

## 第十二章 电动力学与相对论

### §1. 狭义相对论

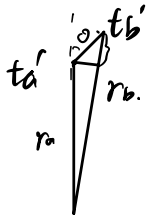
Intro. 在以前, 人们认为相对性原理在电磁学中并不成立, 亦即在不同惯性系间电磁学定律并非全部成立, 只在一个绝对静止参考系中成立, 这样可以对不同惯性系的物理现象作出“统一”的解释.

但爱因斯坦指出所有惯性系等价, 且电磁学定律在 每个参考系中 都有同样形式, 同样成立, 不存在对某现象的“统一”解释.

Att. 对于观测 (observe) 概念的澄清: 观测者的观测结果并非如人一般用眼睛, 而是在空间各点

有标位置的校正时钟, 观测者可以在事后人为处理数据, 但数据是由时钟获得的, 并非观测者则是误差, 而定义上的则是不同

Ex. 视速度可以超光速. (see can, observe not)



$$\begin{cases} t_a = t'_a + \frac{r_a}{c} \\ t_b = t'_b + \frac{r_b}{c} \end{cases}$$

$$\Delta t = t_b - t_a = t'_b - t'_a + \frac{(r_b - r_a)}{c} = t'_b - t'_a + \frac{v(t'_b - t'_a) \cos \theta}{c}$$

$$\Delta t' = t'_b - t'_a$$

$$V_{\text{视}} = \frac{v(t'_b - t'_a) \sin \theta}{\Delta t} = \frac{v \sin \theta}{1 + \frac{v}{c} \cos \theta} \quad \frac{dV_{\text{视}}}{d\theta} = 0 \Rightarrow \theta = \pi - \arccos \frac{v}{c} \quad \text{当 } v \rightarrow c$$

$$V_{\text{视}} = \frac{v \frac{\sqrt{c^2 - v^2}}{c}}{1 - v^2/c^2} = \frac{v}{\sqrt{1 - v^2/c^2}} \quad \text{在 } v \rightarrow c \text{ 时即趋于 } \infty$$

## Ex. 对时间膨胀的解释:

1° S 系中认为 S' 系中的钟变慢

火车为 S' 系中一个事件点, 它在 S 系中经过  $x_1, x_2$ , 被该处的钟记录下  $t_1$  与  $t_2$ , 火车记录下

在 S 系中经过  $x_2$  时, 被该处的钟记录下  $t_1$  与  $t_2$  火车记录下

S 系中的钟全被收, S 系观测到事件间隔  $t_2 - t_1$ , S' 系中观测到  $t_2' - t_1'$

有  $t_2' - t_1' = \frac{1}{\gamma} (t_2 - t_1)$  即 S 系认为 S' 系中的钟变慢

2° 但在 S' 系看来, S 系系了两个未校正的钟的时间间隔.

在 S' 系同时, S 系中位于  $x_1, x_2$  处 = 两钟相差

$$\Delta t = \gamma \frac{v}{c^2} \Delta x = \frac{v}{c^2} (x_2 - x_1) = \frac{v^2}{c^2} (t_2 - t_1), \text{ 所以 S 系中测得的时间间隔}$$

应减去  $\Delta t$ , 亦即 S' 系中认为, S 系中实际上只经过

$$(t_2 - t_1) - \Delta t = (1 - \frac{v^2}{c^2})(t_2 - t_1) = \frac{1}{\gamma^2} (t_2 - t_1), \quad t_2' - t_1' = \frac{1}{\gamma} (t_2 - t_1) = \gamma \left( \frac{1}{\gamma^2} (t_2 - t_1) \right)$$

亦即 S' 系认为 S 系中的钟变慢.

## 洛伦兹变换

Low. 洛伦兹变换: 利用长度收缩于得公式

$$\begin{cases} x' = \gamma(x - vt) \\ y' = y \\ z' = z \end{cases} \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \text{ 且 } v \text{ 正负与 } x \text{ 正负同}$$

$$t' = \gamma(ct - vx/c^2)$$

## 时空结构

Def. 时空四矢量:  $x^\mu$  表示时空四矢量, 具体而言,  $\mu=0,1,2,3$ .

$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z, \quad (\text{若写 } x^i, \text{ 则一般 } i \text{ 取 } 1,2,3)$$

Def. 洛伦兹变换矩阵:  $\Lambda^\mu_\nu$  为洛伦兹变换矩阵,  $\mu, \nu$  均取  $0,1,2,3$

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\Lambda^T = \Lambda)$$

Explain: 由于

$$\begin{cases} (x^0)' = \gamma(x^0 - \beta x^1) \\ (x^1)' = \gamma(x^1 - \beta x^0) \\ (x^2)' = x^2 \\ (x^3)' = x^3 \end{cases} \Rightarrow \begin{pmatrix} (x^0)' \\ (x^1)' \\ (x^2)' \\ (x^3)' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

因而有  $(x^\mu)' = \Lambda^\mu_\nu x^\nu$ , 称  $\Lambda$  称为洛伦兹变换矩阵.

Tip. 又, 4维标量积 即  $x^\mu \cdot x^\mu = x^\mu x_\mu$ . ( $x_\mu = g_{\mu\nu} x^\nu$ ,  $g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ )

$$\text{所以 } x^\mu \cdot x^\mu = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = x^2 + y^2 + z^2 - c^2 t^2$$

Def. 4-矢量: 在不同参考系中, 以洛伦兹变换联系的矢量称为4-矢量.

$$\text{且满足4维标积 } a^\mu \cdot a^\mu = a^\mu a_\mu$$

Ex. 31 进速度:  $\theta = \operatorname{artanh}(v/c)$

1° 用  $\theta$  表示沿  $x$  轴的纯洛伦兹变换

$$\tanh \theta = \frac{\sinh \theta}{\cosh \theta} = \frac{v}{c} = \beta \quad \text{又} \quad \gamma^2 - \gamma^2 \beta^2 = 1, \quad \frac{\gamma \beta}{\gamma} = \beta. \quad \text{因而} \quad \gamma \beta = \sinh \theta, \quad \gamma = \cosh \theta.$$

$$\Lambda = \begin{pmatrix} \cosh \theta & -\sinh \theta & 0 \\ -\sinh \theta & \cosh \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2° 用速度表示爱因斯坦速度叠加

$$u'_x = \frac{u_x - v}{1 - vu_x/c^2} \Rightarrow \frac{u'_x}{c} = \frac{\frac{u_x}{c} - \frac{v}{c}}{1 - \frac{v}{c} \frac{u_x}{c}} \quad \text{代} \quad \frac{u_x}{c} = \tanh \alpha, \quad \frac{u'_x}{c} = \tanh \alpha'$$

$$\Rightarrow \tanh \alpha' = \frac{\tanh \alpha - \tanh \theta}{1 - \tanh \alpha \tanh \theta} \quad (\text{差角公式}) \Rightarrow \tanh \alpha' = \tanh(\alpha - \theta) \Rightarrow \alpha' = \alpha - \theta \quad (\text{单侧})$$

因而有  $\alpha = \alpha' + \theta$  即 速度符合叠加. 绝对速度 = 相对速度 + 牵连速度

Thm. 间隔不变性: 设事件  $A$  坐标  $x_A^\mu$ ,  $B$  坐标  $x_B^\mu$ .

$\Delta x^\mu = x_A^\mu - x_B^\mu$  称为 4-位移矢,  $\Delta x^\mu \Delta x_\mu$  是一个标量不变量, 称为事件  $A$  与  $B$  的间隔

(Tip. 实际上, 若  $x_B^\mu = 0$  则对  $\forall x_A^\mu$ ,  $x_A^\mu x_{A\mu}$  均为标量不变量)

Def. 类时间隔, 类光间隔, 类空间隔: 按如上定义  $g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  有

$$\Delta x_\mu \Delta x^\mu = -(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2$$

若 1°  $\Delta x_\mu \Delta x^\mu < 0$ , 则称为类时间隔 ( $\exists S'$ , s.t.  $\Delta x'^1 = \Delta x'^2 = \Delta x'^3 = 0$ , 即同时)

2°  $\Delta x_\mu \Delta x^\mu = 0$ , 则称为类光间隔

3°  $\Delta x_\mu \Delta x^\mu > 0$ , 则称为类空间隔 ( $\exists S'$ , s.t.  $\Delta x'^0 = 0$ , 即同时)



## §2. 相对论力学.

固有时间和固有速度,

Def. 固有时: 相对参考系静止的时钟测得该系固有时或本标时.

证: 在该参考系内同时事件时间间隔为  $dt$ , 则在别的系内, 时间间隔为  $dt = \gamma dt$

Att: 某参考系(例如 A)中的固有时不随本标系变换而改变, 即不仅在 B 中, C 中, A 的固有时不变.

Def. 固有速度: 设 A 中静止的某粒子在 S 系中运动, 在 A 中用时为  $t$ ,  $dt$  内在 S 系中

经过了  $d\vec{l}$  ( $d\vec{l} = (dx_S^1, dx_S^2, dx_S^3)$ ) 又  $\vec{\eta} = \frac{d\vec{l}}{dt}$ , 称为固有速度.

$$\text{Att. } \frac{1}{\gamma} dt = dt \Rightarrow \vec{u} = \frac{d\vec{l}}{dt} = \sqrt{1-u^2/c^2} \vec{\eta} \Rightarrow \vec{\eta} = \frac{\vec{u}}{\sqrt{1-u^2/c^2}}$$

其次又有, 若将  $d\vec{l}$  改为四矢量  $dx^\mu$ , 则有  $\eta^\mu = \frac{dx^\mu}{d\tau}$ , 也为四矢量, 称为 4-速度

$$\eta^0 = \frac{dx^0}{d\tau} = \frac{cdt}{d\tau} = c\gamma = \frac{c}{\sqrt{1-u^2/c^2}} \quad (\text{Att. } \eta^\mu \eta_\mu = -c^2)$$

Att. 由于  $\eta^\mu$  为四矢量, 所以变换满足洛伦兹变换. 若有 S' 系沿 S 正方向以  $v$  运动

$$(\eta^\mu)' = \Lambda^\mu_\nu \eta^\nu \quad \text{而 正方向 } \vec{u} \text{ 变换较复杂}$$

Ex. 1° 推导出  $\vec{u}(\vec{\eta})$

$$\eta^2 = \frac{u^2}{1-u^2/c^2} \Rightarrow \vec{u} = \frac{\vec{\eta}}{\sqrt{1+\eta^2/c^2}}$$

2° 推导出粒子的快度  $\theta = \text{arctanh}(u/c)$  与  $\vec{\eta}$  的固有速度间关系

$$\tanh \theta = \frac{u}{c}, \quad \eta = \frac{u}{\sqrt{1-u^2/c^2}}, \quad \cosh \theta = \frac{1}{\sqrt{1-u^2/c^2}}, \quad \sinh \theta = \frac{u/c}{\sqrt{1-u^2/c^2}} \Rightarrow \eta = c \sinh \theta.$$

Ex. 考虑粒子在  $S$  中作双曲线运动.  $x(t) = \sqrt{b^2 + (ct)^2}$ ,  $y = z = 0$ .

1° 求以  $t$  为变量的固有时刻  $\tau$ , 且  $t=0$  时  $\tau=0$

$$d\tau = \sqrt{1 - u^2/c^2} dt, \quad u = \frac{dx}{dt} = \frac{c^2 t}{\sqrt{b^2 + (ct)^2}}$$

$$\Rightarrow d\tau = \frac{1}{\sqrt{1 + c^2 t^2/b^2}} dt \quad (\text{不妨令 } b>0) \quad \text{积分得} \quad \tau = \frac{b}{c} \ln\left(\frac{ct}{b} + \sqrt{1 + \frac{c^2 t^2}{b^2}}\right)$$

2° 求以  $\tau$  为变量的  $x$  与  $u$  (匀速运动)

$$m + \sqrt{1 - u^2} = e^{\frac{c\tau}{b}}, \quad m = \frac{ct}{b} \Rightarrow t = \frac{b}{c} (e^{\frac{c\tau}{b}} - e^{-\frac{c\tau}{b}}) = \frac{b}{c} \sinh\left(\frac{c\tau}{b}\right)$$

$$\text{由此, } x(t) = \sqrt{b^2 + (ct)^2} = x(\tau) = \frac{b}{2} (e^{\frac{c\tau}{b}} + e^{-\frac{c\tau}{b}}) = b \cosh\left(\frac{c\tau}{b}\right)$$

$$u = \frac{dx}{dt} = \frac{dx}{d\tau} \left(\frac{d\tau}{dt}\right)^{-1} = c \tanh\left(\frac{c\tau}{b}\right)$$

3° 求以  $t$  为变量的  $\eta^\mu$

$$\eta^\mu = \frac{dx^\mu}{d\tau}, \quad \eta^0 = \frac{c dt}{d\tau} = \frac{c}{\sqrt{1 - u^2/c^2}} = c \sqrt{1 + c^2 t^2/b^2} = c \cosh\left(\frac{c\tau}{b}\right)$$

$$\eta^1 = \frac{dx}{d\tau} = c \sinh\left(\frac{c\tau}{b}\right) = \frac{c^2 t}{b}$$

相对论能量和动量

Def. 能动4矢量:

1° 相对论动量

$$\vec{p} = m\vec{\eta} \quad \text{or} \quad p^i = m\eta^i \quad \text{or} \quad \vec{p} = \frac{m\vec{u}}{\sqrt{1 - u^2/c^2}}$$

Tip:  $p^i = m\eta^i = m \frac{dx^i}{d\tau}$ .  $m$  与  $d\tau$  均为洛伦兹标量, 因而再补上  $m \frac{dx^0}{d\tau}$ , 则予构成四元组.

2° 相对论能量,

$$E = \frac{mc^2}{\sqrt{1-u^2/c^2}} = mc\eta^0 = mc \frac{cd\tau}{dt} = \frac{mc^2}{\sqrt{1-u^2/c^2}}$$

所以将  $E$  作为  $p^0$ , 则有  $p^\mu$  是四矢量, 且有  $p^\mu p_\mu = -\frac{m^2 c^2}{1-u^2/c^2} + \frac{m^2 u^2}{1-u^2/c^2} = -m^2 c^2$

Extra Def:

$$\downarrow$$

$$-E^2 + p^2 c^2 = -m^2 c^4 \Rightarrow E^2 = p^2 c^2 + m^2 c^4$$

相对论质量:  $m_{rel} = \frac{m}{\sqrt{1-u^2/c^2}}$  静止质量:  $m$

静止能量:  $E_0 = mc^2$  动能:  $E_k = E - E_0 = mc^2 \left( \frac{1}{\sqrt{1-u^2/c^2}} - 1 \right)$

Law. 能量是守恒

四矢量中的  $E$  与  $\vec{p}$  对于任何封闭体系均守恒.

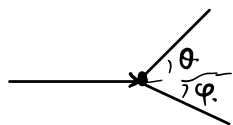
Att: 位置不守恒与不变,  $E, \vec{p}$  守恒非绝对不变,  $m$  不是非守恒,  $\vec{p}$  守恒又守恒.  
 $\downarrow$   
 过程前后不变.

相对论运动学

Def. 相对论中守恒量守恒: 动能守恒的守恒量.

Tip. 相对论中, 位置一般不守恒而能量守恒, 经典力学中, 动能一般不守恒但能量守恒  
 而弹性碰撞中, 能量守恒 + 动能守恒  $\rightarrow$  位置守恒.

Ex. 康普顿散射: 一能量为  $E_0$  光子与静止电子发生碰撞, 求出散射光子能量  $E$  关于光子散射角  $\theta$  的函数



$$\begin{cases} \frac{E}{c} \cos \theta + \check{p}_e \cos \check{\varphi} = \frac{E_0}{c} \\ \frac{E}{c} \sin \theta = \check{p}_e \sin \varphi \end{cases}$$

$$\Rightarrow E = \frac{1}{(E_0)^{-1} + (1 - \cos \theta) (mc^2)^{-1}}$$

$$\sqrt{p^2 c^2 + m^2 c^4} + E = E_0 + m c^2$$

由此  $E = \frac{hc}{\lambda}$ ,  $E_0 = \frac{hc}{\lambda_0} \Rightarrow \lambda = \lambda_0 + (1 - \cos\theta) \frac{h}{mc}$ , 即波长变短, 光子能量

$E_x$ . 对撞的能量增益. (即相对能量增益).

恒量为二 质量为  $m$  粒子均有能量  $E$ .

1° 经典情况.

例 = 若速度相同,  $v_{\text{相}} = 2v$ ,  $\bar{E} = \frac{1}{2} m (2v)^2 = 4E$

2° 相对论情况.

则同有 = 若速度为  $v$ .  $v_{\text{相}} = \frac{-v - v}{1 + v^2/c^2} = -\frac{2v}{1 + v^2/c^2}$

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}}, \quad \bar{E} = \frac{mc^2}{\sqrt{1 - (2v/(1 + v^2/c^2))^2}} = \frac{1 + v^2/c^2}{1 - v^2/c^2} mc^2 = mc^2 \left( \frac{2}{1 - v^2/c^2} - 1 \right) = \frac{2E^2}{mc^2} - mc^2$$

由于与  $E$  成平方关系, 增益极大增益.

例如  $E = 30 \text{ GeV}$  的质子 ( $mc^2 = 1 \text{ GeV}$ )  $\Rightarrow \bar{E} = 1799 \text{ GeV} \approx 60E$

## 相对论动力学

Law. 相对论中的力:  $\vec{F} = \frac{d\vec{p}}{dt}$ ,  $\vec{p} = (p^0, p^1, p^2, p^3)$ .

Def. 功: 在某位移中做的功定义为  $W = \int \vec{F} \cdot d\vec{l}$

Thm. 动能定理:  $dE = dE_k = \vec{F} \cdot \vec{u} dt$ ,  $\Delta E = \Delta E_k = \int \vec{F} \cdot d\vec{l}$

$$\int \vec{F} \cdot d\vec{l} = \int \vec{F} \cdot \vec{u} dt = \int \frac{d\vec{p}}{dt} \cdot \vec{u} dt = \int \frac{m \frac{d\vec{u}}{dt} (1 - u^2/c^2) + \frac{m\vec{u}}{c^2} (\vec{u} \cdot \frac{d\vec{u}}{dt})}{(1 - u^2/c^2)^{3/2}} \cdot \vec{u} dt = \int \frac{m\vec{u}}{(1 - u^2/c^2)^{3/2}} \cdot d\vec{u} = \int d\left(\frac{mc^2}{\sqrt{1 - u^2/c^2}}\right) = \int dE = \int dE_k$$

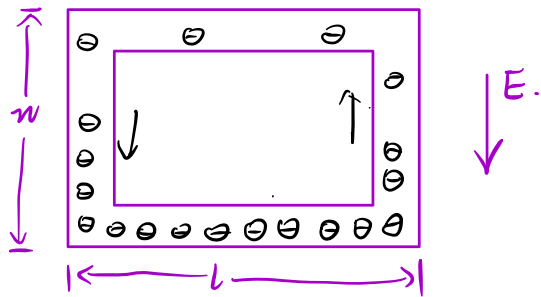
Att. 牛=与功能对应, 牛=不易, 由于在某参考系内同时的力, 若作用的物体空间上分开  
 则在所参考系内, 不一定同时, 违反牛三条件 一般仅在二者相对静止时, 牛三成立  
 力的变换较复杂, 因为  $dp$  与  $dt$  均非洛伦兹不变量

Def. 固有: 某物所受固有  $K^\mu = \frac{dp^\mu}{d\tau}$   $\tau$  为固有时

$K^\mu$  为四矢量,  $K^i = \frac{dp^i}{d\tau} = \frac{dp^i}{dt} \frac{dt}{d\tau} = \frac{1}{\sqrt{1-u^2/c^2}} F^i$

$K^0 = \frac{d(cE)}{d\tau} = \frac{1}{c} \frac{dE}{d\tau}$

Ex. 隐藏能量: 在既有电场又有磁场时, 电磁场能量不为零, 而若其由一个静止电流框产生, 在其静止  
 系的静止系中, 能量应为零, 因而在电荷中隐藏了能量,



考虑左图, 在一个电流框外施加匀强场  $E$ , 且达到稳态,  
 载流子为  $e$ , 电流为  $I$ , 求电子的能量。

在左右侧电子由于电场作用减速或加速, 由于电流处处相等, 上侧快速, 电荷量低, 下侧较慢电荷  
 量高  
 $n_e v_F = n_u e v_{\uparrow} = I$

$p_{\uparrow} = m n_{\uparrow} \times l \times v_{\uparrow}$  ,  $p_{\downarrow} = m n_{\downarrow} \times l \times v_{\downarrow}$  ,  $p_{\uparrow} = p_{\downarrow}$ , 方向相反相互抵消。

但若考虑 相对论能量

(  $p_{\uparrow} = m n_{\uparrow} \times l \times \frac{v_{\uparrow}}{\sqrt{1-v_{\uparrow}^2/c^2}}$  ,  $p_{\downarrow} = m n_{\downarrow} \times l \times \frac{v_{\downarrow}}{\sqrt{1-v_{\downarrow}^2/c^2}}$  )

$$\left\{ -\frac{mc^2}{\sqrt{1-v_x^2/c^2}} + \frac{mc^2}{\sqrt{1-v_z^2/c^2}} = eEw. \right.$$

$$k) \quad \frac{I}{e} \times \frac{l}{c^2} eEw = p_{\perp} - p_{\parallel} \Rightarrow \omega = \frac{EI\omega l}{c^2} \quad \text{向右} \quad \rightarrow \frac{I l V}{c^2}$$

又  $I\omega l = m$  所以  $\vec{p} = \frac{1}{c^2} \vec{m} \times \vec{E}$  (它与电磁场动量相等而相反, 此动量又严格相对论成立)

Ex. 次级固有加速度:  $\alpha^\mu = \frac{d\eta^\mu}{d\tau} = \frac{d^2 x^\mu}{d\tau^2}$

1° 求  $\alpha^0$  和  $\vec{\alpha}$ . 用  $\vec{u}$  与  $\vec{a}$  表示 (正常速度与加速度)

$$\alpha^0 = \frac{d\eta^0}{d\tau} = \frac{d(\frac{c}{\sqrt{1-u^2/c^2}})}{d\tau} = c \frac{d(\frac{1}{\sqrt{1-u^2/c^2}})}{dt} \frac{dt}{d\tau} = \frac{c}{\sqrt{1-u^2/c^2}} \frac{\frac{2\vec{u}}{c^2} \cdot \frac{d\vec{u}}{dt}}{2(1-u^2/c^2)^{3/2}} = \frac{1}{(1-u^2/c^2)^2} \vec{u} \cdot \vec{a}$$

$$\vec{\alpha} = \frac{d\vec{\eta}}{d\tau} = \frac{d(\frac{\vec{u}}{\sqrt{1-u^2/c^2}})}{d\tau} = \frac{d\vec{u}}{d\tau} \frac{1}{\sqrt{1-u^2/c^2}} + \vec{u} \frac{\frac{1}{c^2} \vec{u} \cdot \frac{d\vec{u}}{dt}}{(1-u^2/c^2)^{3/2}} = \frac{\vec{a} + \vec{u} \frac{1}{c^2} (\vec{u} \cdot \vec{a}) \frac{1}{(1-u^2/c^2)}}{1-u^2/c^2} = \frac{1}{(1-u^2/c^2)} \left[ \vec{a} + \frac{\vec{u}(\vec{u} \cdot \vec{a})}{c^2 - u^2} \right]$$

$$= \frac{1}{m} \frac{d\vec{p}}{d\tau} = \frac{1}{m} \vec{F} \frac{1}{\sqrt{1-u^2/c^2}}$$

2° 以  $\vec{u}$  与  $\vec{a}$  表示  $\alpha_\mu \alpha^\mu$

$$\begin{aligned} \alpha_\mu \alpha^\mu &= -(\alpha^0)^2 + |\vec{\alpha}|^2 = \frac{-1}{(1-u^2/c^2)^4} \frac{(\vec{u} \cdot \vec{a})^2}{c^2} + \frac{1}{(1-u^2/c^2)^2} \left[ a^2 + \frac{u^2 (\vec{u} \cdot \vec{a})^2}{(c^2 - u^2)^2} + 2 \frac{(\vec{u} \cdot \vec{a})^2}{c^2 - u^2} \right] \\ &= \frac{1}{(1-u^2/c^2)^4} \left[ \frac{(\vec{u} \cdot \vec{a})^2}{c^2} (-1 + \frac{u^2}{c^2} + 2 - 2 \frac{u^2}{c^2}) + \frac{a^2}{c^2} (c^2 - u^2)^2 \right] \\ &= \frac{1}{(1-u^2/c^2)^2} \left[ \frac{(\vec{u} \cdot \vec{a})^2}{c^2 - u^2} + a^2 \right] \end{aligned}$$

3° 证明  $\eta^\mu \alpha_\mu = 0$

$$\alpha_\mu = \frac{d\eta_\mu}{d\tau}, \quad \eta^\mu \alpha_\mu = \frac{d(\eta^\mu \eta_\mu)}{2 d\tau} = \frac{d(c^2)}{2 d\tau} = 0.$$

4. 写出平动的闵科夫斯基形式, 用  $\alpha^\mu$  表示, 求  $k^\mu \eta_\mu$ .

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d\vec{p}}{d\tau} \frac{d\tau}{dt} = m \frac{d\vec{\eta}}{d\tau} \frac{d\tau}{dt} = m \vec{\alpha} \sqrt{1-u^2/c^2}$$

$$k^\mu = \frac{dp^\mu}{d\tau} = m \alpha^\mu. \quad k^\mu \eta_\mu = m \alpha^\mu \eta_\mu = 0.$$

$$k^\mu k_\mu = \frac{dp^\mu}{d\tau} \frac{dp_\mu}{d\tau} = -\left(\frac{dp^0}{d\tau}\right)^2 + F^2 \left(\frac{dt}{d\tau}\right)^2 = \left(\frac{dt}{d\tau}\right)^2 \left(F^2 - \frac{1}{c^2} u^2 F^2 \cos^2 \theta\right) = F^2 \frac{1-u^2 \cos^2 \theta / c^2}{1-u^2/c^2}$$

$$\frac{dp^0}{d\tau} = \frac{1}{c} \frac{dE}{d\tau} = \frac{1}{c} \vec{u} \cdot \vec{F}$$

### §3. 相对论电动力学

#### 相对论中的磁现象

Intro: 相对论中的电磁现象在不用磁当文件时, 由电当文件 + 洛伦兹变换  $\rightarrow$  磁当文件.

亦即在不同参考系中, 对同一现象解释会不同.

Thm. 电磁场的洛伦兹变换: 假设  $S'$  系相对  $S$  系沿  $x$  轴以  $v$  运动,  $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$

$$\text{则} \begin{cases} E'_x = E_x \\ E'_y = \gamma(E_y - v B_z) \\ E'_z = \gamma(E_z + v B_y) \end{cases} \quad \begin{cases} B'_x = B_x \\ B'_y = \gamma(B_y + \frac{v}{c^2} E_z) \\ B'_z = \gamma(B_z - \frac{v}{c^2} E_y) \end{cases}$$

在  $S$  系中有洛伦兹力公式,

$\vec{F} = q(\vec{E} + \vec{u} \times \vec{B})$ . 据此与力变换, 速度变换与电荷洛伦兹不变性推导变换.

$$\text{即求出 } \vec{F}' = q'(\vec{E}' + \vec{u}' \times \vec{B}') = q(\vec{E}' + \vec{u}' \times \vec{B}')$$

$$\begin{cases} u_x = \frac{u'_x + v}{1 + u'_x v / c^2} \\ u_y = \frac{u'_y}{\gamma(1 + u'_x v / c^2)} \\ u_z = \frac{u'_z}{\gamma(1 + u'_x v / c^2)} \end{cases} \quad \begin{cases} F_x = F'_x + \frac{v u'_y / c^2}{1 + u'_x v / c^2} F'_y + \frac{v u'_z / c^2}{1 + u'_x v / c^2} F'_z \\ F_y = \frac{1}{\gamma(1 + u'_x v / c^2)} F'_y \\ F_z = \frac{1}{\gamma(1 + u'_x v / c^2)} F'_z \end{cases} \quad \frac{1}{\gamma(1 + u'_x v / c^2)} = \delta.$$

$$\begin{cases} F_x = q(E_x + u_y B_z - u_z B_y) \quad \text{--- ①} \\ F_y = q(E_y + u_z B_x - u_x B_z) \quad \text{--- ②} \\ F_z = q(E_z + u_x B_y - u_y B_x) \quad \text{--- ③} \end{cases}$$

$$\begin{aligned} \text{③: } \frac{\delta}{r} F'_z &= q(E_z + (u'_x + v)\delta B_y - \frac{\delta}{r} u'_y B_x) \\ \Rightarrow F'_z &= q\left(\frac{r E_z}{\delta} + r(u'_x + v) B_y - u'_y B_x\right) \\ &= q\left[r(E_z + v B_y) + u'_x(r B_y + \frac{r v}{c^2} E_z) - u'_y B_x\right] \end{aligned} \quad \left. \vphantom{\frac{\delta}{r} F'_z} \right\} \Rightarrow \begin{cases} E'_z = r(E_z + v B_y) \\ B'_y = r(B_y + \frac{v}{c^2} E_z) \\ B'_x = B_x \end{cases}$$

$$\begin{aligned} \text{②: } \frac{\delta}{r} F'_y &= q(E_y + \frac{\delta}{r} u'_z B_x - (u'_x + v)\delta B_z) \\ \Rightarrow F'_y &= q\left(\frac{r}{\delta} E_y + u'_z B_x - r(u'_x + v) B_z\right) \\ &= q\left[r(E_y - v B_z) + r\left(\frac{v}{c^2} E_y - B_z\right) u'_x + u'_z B_x\right] \end{aligned} \quad \left. \vphantom{\frac{\delta}{r} F'_y} \right\} \Rightarrow \begin{cases} E'_y = r(E_y - v B_z) \\ B'_z = r(B_z - \frac{v}{c^2} E_y) \\ B'_x = B_x \end{cases}$$

①: 由于只有  $E'_x$  未知, 只需寻找  $F'_x$  左侧系数以发现与  $u'_x, u'_y, u'_z$  无关的公共因子, 得  $E'_x = E_x$ .

因而有变换式成立.

Tip:  $\gamma$  可以从别的角度理解.

1°  $\vec{B} = 0$  时,  $E'_y = r E_y, E'_z = r E_z$ , 类似于一块无穷大带电板. 沿  $x$  轴收缩,  $\sigma' = \gamma \sigma$ . 因而  $E'_y, E'_z \rightarrow \gamma E_y, \gamma E_z$



$B_y$  与  $B_z$  同理

2°  $\vec{B}=0$  时, 带电无限板 //  $yOz$ , 此时, 板板间距收缩, 但  $\sigma'=\sigma$ , 因而  $E_x'=E_x$

同理, 带电无限线, 沿  $x$  轴运动,  $n' \rightarrow \gamma n$ , 但在  $S'$  中, 通过相同电荷所截长度 (SS 非正)

$$\text{则 } I' = \frac{1}{\gamma} I, \text{ 故上. } B_x' = B_x$$

Att. 特殊情况

$$1^\circ \vec{B}=0, \text{ 则 } \vec{B}' = \gamma \frac{v}{c^2} E_z \hat{y}' - \gamma \frac{v}{c^2} E_y \hat{z}' = \frac{-1}{c^2} \vec{v} \times \vec{E}' \quad (\vec{v}=v\hat{x}')$$

$$2^\circ \vec{E}=0, \text{ 则 } \vec{E}' = -\gamma v B_z \hat{y}' + \gamma v B_y \hat{z}' = \vec{v} \times \vec{B}' \quad (\vec{v}=v\hat{x}')$$

Att. 守恒量:  $\frac{1}{2} \epsilon_0 E^2 - \frac{1}{2\mu_0} B^2$  与  $\vec{E} \cdot \vec{B}$  均为守恒量

$$\begin{aligned} \frac{1}{2} \epsilon_0 [(E')^2 - c^2 (B')^2] &= \frac{1}{2} \epsilon_0 [E_x^2 + \gamma^2 (E_y - v B_z)^2 + \gamma^2 (E_z + v B_y)^2 - c^2 B_x^2 - c^2 \gamma^2 (B_y + \frac{v}{c^2} E_z)^2 - c^2 \gamma^2 (B_z - \frac{v}{c^2} E_y)^2] \\ &= \frac{1}{2} \epsilon_0 [E_x^2 + (r^2 - c^2 r^2 \frac{v^2}{c^4}) E_y^2 + (r^2 - c^2 r^2 \frac{v^2}{c^4}) E_z^2 - c^2 B_x^2 - (c^2 r^2 - r^2 v^2) B_y^2 - (c^2 r^2 - r^2 v^2) B_z^2] \\ &\quad + \epsilon_0 [-r^2 v E_y B_z + r^2 v E_z B_y - r^2 v B_y E_z + r^2 v B_z E_y] \\ &= \frac{1}{2} \epsilon_0 [E_x^2 + E_y^2 + E_z^2 - c^2 B_x^2 - c^2 B_y^2 - c^2 B_z^2] = \frac{1}{2} \epsilon_0 [E^2 - c^2 B^2] \\ \vec{E}' \cdot \vec{B}' &= E_x B_x + r^2 (E_y - v B_z) (B_y + \frac{v}{c^2} E_z) + r^2 (E_z + v B_y) (B_z - \frac{v}{c^2} E_y) \\ &= E_x B_x + r^2 (1 - \frac{v^2}{c^2}) E_y B_y + r^2 (1 - \frac{v^2}{c^2}) E_z B_z \\ &\quad + (r^2 \frac{v}{c^2} - r^2 \frac{v}{c^2}) E_y E_z + (-r^2 v + r^2 v) B_y B_z \\ &= E_x B_x + E_y B_y + E_z B_z \end{aligned}$$

因而  $\vec{E} \cdot \vec{B}$  守恒, 电场与磁场的能量密度差不全随坐标与变换改变.

Extra. 予以通过先求静电场的变换, 证明变换后的  $\vec{E}, \vec{B}$  仍满足 Maxwell Equations

再由不同运动电荷分布产生的场进行矢量叠加, 得  $\vec{E}, \vec{B}$  仍满足 Maxwell Equations

(Att:  $\rho' = \gamma \rho$ )

Ex. 电磁波在不同参考系下的变换

# 场张量

Intro: 六自由度的是可由二阶反对称张量描述.

而场的变换可以用二阶张量的变换描述, 具体上即 设  $t^{\mu\nu}$  是二阶张量

且若  $(t^{\mu\nu})' = \Lambda^\mu_\lambda \Lambda^\nu_\sigma t^{\lambda\sigma}$ , 则称  $t^{\mu\nu}$  是洛伦兹张量.

并且根据  $t^{\lambda\sigma} = -t^{\sigma\lambda}$  与  $\Lambda_\lambda^\mu = \begin{pmatrix} \gamma & -\beta\gamma & 0 \\ -\beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$  推导  $(t^{\mu\nu})'$  与  $t^{\lambda\sigma}$  关系.

$$\begin{aligned} (t^{01})' &= \Lambda^0_\lambda \Lambda^1_\sigma t^{\lambda\sigma} = \Lambda^0_0 \Lambda^1_0 t^{00} + \Lambda^0_0 \Lambda^1_1 t^{01} + \Lambda^0_1 \Lambda^1_0 t^{10} + \Lambda^0_1 \Lambda^1_1 t^{11} \quad (t^{00}=t^{11}=0) \\ &= (\Lambda^0_0 \Lambda^1_1 - \Lambda^0_1 \Lambda^1_0) t^{01} = t^{01} \end{aligned}$$

$$(t^{02})' = \Lambda^0_\lambda \Lambda^2_\sigma t^{\lambda\sigma} = \Lambda^0_1 \Lambda^2_2 t^{12} + \Lambda^0_0 \Lambda^2_2 t^{02} = \gamma(t^{02} - \beta t^{12})$$

$$(t^{03})' = \Lambda^0_\lambda \Lambda^3_\sigma t^{\lambda\sigma} = \Lambda^0_1 \Lambda^3_3 t^{13} + \Lambda^0_0 \Lambda^3_3 t^{03} = \gamma(t^{03} - \beta t^{13})$$

$$(t^{12})' = \Lambda^1_\lambda \Lambda^2_\sigma t^{\lambda\sigma} = \Lambda^1_0 \Lambda^2_2 t^{02} + \Lambda^1_1 \Lambda^2_2 t^{12} = \gamma(t^{12} - \beta t^{02})$$

$$(t^{13})' = \Lambda^1_\lambda \Lambda^3_\sigma t^{\lambda\sigma} = \Lambda^1_0 \Lambda^3_3 t^{03} + \Lambda^1_1 \Lambda^3_3 t^{13} = \gamma(t^{13} - \beta t^{03})$$

$$(t^{23})' = \Lambda^2_\lambda \Lambda^3_\sigma t^{\lambda\sigma} = \Lambda^2_2 \Lambda^3_3 t^{23} = t^{23}$$

$$\text{对于 } \begin{cases} E_x' = E_x \\ E_y' = \gamma(E_y - v B_z) \\ E_z' = \gamma(E_z + v B_y) \end{cases} \quad \begin{cases} B_x' = B_x \\ B_y' = \gamma(B_y + \frac{v}{c^2} E_z) \\ B_z' = \gamma(B_z - \frac{v}{c^2} E_y) \end{cases} \Rightarrow \begin{cases} \frac{E_x}{c} = t^{01} \\ \frac{E_y}{c} = t^{02} \\ \frac{E_z}{c} = t^{03} \end{cases} \quad \begin{cases} B_x = t^{23} \\ -B_y = t^{13} \\ B_z = t^{12} \end{cases} \quad \text{or } \begin{cases} \frac{E_i}{c} \rightarrow B_i \\ B_i \rightarrow -\frac{E_i}{c} \end{cases}$$

例如可将  $(\frac{E}{c}, B)$  or  $(E, Bc)$  作为  $t$  来对应.

记代入  $\underline{E}_i$  与  $B_i$  的二所反对称张量为场强张量

Def. 场强张量:

$$F^{\mu\nu} = \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & B_z & -B_y \\ -\frac{E_y}{c} & -B_z & 0 & B_x \\ -\frac{E_z}{c} & B_y & -B_x & 0 \end{pmatrix} \quad \text{以及其对应的张量} \quad G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -\frac{E_x}{c} & \frac{E_y}{c} \\ -B_y & \frac{E_x}{c} & 0 & -\frac{E_z}{c} \\ -B_z & -\frac{E_y}{c} & \frac{E_z}{c} & 0 \end{pmatrix}$$

且满足洛伦兹变换.  $(F^{\sigma\lambda})' = \Lambda^\sigma_\mu \Lambda^\lambda_\nu (F^{\mu\nu})$ ,  $(G^{\mu\nu})'$  同有

Extr.

$$F^{\mu\nu} F_{\mu\nu} = 2(B^2 - \frac{E^2}{c^2}) = 4\mu_0(\frac{1}{2}\epsilon_0 B^2 - \frac{1}{2}\epsilon_0 E^2), \quad G^{\mu\nu} G_{\mu\nu} = -F^{\mu\nu} F_{\mu\nu}$$

↓  
洛伦兹标量

$$F^{\mu\nu} G_{\mu\nu} = -\frac{4}{c}(\vec{E} \cdot \vec{B})$$

↓  
洛伦兹标量

张量形式的电动力学

Intro: 电流密度四矢量: 在相对电荷静止系中, 记  $l$  为  $l_0$ ,  $\vec{j}_0 = 0$ .

$$l = l_0 \frac{1}{\sqrt{1-u^2/c^2}} \quad (\text{长度收缩} + \text{电荷密度不变}) \quad \vec{j} = \frac{l_0 \vec{u}}{\sqrt{1-u^2/c^2}}$$

$$(lc, \vec{j}) \text{ 记为 } l_0 \eta^\mu, \text{ 即 } l_0 \eta^0 = lc, \quad l_0 \vec{\eta} = \vec{j}$$

Def. 电流密度四矢量:  $J^\mu = (lc, J_x, J_y, J_z)$  定义为电流密度四矢量

(由于  $J^\mu = \rho_0 \eta^\mu$ , 所以  $J^\mu$  符合洛伦兹变换)

Thm. 连续性方程.  $\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \Leftrightarrow \frac{\partial J^\mu}{\partial x^\mu} = 0$  or  $\partial_\mu J^\mu = 0$  ( $\partial_\mu := (\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3})$ )

Thm. Maxwell Equations (场强张量表述):

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu, \quad \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0 \quad (\text{也可写作})$$

(相对论体系,  $\mu$  取 0, 1, 2, 3. 分量方程, 与矢量形式一致)

$$\frac{\partial F^{1\nu}}{\partial x^\nu} = \frac{1}{c} \nabla \cdot \vec{E} = \mu_0 c \rho \Rightarrow \nabla \cdot \vec{E} = \frac{1}{\epsilon_0} \rho \quad (\text{高斯定理})$$

$$\frac{\partial F^{1\nu}}{\partial x^\nu} = -\frac{1}{c^2} \frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 J_x \Leftrightarrow (\nabla \times \vec{B})_x = \left( \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)_x$$

结合  $\mu=2, 3$  得  $\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

于是  $\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu$  包含  $\nabla \cdot \vec{E}$  ( $\mu=0$ ) 与  $\nabla \times \vec{B}$  ( $\mu=1, 2, 3$ )

$$\frac{\partial G^{0\nu}}{\partial x^\nu} = \nabla \cdot \vec{B} = 0$$

$$\frac{\partial G^{1\nu}}{\partial x^\nu} = -\frac{1}{c} \frac{\partial B_x}{\partial t} - \frac{1}{c} \frac{\partial E_z}{\partial y} + \frac{1}{c} \frac{\partial E_y}{\partial z} = 0 \Leftrightarrow (\nabla \times \vec{E})_x = -\left(\frac{\partial \vec{B}}{\partial t}\right)_x$$

结合  $\mu=1, 2, 3$  得  $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

于是  $\frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0$  包含  $\nabla \cdot \vec{B}$  ( $\mu=0$ ) 与  $\nabla \times \vec{E}$  ( $\mu=1, 2, 3$ )

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu$$

$$\begin{array}{l} \mu=0 \downarrow \\ \nabla \cdot \vec{E} = \frac{1}{\epsilon_0} \rho \end{array} \quad \begin{array}{l} \mu=1, 2, 3 \downarrow \\ \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \end{array}$$

$$\frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0$$

$$\begin{array}{l} \mu=0 \downarrow \\ \nabla \cdot \vec{B} = 0 \end{array} \quad \begin{array}{l} \mu=1, 2, 3 \downarrow \\ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \end{array}$$

Def. 电磁场有力:  $K^\mu = q\eta_\nu F^{\mu\nu}$

根据三力对有力定义  $\vec{F} = \frac{d\vec{p}}{dt} = \frac{d\vec{p}}{d\tau} \frac{d\tau}{dt} = \frac{1}{\sqrt{1-u^2/c^2}} \vec{K} \Rightarrow \vec{K} = \frac{1}{\sqrt{1-u^2/c^2}} \vec{F} = \frac{q}{\sqrt{1-u^2/c^2}} (\vec{E} + \vec{u} \times \vec{B})$

又根据  $\frac{dp^0}{d\tau} = \frac{dp^0}{dt} \frac{dt}{d\tau} = \frac{1}{c} \frac{dE}{d\tau} \frac{1}{\sqrt{1-u^2/c^2}} = \frac{1}{c} \vec{F} \cdot \vec{u} \frac{1}{\sqrt{1-u^2/c^2}} = \frac{1}{c} \vec{F} \cdot \vec{\eta} = K^0$  (对所有力成立)

因而  $K^\mu$  理应有  $K^0 = \frac{1}{c} q(\vec{E} + \vec{u} \times \vec{B}) \cdot \frac{\vec{u}}{\sqrt{1-u^2/c^2}} = \frac{1}{c} \frac{q\vec{u} \cdot \vec{E}}{\sqrt{1-u^2/c^2}} = \frac{1}{c} \frac{\vec{J} \cdot \vec{E}}{\sqrt{1-u^2/c^2}} = \frac{1}{c\sqrt{1-u^2/c^2}} \frac{dE}{d\tau}$  一致

下证  $K^\mu = q\eta_\nu F^{\mu\nu}$  正确满足.

$$K^0 = \frac{q}{c} \vec{E} \cdot \vec{u} \frac{1}{\sqrt{1-u^2/c^2}} = \frac{1}{c} \vec{F} \cdot \vec{\eta}, \quad K^1 = \frac{q}{\sqrt{1-u^2/c^2}} (-E_x + B_z u_y - B_y u_z) = \frac{q}{\sqrt{1-u^2/c^2}} (\vec{E} + \vec{u} \times \vec{B})_x = \frac{1}{\sqrt{1-u^2/c^2}} F_x$$

其余同理可得

相对论势

Def. 四维势:  $A^\mu = (\phi/c, A_x, A_y, A_z)$  有  $F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}$  (Att.  $x_0 = -x^0 = -ct$ )

验证其与  $\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$  与  $\vec{B} = \nabla \times \vec{A}$  的协调性

$$F^{01} = \frac{E_x}{c} = \frac{\partial A^1}{\partial x_0} - \frac{\partial A^0}{\partial x_1} = -\frac{\partial A_x}{c \partial t} - \frac{1}{c} \frac{\partial \phi}{\partial x} \Leftrightarrow (\vec{E})_x = (-\frac{\partial \vec{A}}{\partial t} - \nabla\phi)_x$$

其余分量同理.

$$F^{12} = B_z = \frac{\partial A^2}{\partial x_1} - \frac{\partial A^1}{\partial x_2} = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = (\nabla \times \vec{A})_z$$

其余分量同理, 因而电磁场张量  $F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}$

Att. 如前所述, 规范不变性体现在  $A' \rightarrow A + \nabla\lambda$ ,  $\phi' = \phi - \frac{\partial \lambda}{\partial t}$ .

若体现在四维势上, 等价于  $A'^\mu \rightarrow A^\mu + \frac{\partial \lambda}{\partial x_\mu}$ . ( $\phi'/c = \phi/c + \frac{\partial \lambda}{\partial (-ct)} \Leftrightarrow \phi' = \phi - \frac{\partial \lambda}{\partial t}$ ,  $\vec{A}' = \vec{A} + \nabla\lambda$ )

而人人电磁场张量来看, 它确实不改变  $F^{\mu\nu}$ .

$$F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} = \frac{\partial A^\nu}{\partial x_\mu} + \frac{\partial \lambda}{\partial x_\nu \partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} - \frac{\partial \lambda}{\partial x_\mu \partial x_\nu} = \frac{\partial(A^\nu - \frac{\partial \lambda}{\partial x_\nu})}{\partial x_\mu} - \frac{\partial(A^\mu - \frac{\partial \lambda}{\partial x_\mu})}{\partial x_\nu} = \frac{\partial(A^\nu)'}{\partial x_\mu} - \frac{\partial(A^\mu)'}{\partial x_\nu}$$

Att. 洛伦兹规范:  $\nabla \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial \phi}{\partial t} = 0 \Leftrightarrow \partial_\mu A^\mu = 0$  ( $\frac{\partial A^\mu}{\partial x^\mu} = 0$ )

Thm. 势表示的麦克斯韦方程:

$$\frac{\partial}{\partial x_\mu} \left( \frac{\partial A^\nu}{\partial x^\nu} \right) - \frac{\partial}{\partial x_\nu} \left( \frac{\partial A^\mu}{\partial x^\mu} \right) = \mu_0 J^\mu. \xrightarrow[\text{洛伦兹}]{\text{规范}} \square^2 A^\mu = -\mu_0 J^\mu \quad (\square^2 := \frac{\partial}{\partial x_\nu} \left( \frac{\partial}{\partial x^\nu} \right) = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2})$$

Att. 由势表示自动满足  $\frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0$ , 而代入  $\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu$  即得上方程