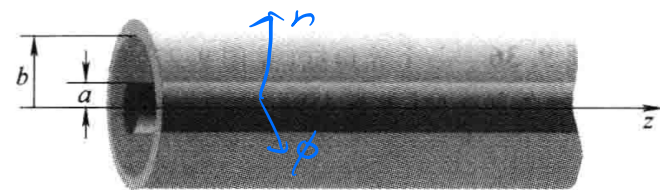
$$\frac{\int_S \langle \vec{S} \rangle \cdot d\vec{a}}{\int_S \langle u \rangle da} = \frac{kc^2}{\omega} = \frac{1}{c} \sqrt{\omega^2 - \omega_{mn}^2} \times \frac{c^2}{\omega} = c \sqrt{1 - (\omega_{mn}/\omega)^2} = v_g$$

Ex. 求矩形波导的 TM 理论

共轴传输线

Intro. 前述, 中空波导无法传输 TEM 波, 但共轴传输线以

半径为 a 长导线与内径为 b 圆柱形导体组成



$$E_z = B_z = 0 \text{ 时, 有 } \begin{cases} E_x = +\frac{\omega}{k} B_y \\ B_y = \frac{\omega}{kc^2} E_x \end{cases} \Rightarrow k = \frac{\omega}{c}$$

不同波长传播速度一致, 无反射 且在空腔表面上传输 ($\vec{E}_n = \vec{0}$)

$$\text{且 } \begin{cases} \partial_x E_y - \partial_y E_x = 0 \\ \partial_x E_x + \partial_y E_y = 0 \quad (\nabla \cdot \vec{E} = 0) \end{cases} \quad \begin{cases} \partial_x B_y - \partial_y B_x = 0 \\ \partial_x B_x + \partial_y B_y = 0 \quad (\nabla \cdot \vec{B} = 0) \end{cases}$$

则平面场为调和场.

1° 柱面传输 E_r 有 1/2-解 $E_r = \frac{A}{r}$ (又 $\vec{B} = \frac{1}{c} \vec{k} \times \vec{E}$, 所以 $B = \frac{A}{c}$)

2° $B_z = 0$, 则用均匀带电长线求出 $B_\phi = \frac{B}{r} = \frac{A}{cr}$

$$\text{综上, } \vec{E}(r, \phi, z, t) = \frac{A}{r} \cos(kz - \omega t) \hat{r}$$

$$\vec{B}(r, \phi, z, t) = \frac{A}{cr} \cos(kz - \omega t) \hat{\phi}$$

假设

$$E(r, \theta, \phi, t) = A \frac{\sin \theta}{r} [\cos(kr - \omega t) - (1/kr) \sin(kr - \omega t)] \hat{\phi}, \quad c = \omega/k$$

$$\nabla u = \frac{\partial u}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial u}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial u}{\partial \phi} \hat{\phi}$$

符

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} v_\phi$$

符

$$\begin{aligned} \nabla \times \mathbf{v} = & \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right] \hat{r} + \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r} (r v_\phi) \right] \hat{\theta} \\ & + \frac{1}{r} \left[\frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta} \right] \hat{\phi} \end{aligned}$$

断算符

$$\nabla^2 u = \nabla \cdot (\nabla u) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2}$$

$$\Rightarrow \vec{B} =$$

$$\nabla \cdot \vec{E} = \frac{1}{r \sin \theta} \quad 0 = 0$$

$$\nabla \times \vec{E} = \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left(\frac{A \sin^2 \theta}{r} [\cos u - \frac{1}{kr} \sin u] \right) \hat{r}$$

$$- \frac{1}{r} \frac{\partial}{\partial r} (A \sin \theta [\cos u - \frac{1}{kr} \sin u]) \hat{\theta}$$

$$= \frac{1}{r \sin \theta} \frac{A}{r} 2 \sin \theta \cos \theta [\cos u - \frac{1}{kr} \sin u] \hat{r}$$

$$- \frac{1}{r} A \sin \theta \left[-k \sin u + \frac{1}{kr^2} \sin u - \frac{1}{r} \cos u \right] \hat{\theta}$$

$$= \frac{2A \cos \theta}{r^2} (\cos u - \frac{1}{kr} \sin u) \hat{r} + \frac{A \sin \theta}{r} \left[\left(k - \frac{1}{kr^2} \right) \sin u + \frac{1}{r} \cos u \right] \hat{\theta}$$

$$= - \frac{\partial \vec{B}}{\partial t}$$