

§1. 势表述

标势和矢势

Intro. 在时变的电磁场中, 再也 $\nabla \times \vec{E} = 0$, 也就无法将 \vec{E} 表为 $-\nabla\phi$

但 $\nabla \cdot \vec{B} = 0$, 所以仍有 $\nabla \times \vec{A} = \vec{B}$

则有 $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\nabla \times \frac{\partial \vec{A}}{\partial t} \Rightarrow \nabla \times (\vec{E} + \frac{\partial \vec{A}}{\partial t}) = 0$ 所以有 $\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla\phi$, $\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla\phi$

Thm. 势表示式: $\begin{cases} \vec{B} = \nabla \times \vec{A} \\ \vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla\phi \end{cases}$ 用此二式给出用势表示的 Maxwell Equations

证: 二式自然满足 $\nabla \cdot \vec{B} = 0$ 与 $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$.

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \Leftrightarrow +\frac{\partial}{\partial t}(\nabla \cdot \vec{A}) + \nabla^2\phi = -\frac{\rho}{\epsilon_0}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \Leftrightarrow \nabla \times (\nabla \times \vec{A}) = \mu_0 \vec{J} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} - \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\nabla\phi)$$

$$\text{or 利用 } \nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \Rightarrow (\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}) - \nabla(\nabla \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial \phi}{\partial t}) = -\mu_0 \vec{J}$$

Tip. 定义 $\square^2 \equiv \nabla^2 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2}$, $\mathcal{L} \equiv \nabla \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial \phi}{\partial t}$

$$\square^2 \phi + \frac{\partial \mathcal{L}}{\partial t} = -\frac{\rho}{\epsilon_0} \quad \dots \textcircled{3}$$

$$\square^2 \vec{A} - \nabla \mathcal{L} = -\mu_0 \vec{J} \quad \dots \textcircled{4}$$

$$\text{证: } \square^2 \phi + \frac{\partial \mathcal{L}}{\partial t} = \nabla^2 \phi - \mu_0 \epsilon_0 \frac{\partial^2 \phi}{\partial t^2} + \nabla \cdot (\frac{\partial \vec{A}}{\partial t}) + \mu_0 \epsilon_0 \frac{\partial^2 \phi}{\partial t^2} = \nabla^2 \phi + \frac{\partial}{\partial t}(\nabla \cdot \vec{A}) = -\frac{\rho}{\epsilon_0}$$

第二式也证.

规范变换

Intro: 规范变换后的方程③④可能不美观, 但若将本解 \vec{B}, \vec{E} 的大含量至少 6 方程 (共 8 方程) 转化为求 ϕ 与 A_x, A_y, A_z 4 个方程, 此时, \vec{A} 与 ϕ 有一次自由度, 在不影响它们决定的 \vec{E} 与 \vec{B} 时的自由度称为 **规范自由度**, 相应变换称为 **规范变换**

Thm. \vec{A} 与 ϕ 的规范变换:
$$\begin{cases} \vec{A}' = \vec{A} + \nabla\lambda \\ \phi' = \phi - \frac{\partial\lambda}{\partial t} \end{cases} \quad \lambda = \lambda(\vec{r}, t) \quad \text{称为规范变换.}$$

证: 若需 $\nabla \times \vec{A}' = \nabla \times \vec{A}$, 则 $\vec{A}' = \vec{A} + \nabla\lambda$ ($\lambda = \lambda(\vec{r}, t)$). 若有 $\phi = \phi + \beta$

$$\text{则需有 } \vec{E} = -\frac{\partial\vec{A}}{\partial t} - \nabla\phi = -\frac{\partial\vec{A}'}{\partial t} - \nabla\phi' = -\frac{\partial\vec{A}'}{\partial t} - \nabla(\phi + \beta)$$

$$\Rightarrow \nabla(\frac{\partial\lambda}{\partial t} + \beta) = 0 \quad \text{则 } \beta = -\frac{\partial\lambda}{\partial t} + f(t), \text{ 从而定义新 } \lambda' = \lambda - \int f(t) dt, \text{ 这不影响 } \nabla\lambda,$$

$$\text{即 } \nabla\lambda' = \nabla\lambda, \text{ 则有 } \vec{A}' = \vec{A} + \nabla\lambda', \quad \phi' = \phi - \frac{\partial\lambda'}{\partial t}, \text{ 保证了 } \vec{E} \text{ 与 } \vec{B} \text{ 不变.}$$

Tip. 作用: 用于美化③④式, 即在解方程时利用规范变换简化方程

在静电学中, 常使 $\nabla \cdot \vec{A} = 0$, 但在其他条件下, 这一点不用了

Ex. 对 $\phi(\vec{r}, t) = 0$, $\vec{A}(\vec{r}, t) = -\frac{1}{4\pi\epsilon_0} \frac{qt}{r^2} \vec{r}$ 求场, 电荷与电流分布

$$\text{若不进行变换, } \vec{B} = \nabla \times \vec{A} = \vec{0}, \quad \vec{E} = -\frac{\partial\vec{A}}{\partial t} - \nabla\phi = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \vec{r}$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \Rightarrow \nabla \cdot \left(\frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \vec{r} \right) = \frac{q}{4\pi\epsilon_0} \nabla \cdot \left(\frac{\vec{r}}{r^2} \right) = \frac{q}{4\pi\epsilon_0} 4\pi\delta^3(\vec{r}) = \frac{\rho}{\epsilon_0} \Rightarrow \rho = q\delta^3(\vec{r})$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial\vec{E}}{\partial t} \Rightarrow \vec{J} = \vec{0}$$

若用 $\lambda = -\frac{1}{4\pi\epsilon_0} \frac{q}{r}$ 进行规范变换

$$\phi' = \phi - \frac{\partial \lambda}{\partial t} = \frac{1}{4\pi\epsilon_0} \frac{q}{r} \quad \vec{A}' = \vec{A} + \nabla \lambda = \vec{0} \quad \vec{E}, \vec{B} \text{ 不变}$$

这相当于 q 的正电荷在 r 上产生的 \vec{E}

库仑规范与洛伦兹规范

1. 库仑规范: 定义 $\nabla \cdot \vec{A} = 0$, 则有 $\nabla^2 \phi = -\frac{\rho}{\epsilon_0}$

Tip. 由泊松方程可知: $\phi = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{r}', t)}{r} d\tau'$

仅由该式看, ϕ 的变化在瞬间反应到 ϕ 上, 但 ϕ 非守恒则

$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla \phi$ 为了守恒, 且 \vec{E} 有时间延迟 (亦即在瞬间反应 ϕ 后 \vec{A} 才会随时间变化)

Tip. 库仑规范优点: ϕ 守恒方便

缺点: $(\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}) - \nabla (\nabla \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial \phi}{\partial t}) = -\mu_0 \vec{J}$ 计算 \vec{A} 困难

2. 洛伦兹规范: 定义 $\nabla \cdot \vec{A} = -\mu_0 \epsilon_0 \frac{\partial \phi}{\partial t}$ (设 $c=1$)

则有 $\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$ (消除 ∇ 的麻烦)

同时 $\nabla^2 \phi - \mu_0 \epsilon_0 \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}$

Tip. 利用达朗贝尔方程 $\square^2 \equiv \nabla^2 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \Rightarrow \begin{cases} \square^2 \vec{A} = -\mu_0 \vec{J} \\ \square^2 \phi = -\frac{\rho}{\epsilon_0} \end{cases}$

定义四矢量 $A^\mu = (\vec{A}, \phi)$, 与 $J^\mu = (\vec{J}, \rho)$ 有 $\square^2 A^\mu = \mu_0 J^\mu$

达朗贝尔方程 (\square^2) 可看作四维中的波动方程 (∇^2)

$\square^2 A^\mu = \mu_0 J^\mu$ 可看作四维泊松方程

同理 以 C 连接的回路 波方程 $\square^2 f = 0$ 可称为回路中的拉普拉斯方程

所以. 在洛伦兹规范下, (\vec{A}, ϕ) 满足非齐次的波方程

Tip. 下全使用洛伦兹规范, 问题变为求解给定源下 (\vec{J}^μ) 的波方程的解 ($\square^2 A^\mu = \mu_0 J^\mu$)

Tip. 一定存在 \vec{A} , s.t. $\nabla \cdot \vec{A} = \mu_0 \epsilon \frac{\partial \phi}{\partial t}$.

证: 若 $\nabla \cdot \vec{A} + \mu_0 \epsilon \frac{\partial \phi}{\partial t} = 0 \neq 0$.

$$\text{则设 } \begin{cases} \vec{A}' = \vec{A} + \nabla \lambda \\ \phi' = \phi - \frac{\partial \lambda}{\partial t} \end{cases}, \quad \nabla \cdot \vec{A}' + \mu_0 \epsilon \frac{\partial \phi'}{\partial t} = 0 \Rightarrow \nabla \cdot \vec{A} + \mu_0 \epsilon \frac{\partial \phi}{\partial t} + \nabla^2 \lambda - \mu_0 \epsilon \frac{\partial^2 \lambda}{\partial t^2} = 0$$

$\Rightarrow \square^2 \lambda = -\square \cdot \vec{A}$, 该方程必定有解 (V 电荷分布均产生一个电场与磁场, 因而有一个势)

Tip. 虽以经过规范变换, s.t. $\phi = 0$, 但不一定使 $\vec{A} = \vec{0}$

证. 只要取 $\lambda = \int_0^t \phi(t') dt'$ 即可使 $\phi = 0$.

但非所有 \vec{A} 均可表示为 $-\nabla \lambda$ (非所有向量场为梯度场)

且若 $\vec{A} = \vec{0}$, $\vec{B} = \nabla \times \vec{A} = \vec{0}$ 也并非 \vec{B} 恒为零

§2. 连续分布

推迟势

Intro: 对于静态情形, 即 $\vec{E} = \vec{E}(\vec{x})$, $\vec{B} = \vec{B}(\vec{x})$, 方程组变为

$$\begin{cases} \nabla^2 \phi = -\frac{\rho}{\epsilon_0} \\ \nabla \cdot \vec{A} = -\mu_0 \vec{J} \end{cases}$$

$$\vec{r} \text{ 的逆解为 } \phi(\vec{r}) = \frac{1}{4\pi\epsilon} \int_V \frac{\rho(\vec{r}')}{r} d\tau'$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}(\vec{r}')}{r} d\tau'$$

$\vec{r} = \vec{r} - \vec{r}'$, 若使 $t_r = t - \frac{r}{c}$ 称为推迟时间,

可能有 $\phi(\vec{r}, t) = \frac{1}{4\pi\epsilon} \int_V \frac{\rho(\vec{r}', t - \frac{r}{c})}{r} d\tau'$

同理 $\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}(\vec{r}', t - \frac{r}{c})}{r} d\tau'$

Def. 推迟势: $\phi(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{r}', t - \frac{r}{c})}{r} d\tau'$ 称为推迟势

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}(\vec{r}', t - \frac{r}{c})}{r} d\tau'$$

★★ Thm. 推迟势满足 $\square^2 \phi = -\frac{\rho}{\epsilon_0}$ 与 $\square^2 \vec{A} = -\mu_0 \vec{J}$ 与 $\nabla \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial \phi}{\partial t} = 0$

证: ① 由于 \vec{A} 的分量满足的方程与 ϕ 并无差异, 所以只需证 $\square^2 \phi = -\frac{\rho}{\epsilon_0}$.

$$\square^2 \phi = \nabla^2 \phi - \mu_0 \epsilon_0 \frac{\partial^2 \phi}{\partial t^2}$$

$$\nabla \phi = \frac{1}{4\pi\epsilon_0} \int_V (\nabla \ell) \frac{1}{r} + \ell (\nabla \frac{1}{r}) d\tau'$$

$$\begin{cases} \nabla \ell = \frac{\partial \ell}{\partial (t - \frac{r}{c})} \nabla (t - \frac{r}{c}) = \frac{\partial \ell}{\partial t} \left(\frac{\partial (t - \frac{r}{c})}{\partial t} \right) \left[-\frac{1}{c} \nabla (|\vec{r} - \vec{r}'|) \right] = \frac{\partial \ell}{\partial t} \left(-\frac{1}{c} \right) \hat{r} \\ \nabla \left(\frac{1}{r} \right) = -\frac{\hat{r}}{r^2} \end{cases}$$

$$\text{则 } \nabla \phi = \frac{1}{4\pi\epsilon_0} \int_V \left(\frac{\ell}{r^2} - \frac{1}{c} \frac{\partial \ell}{\partial t} \hat{r} \right) d\tau'$$

$$\nabla^2 \phi = \nabla \cdot (\nabla \phi) = \frac{1}{4\pi\epsilon_0} \int_V -\nabla \cdot \left(\frac{\ell}{r^2} \hat{r} \right) - \nabla \cdot \left(\frac{1}{c} \frac{\partial \ell}{\partial t} \hat{r} \right) d\tau'$$

$$\nabla \cdot \left(\frac{\ell}{r^2} \hat{r} \right) = \nabla \ell \cdot \frac{\hat{r}}{r^2} + 4\pi \ell \delta^3(\vec{r} - \vec{r}') = -\frac{1}{c} \frac{\partial \ell}{\partial t} \frac{1}{r^2} + 4\pi \ell \delta^3(\vec{r} - \vec{r}')$$

$$\nabla \cdot \left(\frac{1}{c} \frac{\partial \ell}{\partial t} \hat{r} \right) = \frac{1}{c} \frac{\partial \ell}{\partial t} (\nabla \ell) \cdot \hat{r} + \frac{1}{c} \frac{\partial \ell}{\partial t} \nabla \cdot \left(\frac{\hat{r}}{r^2} \right) = \frac{1}{c^2} \frac{\partial^2 \ell}{\partial t^2} - \frac{1}{c^2 r} \frac{\partial^2 \ell}{\partial t^2}$$

$$\Rightarrow \nabla^2 \phi = \frac{1}{4\pi\epsilon_0} \int_V \left(\frac{1}{c^2} \frac{\partial^2 \rho}{\partial t^2} - 4\pi \rho \delta^3(\vec{r}-\vec{r}') \right) d\vec{r}' = \mu_0 \epsilon_0 \frac{\partial^2 \phi}{\partial t^2} - \frac{1}{\epsilon_0} \rho(\vec{r}, t - \frac{r}{c})$$

$$\text{即 } \nabla^2 \phi - \mu_0 \epsilon_0 \frac{\partial^2 \phi}{\partial t^2} = \square^2 \phi = -\frac{\rho}{\epsilon_0}$$

$$\textcircled{2} \quad \nabla \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial \phi}{\partial t} = 0$$

$$\vec{A} = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}(\vec{r}', t - \frac{r}{c})}{r} d\vec{r}' \quad \phi = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{r}', t - \frac{r}{c})}{r} d\vec{r}'$$

$$\nabla \cdot \left(\frac{\vec{J}}{r} \right) = (\nabla \cdot \vec{J}) \frac{1}{r} + \vec{J} \cdot \nabla \left(\frac{1}{r} \right) \quad \nabla' \cdot \left(\frac{\vec{J}}{r} \right) = (\nabla' \cdot \vec{J}) \frac{1}{r} + \vec{J} \cdot \nabla' \left(\frac{1}{r} \right)$$

$$\text{而 } \nabla \left(\frac{1}{r} \right) = -\nabla' \left(\frac{1}{r} \right) \quad (r = |\vec{r} - \vec{r}'|) \quad \text{则有 } \nabla \left(\frac{\vec{J}}{r} \right) = (\nabla \cdot \vec{J}) \frac{1}{r} + (\nabla' \cdot \vec{J}) \frac{1}{r} - \nabla' \cdot \left(\frac{\vec{J}}{r} \right)$$

$$\nabla \cdot \vec{J} = \frac{\partial \vec{J}}{\partial t} \cdot \nabla \left(-\frac{r}{c} \right) \quad \nabla' \cdot \vec{J} = -\frac{\partial \rho}{\partial t} + \frac{\partial \vec{J}}{\partial t} \cdot \nabla' \left(-\frac{r}{c} \right)$$

$$\downarrow \text{ 将 } r \text{ 看成与 } \vec{r}' \text{ 无关, } \nabla' \cdot \vec{J} = -\frac{\partial \rho}{\partial t - \frac{r}{c}} = -\frac{\partial \rho}{\partial t}$$

$$\text{则 } \nabla \cdot \left(\frac{\vec{J}}{r} \right) = -\frac{\partial \vec{J}}{\partial t} \cdot \frac{\vec{r}}{cr} - \frac{\partial \rho}{\partial t} \frac{1}{r} + \frac{\partial \vec{J}}{\partial t} \cdot \frac{1}{r} \frac{\vec{r}}{c} - \nabla' \cdot \left(\frac{\vec{J}}{r} \right)$$

$$= -\frac{\partial \rho}{\partial t} \frac{1}{r} - \nabla' \cdot \left(\frac{\vec{J}}{r} \right) \quad (\nabla' \cdot d\vec{r}' \text{ 为封闭曲面})$$

$$\text{则 } \nabla \cdot \vec{A} = \frac{\mu_0}{4\pi} \int_V -\frac{\partial \rho}{\partial t} \frac{1}{r} - \nabla' \cdot \left(\frac{\vec{J}}{r} \right) d\vec{r}'$$

$$= -\mu_0 \epsilon_0 \frac{\partial \phi}{\partial t} - \frac{\mu_0}{4\pi} \oint_S \frac{\vec{J}}{r} \cdot d\vec{S}' \quad (\text{由于 } S' \text{ 包了所有电荷, 或 } S' \rightarrow \infty, \text{ 则 } S' \perp \vec{J} = \vec{0})$$

$$= -\mu_0 \epsilon_0 \frac{\partial \phi}{\partial t}$$

Tips. 该结论也适用于超前势, 即 $\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}(\vec{r}', t + \frac{r}{c})}{r} d\vec{r}'$ 与 $\phi(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{r}', t + \frac{r}{c})}{r} d\vec{r}'$

$t_a := t + \frac{r}{c}$ 为超前时间,

虽然满足波动方程 (Maxwell 方程) 但它违反了因果律。

杰斐逊方程.

Intro: 由已知 $\rho(\vec{r}, t)$ 与 $\vec{J}(\vec{r}, t)$, 利用推迟势理论上可得出 \vec{A} 与 ϕ , 从而 \vec{A} 与 ϕ

理论上可以计算 \vec{E} 与 \vec{B} . 如此计算的两个方程称为杰斐逊方程

Thm. Jefimenko Equation: $t_r := t - \frac{r}{c}$

$$\vec{E}(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int_V \left[\frac{\rho(\vec{r}', t_r)}{r^2} \hat{r} + \frac{\partial \rho(\vec{r}', t_r)}{\partial t} \frac{\hat{r}}{c^2} - \frac{\partial \vec{J}(\vec{r}', t_r)}{\partial t} \frac{1}{c^2 r} \right] d\tau' \quad (\text{普遍库仑定律})$$

$$\vec{B}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int_V \left[\frac{\vec{J}(\vec{r}', t_r)}{r^2} + \frac{\partial \vec{J}(\vec{r}', t_r)}{\partial t} \frac{1}{c^2 r} \right] \times \hat{r} d\tau' \quad (\text{普遍毕奥-萨伐尔定律})$$

$$\text{证: } \vec{A} = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}(\vec{r}', t_r)}{r} d\tau', \quad \phi = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{r}', t_r)}{r} d\tau'$$

$$\begin{aligned} \vec{E} &= -\frac{\partial \vec{A}}{\partial t} - \nabla \phi = -\frac{\mu_0}{4\pi} \int_V \frac{\partial \vec{J}(\vec{r}', t_r)}{\partial t} \frac{1}{r} d\tau' + \frac{1}{4\pi\epsilon_0} \int_V \frac{1}{r} \frac{\partial \rho}{\partial t} \frac{\hat{r}}{r} + \frac{\rho}{r^2} \hat{r} d\tau' \\ &= \frac{1}{4\pi\epsilon_0} \int_V \left[\frac{\rho}{r^2} \hat{r} + \frac{1}{cr} \frac{\partial \rho}{\partial t} \hat{r} - \frac{1}{c^2 r} \frac{\partial \vec{J}}{\partial t} \right] d\tau' \end{aligned}$$

$$\begin{aligned} \vec{B} &= \nabla \times \vec{A} = \frac{\mu_0}{4\pi} \int_V \left[(\nabla \times \vec{J}) \frac{1}{r} - \vec{J} \times \nabla \left(\frac{1}{r} \right) \right] d\tau' \\ &= \frac{\mu_0}{4\pi} \int_V \left[-\frac{1}{r} (\nabla r \times \frac{\partial \vec{J}}{\partial t}) \frac{1}{r} + \frac{\vec{J}}{r^2} \times \hat{r} \right] d\tau' \\ &= \frac{\mu_0}{4\pi} \int_V \left[\frac{1}{cr} \frac{\partial \vec{J}}{\partial t} + \frac{\vec{J}}{r^2} \right] \times \hat{r} d\tau' \end{aligned}$$

Tip: 若为静电场, 也即 $\rho = \rho(\vec{r})$, $\vec{J} = \vec{J}(\vec{r})$ 时, Jefimenko Equation 退化为普遍方程

且有, 势可以透过推迟因子得到, 但场不行. 其中又给出 \vec{J} 与 ρ 对时间的偏微分

Tip: Jefimenko 方程对静电场近似有用.

Thm. 若 $\vec{J} = \vec{J}(\vec{r})$ (即 \vec{J} 与时间无关) 但 ρ 不恒定, 即 $\rho = \rho(\vec{r}, t)$, 则有

1° $\rho(\vec{r}, t)$ 是时间的线性函数

$$\phi(\vec{r}, t) = \phi(\vec{r}, 0) + \frac{\partial \phi}{\partial t}(\vec{r}, 0) t$$

Tip: 做问答题中, \vec{B} 与 t 关, 但不宜看成静磁学问题

因为 $\nabla \cdot \vec{J} = 0$ 与 $\nabla \times \vec{B} = \mu_0 \vec{J}$ 不再适用, 又有 $\phi = \phi(\vec{r})$ 时成立

$$2^\circ \text{ 在此情况下 } \vec{E}(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\phi(\vec{r}', t) \hat{r}}{r^2} d\tau'$$

$$\vec{B}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}(\vec{r}') \times \hat{r}}{r^2} d\tau'$$

$$\text{证: } \vec{J} \text{ 相称 } \nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \Rightarrow \rho(\vec{r}, t) = \rho(\vec{r}) - (\nabla \cdot \vec{J})_{\vec{r}} t \quad \text{取 } t=0 \Rightarrow \rho(\vec{r}) = \rho(\vec{r}, 0)$$

$$\therefore \phi(\vec{r}, t) = \phi(\vec{r}, 0) - \nabla \cdot \vec{J}(\vec{r}) t = \phi(\vec{r}, 0) + \frac{\partial \phi(\vec{r}, t)}{\partial t} t$$

$$\begin{aligned} 2^\circ. \vec{E}(\vec{r}, t) &= \frac{1}{4\pi\epsilon_0} \int_V \left[\frac{\rho(\vec{r}', t_r)}{r^2} \hat{r} + \frac{\partial \rho(\vec{r}', t_r)}{\partial t} \frac{\hat{r}}{c^2} - \frac{\partial \vec{J}(\vec{r}', t_r)}{\partial t} \frac{1}{c^2} \right] d\tau' \\ &= \frac{1}{4\pi\epsilon_0} \int_V \left[\frac{\rho(\vec{r}', 0) + \frac{\partial \rho(\vec{r}', t_r)}{\partial t} (t - \frac{r}{c}) + \frac{\partial \rho(\vec{r}', t_r)}{\partial t} \frac{r}{c}}{r^2} \hat{r} \right] d\tau' \end{aligned}$$

$$= \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{r}', t)}{r^2} \hat{r} d\tau'$$

$$\vec{B}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int_V \left[\frac{\vec{J}(\vec{r}')}{r^2} + \frac{\partial \vec{J}(\vec{r}')}{\partial t} \frac{1}{c^2} \right] \times \hat{r} d\tau' = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}(\vec{r}')}{r^2} \times \hat{r} d\tau'$$

接上, 假设电流变化十分缓慢, 使 \vec{J} 的关于时间在 t 处的泰勒展开可以近似到一阶

$$\vec{J}(\vec{r}, t_r) = \vec{J}(\vec{r}, t) + \frac{\partial \vec{J}(\vec{r}, t)}{\partial t} (t_r - t) + \dots \quad \frac{\partial^2 \vec{J}}{\partial t^2} \times (\frac{r}{c})^2 \Rightarrow \text{二}$$

$$\text{证明在一阶近似下, } \vec{B}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}(\vec{r}', t)}{r^2} \times \hat{r} d\tau' \quad (\text{即不用非推迟时间代替})$$

$$\text{证: 代入 } \vec{J}(\vec{r}, t_r) = \vec{J}(\vec{r}, t) - \frac{\partial \vec{J}(\vec{r}, t)}{\partial t} \frac{r}{c}, \text{ 有.}$$

$$\begin{aligned}\vec{B}(\vec{r}, t) &= \frac{\mu_0}{4\pi} \int \left[\frac{\vec{j}(\vec{r}', t) - \frac{\partial \vec{j}(\vec{r}', t)}{\partial t} \frac{r}{c}}{r^2} + \left(\frac{\partial \vec{j}(\vec{r}', t)}{\partial t} - \frac{\partial^2 \vec{j}(\vec{r}', t)}{\partial t^2} \frac{r}{c} \right) \frac{1}{c} \right] \times \vec{r}' d\tau' \\ &= \frac{\mu_0}{4\pi} \int \left[\frac{\vec{j}(\vec{r}', t) - \frac{\partial \vec{j}(\vec{r}', t)}{\partial t} \frac{r}{c} + \frac{r}{c} \frac{\partial \vec{j}(\vec{r}', t)}{\partial t} - \frac{\partial^2 \vec{j}(\vec{r}', t)}{\partial t^2} \frac{r^2}{c^2}}{r^2} \right] \times \vec{r}' d\tau' \\ &\stackrel{\text{一阶近似}}{\approx} \frac{\mu_0}{4\pi} \int \frac{\vec{j}(\vec{r}', t) \times \vec{r}'}{r^2} d\tau'\end{aligned}$$

Conclusion: \vec{j} 不含 t 时, \vec{E}, \vec{B} 均用非推迟时间计算 (库仑 + 毕安-萨伐尔)

\vec{j} 变化较慢的一阶近似下 (例如 $\frac{v}{c} \ll 1$), \vec{B} 用非推迟时间计算 (毕安-萨伐尔)


§ 3. 点电荷

李纳-维谢尔势 (Liénard-Wiechert Potential)

Intro. 需计算特定轨迹 $\vec{r} = \vec{w}(t)$ 运动的带电粒子产生的推迟势 \vec{A} 与 ϕ

Att. $\vec{r} = \vec{r} - \vec{w}(t)$, 则 $t_r = t - \frac{|\vec{r} - \vec{w}(t_r)|}{c}$, 给定 t_r , 可解出唯一 t_r (带电粒子 $v < c$)

若可解出 t_r , 不妨设为 t_1, t_2 , 则 $(t_1 - t)c = r_1, (t_2 - t)c = r_2$, 相减

$(t_1 - t_2)c = r_1 - r_2$  $\alpha > \frac{\pi}{2}$, 则 $|\vec{r}_1 - \vec{r}_2| \geq r_1 - r_2$, 则 $\frac{|\vec{r}_1 - \vec{r}_2|}{t_1 - t_2} \geq c$, 这说明间距 $\geq c$, 矛盾

推迟势公式 $\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}', t_r)}{r} d\tau'$

似乎有 $\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$

但当心! 若 q 非点电荷, 而是连续电荷分布, 则在计算时 $\int \rho(\vec{r}', t_r)$ 非所有电荷, 因为 t_r 与 \vec{r}' 有关

这让我们计算了不同时间的电荷在空间中的分布

而对于点电荷, 会有类似问题, 可以近似

1°. 物理等效理解

2°. 将 $\rho(\vec{r}', t_r)$ 用 $q\delta^3(\vec{w}(t_r) - \vec{r}')$ 表示, 也即

Thm. 李纳-维谢尔 (Liénard-Wiechert) 势: 即运动的带电粒子的势

带电粒子的位置矢量为 $\vec{w}(t)$. \vec{r} 满足 $r = |\vec{r} - \vec{w}(t_r)|$

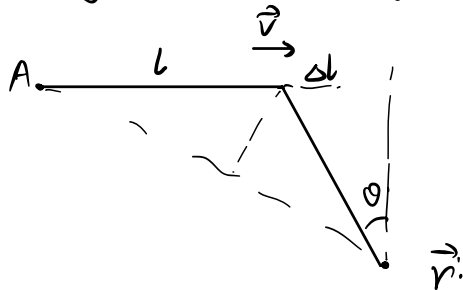
且 $|\vec{r} - \vec{w}(t_r)| = c(t - t_r)$ 由此给出 t 求得出 t_r , 进而得出 $\vec{w}(t_r)$, 进而得出 $\vec{v}(t_r) = \dot{\vec{w}}(t_r)$

$$\text{则有 } \begin{cases} \phi(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{(rc - \vec{r} \cdot \vec{v})} \\ \vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{qc\vec{v}}{(rc - \vec{r} \cdot \vec{v})} = \frac{\phi}{c^2} \vec{v} \end{cases}$$

证: 1°. 物理理解

点电荷可以看成连续分布电荷极限, 且由于点电荷并非结构, 所以取极限时的电荷分布

可近乎任意取, 例如取为均匀的棒状



假设求 $\phi(\vec{r}, t)$ 对于 l 左端点, 在 $t_{r\text{左}}$ 时刻位于 A 点.

在 \vec{r}' 向右移动时, $t_r = t - \frac{r}{c}$ 在逐渐变大,

因而在 t_r 时刻无电荷的地方, 可能在 \vec{r}' 移动有了电荷, 给出出一段 Δl

$$\Delta z = -(l + \Delta l) \sin \theta. \quad \Delta l = v \Delta t_r = v \times \frac{l + \Delta l}{c} \sin \theta$$

$$\Rightarrow \Delta l = \frac{v \sin \theta}{c - v \sin \theta} l, \text{ 也即 等效的 } q' = \frac{c}{c - v \sin \theta} q$$

$$\hat{n} \cdot \vec{v} = v \cos(\frac{\pi}{2} - \theta) = + \sin \theta \cdot v \quad (2) \quad q' = \frac{1}{1 - \hat{n} \cdot \vec{v}/c} q. \quad (\text{At, 当棒} \rightarrow \text{点时, } t_{r\text{左}} \rightarrow t_{r\text{右}}, \vec{v} \rightarrow \vec{v}(t_r), \theta_{\text{左}} \rightarrow \theta_{\text{右}})$$

$$\text{则 } \phi(\vec{r}, t) = \frac{q'}{4\pi\epsilon_0 r} = \frac{1}{4\pi\epsilon_0} \frac{qc}{rc - \vec{r} \cdot \vec{v}}$$

而 $\vec{j} = \rho \vec{v}$ 同样有 t 的变化, 则 $\vec{j} = \frac{\mu_0}{4\pi} \frac{qc\vec{v}}{rc - \vec{r} \cdot \vec{v}} = \frac{\vec{v}}{c^2} \phi$

2° 以 δ 函数表示,

$$\phi(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{q \delta^3(\vec{r}' - \vec{w}(t_r))}{r} d\tau' \quad , \text{ 这里 } \delta \text{ 对 } \vec{r}' \text{ 的依赖点与变数, 故让变数为 } \delta^3(\vec{r}'')$$

1) 进行变量代换, 令 $\vec{r}'' = \vec{r}' - \vec{w}(t_r) = \vec{r}' - \vec{w}(t - \frac{|\vec{r} - \vec{r}'|}{c})$

2) $d\tau' = \frac{\partial(x', y', z')}{\partial(x'', y'', z'')} dz'' = \left(\frac{\partial(x', y', z')}{\partial(x'', y'', z'')} \right)^{-1} d\tau''$

$$\frac{\partial(x', y', z')}{\partial(x'', y'', z'')} = |\nabla'(\vec{r}'')| = |\nabla(\vec{r}') - \nabla'(\vec{w}(t - \frac{|\vec{r} - \vec{r}'|}{c}))| = |\mathbf{I}_{3 \times 3} + \frac{d\vec{w}}{cdt} \nabla |\vec{r} - \vec{r}'|| \quad (\vec{a} \cdot \vec{b} = a_i b_i)$$

$$= |\mathbf{I}_{3 \times 3} - \frac{1}{c} \vec{v} \hat{r}| = -\frac{1}{c} \vec{v} \cdot \hat{r} + 1$$

$$\text{由此, } \phi(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{q \delta^3(\vec{r}'')}{r} \frac{d\tau''}{1 - \frac{1}{c} \vec{v} \cdot \hat{r}} = \frac{1}{4\pi\epsilon_0} \frac{q}{r} \times \frac{1}{1 - \frac{1}{c} \vec{v} \cdot \hat{r}} = \frac{1}{4\pi\epsilon_0} \frac{qc}{rc - \vec{v} \cdot \vec{r}} \quad (\text{这里 } \vec{r} = \vec{r} - \vec{w}(t_r))$$

同理有 $\vec{A}(\vec{r}, t) = \frac{\vec{v}}{c^2} \phi(\vec{r}, t)$

Ex. 求匀速运动电荷的势.

$$\phi(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{rc - \vec{v} \cdot \vec{r}} \quad , \quad \vec{r} = \vec{r} - \vec{w}(t_r) \quad , \quad |\vec{r} - \vec{w}(t_r)| = c(t - t_r)$$

$$\vec{w}(t) = \vec{v}t \quad \text{则} \quad |\vec{r} - \vec{v}t_r| = c(t - t_r) \Rightarrow \sqrt{(\vec{r} - \vec{v}t_r) \cdot (\vec{r} - \vec{v}t_r)} = c(t - t_r)$$

$$\Leftrightarrow (c^2 - v^2) t_r^2 + 2(\vec{r} \cdot \vec{v} - ct) t_r + c^2 t^2 - r^2 = 0$$

$$\Rightarrow t_r = \frac{(c^2 t - \vec{r} \cdot \vec{v}) \pm c |\vec{r} - \vec{v}t| \sqrt{1 - (v^2/c^2) \sin^2 \theta}}{c^2 - v^2} \quad , \quad t=0 \text{ 时, } t_r < 0, \text{ 因而取 } -, \theta \text{ 为 } \vec{r} \text{ 与 } \vec{v} \text{ 夹角}$$

$$\vec{r} = \vec{r} - \vec{v}t_r \quad , \quad r = c(t - t_r)$$

$$\Rightarrow \phi(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{c^2 t - \vec{v} \cdot \vec{r} - (c^2 - v^2) t_r} = \frac{qc}{4\pi\epsilon_0} \frac{1}{c^2 t - \vec{v} \cdot \vec{r} - (c^2 - v^2) t_r}$$

$$= \frac{q}{4\pi\epsilon_0} \frac{1}{|\vec{r}-\vec{v}t| \sqrt{1-v^2/c^2 \sin^2\theta}}$$

$$\vec{A} = \frac{\vec{v}}{c^2} \phi = \frac{\mu_0}{4\pi} \frac{q \vec{v}}{|\vec{r}-\vec{v}t| \sqrt{1-v^2/c^2 \sin^2\theta}} \quad (\text{Tip: 依然满足 Lorentz 规范, 即 } \nabla \cdot \vec{A} + \mu_0 \epsilon \frac{\partial \phi}{\partial t} = 0)$$

运动点电荷的场

Intro: 利用李纳-维谢尔势计算运动电荷产生的场 (当然, Jefimenko 方程 + δ 函数也行到场的)

Thm. 运动点电荷的场: 点电荷的位置函数为 $\vec{w}(t)$. 式中, $\vec{v} = \dot{\vec{w}}(t_r)$, $\vec{a} = \ddot{\vec{w}}(t_r)$, $\vec{r} = \vec{r} - \vec{w}(t_r)$,

t_r 由 $|\vec{r} - \vec{w}(t_r)| = c(t - t_r)$ 决定, 因而作为 $t_r = t_r(t, \vec{r})$, 且 $\vec{u} \equiv c\hat{r} - \vec{v}$

$$\text{则有 } \vec{E}(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\vec{r}}{(\vec{r} \cdot \vec{u})^3} [(c^2 - v^2)\vec{u} + \vec{r} \times (\vec{u} \times \vec{a})]$$

$$\vec{B}(\vec{r}, t) = \frac{1}{c} \hat{r} \times \vec{E}(\vec{r}, t)$$

Tip: \vec{E} 第一项, 以反比于 r^2 的速度衰减, \vec{E} 在 $\vec{v} = \vec{a} = \vec{0}$ 时恢复静场

由于此, \vec{E} 第一项也称为 **库仑场** (不依赖于 \vec{a} , 也称速度场)

\vec{E} 第二项, 以反比于 r 的速度衰减, 在远距时起主要作用,

并且引起辐射效应, 称为 **辐射场** ($\propto \vec{a}$, 也称加速度场)

以上同样适用于磁场

$$\text{Tip: 根据 } \vec{F} = q(\vec{v} \times \vec{B} + \vec{E}) = \frac{qQ}{4\pi\epsilon_0} \frac{\vec{r}}{(\vec{r} \cdot \vec{u})^3} \left\{ [(c^2 - v^2)\vec{u} + \vec{r} \times (\vec{u} \times \vec{a})] + \frac{\vec{v}}{c} \times [\hat{r} \times [(c^2 - v^2)\vec{u} + \vec{r} \times (\vec{u} \times \vec{a})]] \right\}$$

即 \vec{v} 相量 能加在电荷上的力

$$\text{证: } \phi(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{q}{r - \vec{r} \cdot \vec{v}}, \quad \vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{qc\vec{v}}{r - \vec{r} \cdot \vec{v}}$$

并由 $\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla\phi$ 与 $\vec{B} = \nabla \times \vec{A}$ 得 $\vec{E} \cdot \vec{B} = 0$, $\vec{v}(t_r)$, $\vec{r} = \vec{r} - \vec{w}(t_r)$, $r = c(t - t_r)$ 而 $t_r = t_r(\vec{r}, t)$,

$$\nabla \phi = \frac{qc}{4\pi\epsilon_0} \frac{-1}{(rc - \vec{r} \cdot \vec{v})^2} \nabla (rc - \vec{r} \cdot \vec{v})$$

$$\nabla(rc) = c \nabla r = -c^2 \nabla t_r$$

$$\nabla(\vec{r} \cdot \vec{v}) = (\vec{r} \cdot \nabla) \vec{v} + \vec{v} \cdot \nabla \vec{r} + \vec{r} \times (\nabla \times \vec{v}) + \vec{v} \times (\nabla \times \vec{r})$$

$$\left\{ \begin{array}{l} (\vec{r} \cdot \nabla) \vec{v} = \vec{a} (\vec{r} \cdot \nabla t_r) \\ (\vec{v} \cdot \nabla) \vec{r} = (\vec{v} \cdot \nabla) \vec{r} - (\vec{v} \cdot \nabla) \vec{w} = \vec{v} - \vec{v} (\vec{v} \cdot \nabla t_r) \\ \nabla \times \vec{v} = (\nabla t_r) \times \vec{a} \\ \nabla \times \vec{r} = -\nabla t_r \times \vec{v} \end{array} \right\} \rightarrow \text{Hilf } \nabla(\vec{r} \cdot \vec{v})$$

$$\begin{aligned} \text{d) } \nabla(\vec{r} \cdot \vec{v}) &= \vec{a} (\vec{r} \cdot \nabla t_r) + \vec{v} - \vec{v} (\vec{v} \cdot \nabla t_r) + \vec{r} \times [(\nabla t_r) \times \vec{a}] + \vec{v} \times (\vec{v} \times \nabla t_r) \\ &= \vec{v} + (\vec{r} \cdot \vec{a} - v^2) \nabla t_r \end{aligned}$$

$$-c \nabla t_r = \nabla r = \nabla(\sqrt{\vec{r} \cdot \vec{r}}) = \frac{1}{2r} \nabla(\vec{r} \cdot \vec{r}) = \frac{1}{2} [(\vec{r} \cdot \nabla) \vec{r} + \vec{r} \times (\nabla \times \vec{r})]$$

$$(\vec{r} \cdot \nabla) \vec{r} = \vec{r} - \vec{v} (\vec{r} \cdot \nabla t_r), \quad \nabla \times \vec{r} = \vec{v} \times \nabla t_r \quad \text{b.)}$$

$$\nabla r = \hat{r} - \frac{1}{r} (\vec{r} \cdot \vec{v}) \nabla t_r = -c \nabla t_r \Rightarrow \nabla t_r = \frac{\vec{r}}{\vec{r} \cdot \vec{v} - cr}$$

$$\begin{aligned} \text{Hilf } \nabla \phi &= \frac{qc}{4\pi\epsilon_0} \frac{-1}{(rc - \vec{r} \cdot \vec{v})^2} (-c^2 \nabla t_r - \vec{v} + v^2 \nabla t_r - (\vec{r} \cdot \vec{a}) \nabla t_r) \\ &= \frac{qc}{4\pi\epsilon_0} \frac{1}{(rc - \vec{r} \cdot \vec{v})^3} [(rc - \vec{r} \cdot \vec{v}) \vec{v} - (c^2 - v^2 + \vec{r} \cdot \vec{a}) \vec{r}] \end{aligned}$$

$$\frac{\partial \vec{A}}{\partial t} = \frac{\partial}{\partial t} \left(\frac{\mu_0}{4\pi} \frac{qc\vec{v}}{rc - \vec{r} \cdot \vec{v}} \right) = \frac{\mu_0 qc}{4\pi} \left(\frac{\frac{\partial \vec{v}}{\partial t} (rc - \vec{r} \cdot \vec{v}) - \vec{v} \frac{\partial}{\partial t} (rc - \vec{r} \cdot \vec{v})}{(rc - \vec{r} \cdot \vec{v})^2} \right)$$

$$\frac{\partial \vec{v}}{\partial t} = \frac{\partial \vec{v}}{\partial t_r} \frac{\partial t_r}{\partial t} = \vec{a}(t_r) \frac{\partial t_r}{\partial t}, \quad \text{又 } r = c(t - t_r), \quad \vec{v} = \vec{r} - \vec{w}(t_r)$$

$$\frac{\partial r}{\partial t} = c - c \frac{\partial t_r}{\partial t} = \frac{1}{2} \frac{\partial(\vec{r} \cdot \vec{v})}{\partial t} = \frac{\partial \vec{r}}{\partial t} \cdot \vec{v} \quad \text{又} \quad \frac{\partial \vec{r}}{\partial t} = -\frac{\partial \vec{w}}{\partial t_r} \frac{\partial t_r}{\partial t} = -\vec{v} \frac{\partial t_r}{\partial t}$$

$$\vec{R}^* = \vec{r} - \vec{r}^* \quad \text{fix fix 20} \quad \vec{r}_e(t^*) = \vec{r}_e(t^*, \vec{r}^*, t_r)$$

fix fix t.

$$\times] \quad c(1 - \frac{\partial t_r}{\partial t}) = - \vec{v} \cdot \vec{r} \frac{\partial t_r}{\partial t} \Rightarrow \frac{\partial t_r}{\partial t} = \frac{c}{c - \vec{v} \cdot \vec{r}} = \frac{c\vec{r}}{\vec{r} \cdot \vec{u}}$$

$$\text{则有 } \frac{\partial \vec{v}}{\partial t} = \vec{a} \frac{\partial t_r}{\partial t}, \quad \frac{\partial}{\partial t}(rc) = c^2(1 - \frac{\partial t_r}{\partial t}), \quad \frac{\partial}{\partial t}(\vec{r} \cdot \vec{v}) = (\frac{\partial \vec{r}}{\partial t} \cdot \vec{v}) + \vec{r} \cdot (\frac{\partial \vec{v}}{\partial t}) = -v^2 \frac{\partial t_r}{\partial t} + \vec{r} \cdot \vec{a} \frac{\partial t_r}{\partial t}$$

$$\Rightarrow \frac{\partial \vec{v}}{\partial t} (\vec{r} \cdot \vec{v}) - \vec{v} \frac{\partial(\vec{r} \cdot \vec{v})}{\partial t} = \vec{a} (\vec{r} \cdot \vec{v}) \frac{\partial t_r}{\partial t} - \vec{v} (c^2(1 - \frac{\partial t_r}{\partial t}) + v^2 \frac{\partial t_r}{\partial t} - \vec{r} \cdot \vec{a} \frac{\partial t_r}{\partial t}) =$$

$$\Rightarrow \frac{\partial \vec{A}}{\partial t} = \frac{1}{4\pi\epsilon_0} \frac{qc}{(rc - \vec{r} \cdot \vec{v})^3} \left[\underbrace{\frac{(rc - \vec{r} \cdot \vec{v})}{c^2} \vec{a} (\vec{r} \cdot \vec{v}) \frac{c\vec{r}}{\vec{r} \cdot \vec{u}}}_{\text{purple}} - \underbrace{c^2 \vec{v} \frac{rc - \vec{r} \cdot \vec{v}}{c^2}}_{\text{purple}} + \underbrace{\frac{rc - \vec{r} \cdot \vec{v}}{c^2} \frac{c\vec{r}}{\vec{r} \cdot \vec{u}} \vec{v} (c^2 - v^2 + \vec{r} \cdot \vec{a})}_{\text{purple}} \right]$$

$$\frac{\partial \vec{A}}{\partial t} = \frac{1}{4\pi\epsilon_0} \frac{qc}{(rc - \vec{r} \cdot \vec{v})^3} \left[(rc - \vec{r} \cdot \vec{v}) (\vec{c} - \vec{v} + \frac{\vec{r}\vec{a}}{c}) + \vec{r} (c^2 - v^2 + \vec{r} \cdot \vec{a}) \vec{v} \right]$$

由上二式得 $\vec{E}(\vec{r}, t)$

$$\nabla \times \vec{A} = \frac{1}{c^2} \nabla \times (\phi \vec{v}) = \frac{1}{c^2} [(\nabla \phi) \times \vec{v} + \phi \nabla \times \vec{v}]$$

$$\nabla \times \vec{v} = (\nabla t_r) \times \vec{a} = \frac{\vec{a} \times \vec{r}}{rc - \vec{r} \cdot \vec{v}}, \quad \nabla \phi \text{ 由上.}$$

$$\text{得 } \nabla \times \vec{A} = -\frac{1}{c} \frac{q}{4\pi\epsilon_0} \frac{1}{(\vec{r} \cdot \vec{u})^3} \vec{r} \times [(c^2 - v^2)\vec{v} + (\vec{v} \cdot \vec{a})\vec{v} + (\vec{r} \cdot \vec{a})\vec{a}]$$

$$\vec{u} = c\vec{r} - \vec{v}$$

\vec{E} 中 $(c^2 - v^2)\vec{u} + \vec{r} \times (\vec{u} \times \vec{a}) = (c^2 - v^2)\vec{u} + (\vec{r} \cdot \vec{a})\vec{u} - (\vec{r} \cdot \vec{u})\vec{a}$ 若将 $\nabla \times \vec{A}$ 中 \vec{v} 都改为 \vec{u} , 则仅改一个 \vec{r} 后即为 \vec{E} 中的 \vec{u}

$$\text{因此 } \vec{B} = \nabla \times \vec{A} = \frac{1}{c} \vec{r} \times \vec{E}(\vec{r}, t)$$

Ex. 计算匀加速运动带电粒子的电磁场

Thm. 匀加速运动带电粒子的电磁场, $\vec{R} := \vec{r} - \vec{v}t$. \vec{v} 为 q 电荷运动速度, $\theta = \langle \vec{R}, \vec{v} \rangle$

$$\text{则有 } \vec{E}(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{[1 - (v^2/c^2)\sin^2\theta]^{\frac{3}{2}}} \frac{\vec{R}}{R^3}$$

$$\vec{B}(\vec{r}, t) = \frac{1}{c} (\vec{r} \times \vec{E}) = \frac{1}{c} (\vec{v} \times \vec{E})$$

$$\text{证: 此时 } \vec{a} = \vec{0}, \text{ 则 } \vec{E}(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{v}{(\vec{r} \cdot \vec{u})^3} (c^2 - v^2) \vec{u}, \quad \vec{u} = c\vec{r} - \vec{v}$$

$$\text{由匀加速运动电荷的场中可知 } c\vec{u} - \vec{r} \cdot \vec{v} = c|\vec{r} - \vec{v}t| \sqrt{1 - (v/c)^2 \sin^2\theta}, \quad \theta \text{ 为 } \vec{r} \text{ 与 } \vec{v} \text{ 夹角}$$

$$(7) \quad c\vec{r} - r\vec{v} = c(\vec{r} - \vec{v}(t - \frac{r}{c})) - r\vec{v} = c\vec{r} - c\vec{v}t = c(\vec{r} - \vec{v}t), \text{ 令 } \vec{R} = \vec{r} - \vec{v}t$$

$$(8) \quad \vec{E}(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{(c^2 - v^2)}{c^3 R^3 (1 - (v/c)^2 \sin^2 \theta)^{\frac{3}{2}}} = \frac{q}{4\pi\epsilon_0} \frac{1 - (v/c)^2}{(1 - (v/c)^2 \sin^2 \theta)^{\frac{3}{2}}} \frac{\vec{R}}{R^3}$$

$$\text{对于 } \vec{B}, \quad \vec{B} = \frac{1}{c} (\hat{r} \times \vec{E})$$

$$\hat{r} = \frac{\vec{r} - \vec{v}(t - \frac{r}{c})}{r} = \frac{\vec{r} - \vec{v}t}{r} + \frac{\vec{v}}{c} = \frac{\vec{R}}{R} + \frac{\vec{v}}{c} \quad \text{Att. } \vec{R} \times \vec{E} = 0$$

$$\therefore \vec{B} = \frac{1}{c^2} \vec{v} \times \vec{E}$$

Conclusion. 电动力学中, 电, 磁, 场的关系. [] 表示在 (\vec{r}, t) 处取值

$$\begin{array}{l} \vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{[\vec{J}]}{r} d\tau' \\ \phi(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{[\rho]}{r} d\tau' \\ \square\phi = -\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}, \quad \square\vec{A} = -\mu_0 \vec{J} \end{array} \quad \begin{array}{l} \vec{E}(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \left(\frac{[\rho]}{r^2} + \frac{[\partial^2 \phi]}{c^2 r} \right) + \vec{v} d\tau' \\ \vec{B}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \left(\frac{[\vec{J}]}{r^2} + \frac{[\partial^2 \vec{A}]}{c^2 r} \right) d\tau' \end{array}$$

上两式是由于场元是推迟势

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla\phi, \quad \vec{B} = \nabla \times \vec{A}$$

$$\nabla \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial \phi}{\partial t} = 0 \quad (\text{Lorenz})$$

单频电磁波的表示: $\vec{J} = \vec{J}_0(\vec{r}) e^{-i\omega t}, \quad \rho = \rho_0(\vec{r}) e^{-i\omega t}$ 并有 $\nabla \times \vec{B} = d \ll \nabla \times \vec{E}$

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{J_0(\vec{r}')}{R} e^{i\omega \frac{R}{c}} d\tau' e^{-i\omega t}, \quad R = |\vec{r} - \vec{r}'|$$

近似下有 $\frac{1}{R} = \frac{1}{r} + \frac{\vec{r} \cdot \vec{r}'}{r^3} + \dots$, $\frac{\omega R}{c} = k|\vec{r} - \vec{r}'| = k r - \underbrace{k \vec{r} \cdot \vec{r}'}_{\text{长波近似}} - \frac{(\vec{r} \cdot \vec{r}')^2 + \dots}{2r^3} + \dots$ (\vec{k} 与 \vec{r} 夹角 θ)

(A世. 即只要在长波近似中也有 $r \gg d$, $r \gg \lambda$, 即 $d \ll \lambda \ll r$)

$$\Rightarrow \vec{A}(\vec{r}, t) = \vec{A}_0(\theta, \varphi) \frac{e^{i(kr - \omega t)}}{r}, \quad \vec{A}_0(\theta, \varphi) = \frac{\mu_0}{4\pi} \int J_0(\vec{r}') e^{-i\vec{k} \cdot \vec{r}'} d\tau'$$

长波近似, 振幅 $\propto \frac{1}{r}$. 依 \vec{r} 的 (θ, φ) , 方向并化

(Len: $\nabla \times \vec{F}(\vec{r}, t) = i\vec{k} \times \vec{F}(\vec{r}, t)$, $\vec{F}(\vec{r}, t) = \vec{F}_0(\theta, \varphi) \frac{e^{i(kr - \omega t)}}{r}$, $\vec{k} \parallel \vec{r}$.)

由 $\vec{B} = \nabla \times \vec{A}(\vec{r}, t) = \vec{B}_0(\theta, \varphi) \frac{e^{i(kr - \omega t)}}{r}$, $\vec{B}_0(\theta, \varphi) = i\vec{k} \times \vec{A}_0$

再利用其中 $\vec{E} = c \vec{B} \times \hat{k}$, 则有 $\vec{E}(\vec{r}, t) = \vec{E}_0(\theta, \varphi) \frac{e^{i(kr - \omega t)}}{r}$

$$\vec{S} = \frac{1}{\mu_0} (\text{Re } \vec{E} \times \text{Re } \vec{B}) = \frac{c}{\mu_0} (\text{Re } \vec{B} \cdot \text{Re } \vec{B}) \hat{k}$$

则 $\langle \vec{S} \rangle = \frac{c}{2\mu_0} (\vec{B} \cdot \vec{B}^*) \hat{k}$, $P = \oint \langle \vec{S} \rangle \cdot d\vec{\sigma} = \frac{c}{2\mu_0} \iint \vec{B} \cdot \vec{B}^* r^2 d\Omega = \frac{c}{2\mu_0} \iint \vec{B} \cdot \vec{B}^* d\Omega$

$$\frac{dP}{d\Omega} = \frac{c}{2\mu_0} \vec{B} \cdot \vec{B}^*, \quad \text{即辐射功率分布}$$

Ex. 中心电流天线, $\vec{J}(\vec{r}, t) = I(z) \delta(x) \delta(y) e^{-i\omega t} \hat{z}$, $I(z) = I_0 \sin[k(\frac{d}{2} - |z|)]$

$$\begin{aligned} \vec{A}_0(\theta, \varphi) &= \frac{\mu_0}{4\pi} \int \vec{J}_0(\vec{r}') e^{-i\vec{k} \cdot \vec{r}'} d\tau' = \frac{\mu_0 I_0}{4\pi} \int_{-d/2}^{d/2} \sin[k(\frac{d}{2} - |z'|)] e^{-i\vec{k} \cdot \vec{z}'} dz' \\ &= \frac{\mu_0 I_0}{2\pi k} \left[\frac{\cos(\frac{kd}{2} \cos\theta) - \cos(\frac{kd}{2})}{\sin^2\theta} \right] \hat{z} \end{aligned}$$

$$\vec{B}_0 = i\vec{k} \times \vec{A}_0, \quad B_0 = k|A_0| \sin\theta, \quad \frac{dP}{d\Omega} = \frac{c}{2\mu_0} \vec{B} \cdot \vec{B}^* = \frac{\mu_0 c I_0^2}{8\pi^2} \left[\right] \sin^2\theta$$

① 短天线 $\lambda \gg d$ 时, $\frac{dP}{d\Omega} = \frac{\mu_0 I_0^2}{128\pi^2 c^3} \omega^4 d^4 \sin^2\theta \propto \sin^2\theta$, $P = \int dP = \frac{\mu_0 I_0^2}{48\pi c^3} \omega^4 d^4 \propto \lambda^{-4} \propto \left(\frac{d}{\lambda}\right)^4$

($\theta = 0, \pi$ 处为 0)

② 半波天线, $d = \frac{\lambda}{2}$ 时, $\frac{dP}{d\Omega} = \frac{\mu_0 c I_0^2}{8\pi^2} \left(\frac{\cos(\frac{\pi}{2} \cos\theta)}{\sin\theta} \right)^2$ (5 $\sin^2\theta$ 近似!) (因为 $\theta = 0$ 或 π 无意义)

$$P = \int dP = 1.22 \frac{\mu_0 c I_0^2}{4\pi} \cdot 5 \omega \pi \lambda^2$$

电场的辐射场的多级展开.

$\vec{A}_0(\vec{r}, \varphi) = \frac{\mu_0}{4\pi} \int \vec{J}_0(\vec{r}') e^{-i\vec{k} \cdot \vec{r}'} d\tau'$ (使用 $r \gg d, r \gg \lambda$, 远场近似) (因为远场近似, $d \ll r \ll r$) ($\vec{A} = \vec{A}_0(\vec{r}, \varphi) \frac{e^{i(kr - \omega t)}}{r}$)

对 $e^{-i\vec{k} \cdot \vec{r}'}$ 展开 $e^{-i\vec{k} \cdot \vec{r}'} = 1 - i\vec{k} \cdot \vec{r}' + \dots$

电场的辐射场:

$$\vec{A}_0^{(ic)}(\vec{r}, \varphi) = \frac{\mu_0}{4\pi} \int \vec{J}_0(\vec{r}') d\tau' = -\frac{\mu_0}{4\pi} i\omega \vec{p}_0$$

电场的辐射场:

$$\vec{A}_0^{(2e)}(\vec{r}, \varphi) = -\frac{\mu_0}{4\pi} i\omega \int \vec{r}' \cdot \vec{J}_0(\vec{r}') d\tau' = -\frac{\mu_0}{4\pi} i\omega \vec{p}_0$$

$\nabla' \cdot (\vec{J}_0 \vec{r}') = (\nabla' \cdot \vec{J}_0) \vec{r}' + (\vec{J}_0 \cdot \nabla') \vec{r}'$ (利用 $\nabla' \cdot \vec{J}_0 = 0$)

令 $\vec{r}' = 0$, $0 = \int i\omega \vec{r}' \cdot \vec{J}_0(\vec{r}') d\tau' + \int (\vec{J}_0 \cdot \nabla') \vec{r}' d\tau' = \int \vec{J}_0 d\tau'$

$\Rightarrow \vec{A}_0^{(2e)}(\vec{r}, \varphi) = -\frac{\mu_0}{4\pi} i\omega \int \vec{r}' \cdot \vec{J}_0(\vec{r}') d\tau' = -\frac{\mu_0}{4\pi} i\omega \vec{p}_0$

$\Rightarrow \vec{B}^{(2e)}(\vec{r}, t) = \frac{\mu_0 \omega}{4\pi} \vec{k} \times \vec{p}_0 \frac{e^{i(kr - \omega t)}}{r}$, $\frac{dP^{(2e)}}{d\Omega} = \frac{c}{2\mu_0} \vec{B}_0 \vec{B}_0^* = \frac{\mu_0 \omega^4 p_0^2}{32\pi^2 c} \sin^2\theta$, $P^{2e} = \frac{\mu_0 \omega^4 p_0^2}{12\pi c}$

磁场的辐射场:

电场的辐射场:

$$\vec{A}_0^{(2m)} + \vec{A}_0^{(4e)} = -i \frac{\mu_0}{4\pi} \int \vec{k} \cdot \vec{r}' \vec{J}_0(\vec{r}') d\tau'$$

$\int \vec{k} \cdot \vec{r}' \vec{J}_0(\vec{r}') d\tau' = \int \vec{k} \cdot (\vec{r}' \vec{J}_0(\vec{r}')) d\tau'$, $\vec{r}' \vec{J}_0(\vec{r}') = \frac{1}{2} (\vec{r}' \vec{J}_0 - \vec{J}_0 \vec{r}') + \frac{1}{2} (\vec{r}' \vec{J}_0 + \vec{J}_0 \vec{r}')$

① 磁场的辐射场:

再根据 $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C}) \vec{B} - (\vec{A} \cdot \vec{B}) \vec{C} = \vec{A} \cdot (\vec{C} \vec{B} - \vec{B} \vec{C})$

由 $\vec{A}_0^{(2m)} = i \frac{\mu_0}{4\pi} \times \frac{1}{2} \int \vec{k} \times [\vec{r}' \times \vec{J}_0(\vec{r}')] d\tau' = \frac{i\mu_0}{4\pi} \vec{k} \times [\frac{1}{2} \int \vec{r}' \times \vec{J}_0(\vec{r}') d\tau'] = \frac{i\mu_0}{4\pi} \vec{k} \times \vec{m}_0$

$$\vec{B}_0^{(2m)} = i\vec{k} \times \vec{A}_0^{(2m)}, \quad |\vec{B}_0^{(2m)}| = \frac{\mu_0}{4\pi} k^2 m_0 \sin\theta.$$

② ④

$$\nabla' \cdot (\vec{J}_0 \vec{r}' \vec{r}') = (\nabla' \cdot \vec{J}_0) \vec{r}' \vec{r}' + \vec{J}_0 \cdot (\nabla' \vec{r}') \vec{r}' + \vec{J}_0 \vec{r}' \cdot (\nabla' \vec{r}') \quad (\nabla' \vec{r}' = \mathbf{I}, \nabla' \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \Rightarrow \nabla' \cdot \vec{J}_0 = i\omega \rho_0)$$

$$\Rightarrow \nabla' \cdot (\vec{J}_0 \vec{r}' \vec{r}') = i\omega \rho_0 \vec{r}' \vec{r}' + \underbrace{\vec{J}_0 \vec{r}' + \vec{r}' \vec{J}_0}_{\text{全为反对称?}}$$

$$\Rightarrow \vec{A}_0^{(4e)} = -i \frac{\mu_0}{4\pi} \vec{k} \cdot \int \frac{1}{2} [\nabla' \cdot (\vec{J}_0 \vec{r}' \vec{r}') - i\omega \rho_0 \vec{r}' \vec{r}'] d\tau' = -\frac{\mu_0}{8\pi} \vec{k} \cdot \int \underbrace{\vec{D}_0}_{\vec{D}_0} d\tau' = -\frac{\mu_0}{8\pi} \vec{k} \cdot \vec{D}_0$$

$$\text{可以用艾里逊定理} \quad \vec{D}_0 = 3\vec{D}_0 - \text{tr}(\vec{D}_0) \mathbf{I} \Rightarrow \vec{A}_0^{(4e)} = -\frac{\mu_0 \omega^2}{24\pi c} \vec{k} \cdot \vec{D}_0 + \frac{\mu_0 \omega^2}{24\pi c} \text{Tr}(\vec{D}_0) \vec{k}$$

$$\Rightarrow \vec{B}_0^{(4e)} = i\vec{k} \times \vec{A}_0^{(2m)}, \quad \left(\frac{dP}{d\Omega}\right)^{(4e)} = \frac{c}{4\pi_0} (\vec{B}_0^{(4e)} \cdot \vec{B}_0) = \frac{\mu_0 \omega^6}{1152 \pi^2 c^3} \underbrace{|\hat{k} \cdot (\vec{k} \cdot \vec{D}_0)|^2}_{\text{校正}} \quad \rho^{(4e)} = \frac{\mu_0 \omega^6}{1440 \pi^2 c^3} (\vec{D}_0 : \vec{D}_0) \quad \text{校正 (平行于 } \vec{k}), \nabla \times \mathbf{O} = \vec{0}$$

磁回与电八 (可以用艾里逊定理从电回推得磁回)

$$\vec{p} \leftrightarrow m/c, \quad c\vec{B} \leftrightarrow -\vec{E}, \quad \vec{E} \leftrightarrow c\vec{B} \quad (\text{对 } \vec{D}_0 \text{ 的变换需排斥})$$

§4. 电石良良良良良

$$\text{并化 } \sigma_T = \frac{8}{3} \pi r_0^2, \quad \sigma = \frac{P}{S} \rightarrow \text{辐射截面}$$

σ_T 为 Thomson 散射的截面

$$\frac{d\sigma}{d\Omega}(\vec{k}_s, \hat{e}_s; \vec{k}_i, \hat{e}_i)$$

↓ ↓ ↓ ↓
辐射波 辐射偏振 入射波 入射偏振

$$= r^2 \frac{(\vec{2Z}_0)^{-1}}{(\vec{2Z}_0)^{-1}} \frac{|\hat{e}_s^* \cdot \vec{E}_s|^2}{|\hat{e}_i^* \cdot \vec{E}_i|^2} = r^2 \frac{|\hat{e}_s^* \cdot \vec{E}_s|^2}{|\hat{e}_i^* \cdot \vec{E}_i|^2} = r^2 \left\{ \hat{e}_s^* \left[\hat{k}_s \times (\hat{k}_s \times \hat{e}_i) \right] \right\}^2$$

校正了 \hat{e}_s 与 \vec{E}_s 的相位
↑ ↓
辐射波 辐射偏振