

## Necessity of Quantization

Let  $\varphi$  such that

$$i\hbar\dot{\varphi} = [\varphi, H]$$

Consider a harmonic oscillator, we have

$$\ddot{\varphi} = -\omega^2\varphi$$

Introduce

$$A = \frac{1}{2}\left(\varphi + \frac{i}{\omega} \frac{d\varphi}{dt}\right), \quad A^\dagger = \frac{1}{2}\left(\varphi - \frac{i}{\omega} \frac{d\varphi}{dt}\right)$$

We have

$$\varphi = A + A^\dagger, \quad \frac{dA}{dt} = -i\omega A$$

thus

$$A(t) = A(0)e^{-i\omega t}$$

which means positive-frequency mode if  $\omega > 0$ . Similarly,

$$A^\dagger(t) = A^\dagger(0)e^{i\omega t}$$

which means negative-frequency mode if  $\omega > 0$ .

With  $i\hbar\dot{\varphi} = [\varphi, H]$ , we have

$$i\hbar\dot{A} = [A, H]$$

while using  $A(t) = A(0)e^{-i\omega t}$  we have

$$[H, A] = -\hbar\omega A$$

Thus for any eigenstate  $|E\rangle$  of  $H$ ,  $H|E\rangle = E|E\rangle$ ,

$$H(A|E\rangle) = ([H, A] + AH)|E\rangle = (E - \hbar\omega)A|E\rangle$$

meaning that  $A|E\rangle$  is eigenstate of  $H$ , with eigenvalue  $E - \hbar\omega$  unless  $A|E\rangle = 0$ .

Similarly,

$$[H, A^\dagger] = \hbar\omega A^\dagger$$

meaning that  $A^\dagger|E\rangle$  is eigenstate with eigenvalue  $E + \hbar\omega$ .

To summarize, with

$$\dot{A} = -i\omega A$$

we conclude that  $A$  lowers energy by  $\hbar\omega$ .

## Special Relativity

Event:

$$(x^0 = ct, x^1, x^2, x^3)$$

The spacetime interval between  $x$  and the origin

$$x^2 = -x^{0^2} + x^{1^2} + x^{2^2} + x^{3^2}$$

Let

$$g_{\mu\nu} = \begin{cases} -1, & \mu = \nu = 0 \\ +1, & \mu = \nu = 1, 2, 3, \\ 0, & \mu \neq \nu \end{cases} \quad g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

We make a convention on indices,

$$\mu, \nu, \rho \in \{0, 1, 2, 3\}, \quad i, j, k \in \{1, 2, 3\}$$

thus

$$x^2 = g_{\mu\nu} x^\mu x^\nu$$

and  $x$  is a 4-vector. Define

$$x_\mu = g_{\mu\nu} x^\nu$$

we have

$$x_0 = -x^0, \quad x_1 = x^1, \quad x_2 = x^2, \quad x_3 = x^3$$

thus

$$x^2 = x^\mu x_\mu$$

A Lorentz transformation is a linear homogeneous change of coordinates from  $x^\mu$  to  $\tilde{x}^\mu$ ,

$$\tilde{x}^\mu = \Lambda^\mu_\nu$$

that preserves the spacetime interval, namely

$$\tilde{x}^\mu \tilde{x}_\mu = x^\mu x_\mu, \quad g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = g_{\rho\sigma}$$

The inverse of  $\Lambda$  is  $\Lambda^{-1}$ ,

$$\Lambda_\nu^\mu = (\Lambda^{-1})^\mu_\nu = g^{\mu\rho} g_{\nu\sigma} \Lambda^\sigma_\rho$$

using

$$g^{\mu\rho} g_{\rho\nu} = \delta^\mu_\nu$$

we have

$$(\Lambda^{-1})^\mu_\alpha \Lambda^\alpha_\nu = \delta^\mu_\nu$$

For an infinitesimal LT (Lorentz Transformation),

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \theta^\mu_\nu$$

where  $\theta^\mu_\nu$  is infinitesimal, with

$$\theta_{\mu\nu} = g_{\mu\rho} \theta^\rho_\nu$$

Since

$$0 = g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma - g_{\rho\sigma} = g_{\mu\sigma} \theta^\mu_\rho + g_{\rho\nu} \theta^\nu_\sigma + O(\theta^2) = \theta_{\sigma\rho} + \theta_{\rho\sigma} + O(\theta^2)$$

we have

$$\theta_{\mu\nu} = -\theta_{\nu\mu}, \quad \text{to leading order in } \theta$$

Also we need

$$\det \Lambda = \pm 1, \quad \begin{cases} +1 : & \text{proper LT} \\ -1 : & \text{improper LT} \end{cases}$$

In  $g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = g_{\rho\sigma}$  we set  $\rho = \sigma = 0$  and find

$$(\Lambda^0_0)^2 = 1 + (\Lambda^1_0)^2 + (\Lambda^2_0)^2 + (\Lambda^3_0)^2 \geq 1$$

i.e.

$$\Lambda^0_0 \geq +1 \text{ or } \Lambda^0_0 \leq -1$$

LT's with  $\Lambda^0_0 \geq +1$  are orthochronous LT's ,while LT's with  $\Lambda^0_0 \leq -1$  are nonorthochronous LT's.

Parity:

$$\mathcal{P} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

which is an orthochronous improper LT. While time reversal:

$$\mathcal{T} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}$$

which is an nonorthochronous improper LT.

## General principles of relativistic QFT

1. At every point  $x$  in spacetime, there is one or a number of fundamental variables  $\varphi(x)$  or  $\varphi_1(x), \dots, \varphi_N(x)$ .
2. Observables in two distinct regions with spacelike interval we have  $FG = GF$ . And we require  $\varphi_a(x)\varphi_b(y) = \pm\varphi_b(y)\varphi_a(x)$ , if  $(x - y)^2 > 0$ .
3. For a state  $|\alpha\rangle$  with  $\langle\alpha|\alpha\rangle$  and a set of orthonormal base  $\langle\beta_i|\beta_j\rangle$ , then the probability to be measured at  $|\beta_i\rangle$  is  $P_i = |\langle\beta_i|\alpha\rangle|^2$ , which is the Born's rule.
4.  $\mathcal{U}(\Delta t)$ : unitary operators satisfying  $\mathcal{U}(t_1)\mathcal{U}(t_2) = \mathcal{U}(t_1 + t_2)$ . If  $|\psi\rangle$  is a state, then  $\mathcal{U}(\Delta t)|\psi\rangle$  is also a state, and if  $\varphi_a(x)|\psi\rangle = \lambda|\psi\rangle$  then  $\varphi_a(x + \Delta t)[\mathcal{U}(\Delta t)|\psi\rangle] = \lambda[\mathcal{U}(\Delta t)|\psi\rangle]$ , with

$$\varphi_a(x + \Delta t) = \varphi_a(x^0 + \Delta t, x^1, x^2, x^3)$$

and we find

$$\varphi_a(x + \Delta t) = \mathcal{U}(\Delta t)\varphi_a(x)\mathcal{U}(\Delta t)^{-1}$$

For  $\varepsilon$  infinitesimal,  $\mathcal{U}(\varepsilon) = 1 + \frac{iG}{\hbar}\varepsilon$ , while  $G$  is Hermitian. We have

$$\mathcal{U}(N\varepsilon) = (\mathcal{U}(\varepsilon))^N = e^{\frac{iG}{\hbar}\varepsilon N}$$

thus

$$\mathcal{U}(\Delta t) = e^{\frac{iG\Delta t}{\hbar}}, \quad \varphi_a(x + \Delta t) = e^{\frac{iG\Delta t}{\hbar}}\varphi_a(x)e^{-\frac{iG\Delta t}{\hbar}}$$

meaning that

$$\dot{\varphi}_a = \frac{1}{i\hbar}[G, \varphi_a]$$

therefore

$$G = H$$

thus

$$\mathcal{U}(\Delta t) = e^{\frac{iHt}{\hbar}}$$

Let  $\vec{p} = (p^1, p^2, p^3)$  and  $p^\mu = (H, \vec{p})$ , together with translation we have

$$\mathcal{U}(l^\mu) = e^{\frac{iHl^0 - ip^1l^1 - ip^2l^2 - ip^3l^3}{\hbar}} = e^{-\frac{ig_{\mu\nu}p^\mu p^\nu}{\hbar}} = e^{-\frac{ip \cdot l}{\hbar}}$$

where

$$u \cdot v = u^\mu v_\mu = u_\mu v^\mu = g^{\mu\nu}u_\mu v_\nu$$

thus

$$\varphi_a(x+l) = e^{\frac{iH^0 - i\vec{p}\cdot\vec{l}}{\hbar}} \varphi_a(x) e^{-\frac{iH^0 - i\vec{p}\cdot\vec{l}}{\hbar}} = e^{-\frac{i\vec{p}\cdot\vec{l}}{\hbar}} \varphi_a(x) e^{\frac{i\vec{p}\cdot\vec{l}}{\hbar}}$$

Therefore we have the derivatives

$$\begin{cases} \frac{\partial \varphi_a(x)}{\partial t} = +\frac{i}{\hbar} [H, \varphi_a(x)] = -\frac{i}{\hbar} [p_0, \varphi_a(x)] \\ \frac{\partial \varphi_a(x)}{\partial x^i} = -\frac{i}{\hbar} [p^i, \varphi_a(x)] = -\frac{i}{\hbar} [p_i, \varphi_a(x)] \end{cases}$$

or to be written as

$$\partial_\mu \varphi_a(x) = -\frac{i}{\hbar} [p_\mu, \varphi_a(x)]$$

$$\text{with } \partial_\mu = \frac{\partial}{\partial x^\mu}.$$

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## Scalar fields

For an event A, we note as

$$\begin{cases} x \\ \bar{x} \end{cases} \text{ in another frame}$$

For scalar fields we let

$$\bar{\varphi}_a(\bar{x}) = \varphi_a(x)$$

and Lorentz transformation

$$\bar{x}^\mu = \Lambda^\mu_\nu x^\nu$$

or

$$\bar{x} = \Lambda x, \quad x = \Lambda^{-1} \bar{x}$$

thus

$$\bar{\varphi}_a(\bar{x}) = \varphi_a(\Lambda^{-1} \bar{x}), \quad \bar{\varphi}_a(x) = \varphi_a(\Lambda^{-1} x)$$

Consider an infinitesimal LT:

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \theta^\mu_\nu, \quad (\Lambda^{-1})^\mu_\nu = \delta^\mu_\nu - \theta^\mu_\nu$$

then

$$\bar{\varphi}_a(x) - \varphi_a(x) = \varphi_a(x^\mu - \theta^\mu_\nu x^\nu) - \varphi_a(x^\mu) = -\theta^\mu_\nu x^\nu \partial_\mu \varphi_a = -\theta_{\mu\nu} x^\nu \partial^\mu \varphi_a = \frac{\theta_{\mu\nu}}{2} (x^\mu \partial^\nu - x^\nu \partial^\mu) \varphi_a$$

Let

$$\mathcal{L}^{\mu\nu} = -i(x^\mu \partial^\nu - x^\nu \partial^\mu)$$

then

$$\delta \varphi_a(x) = \bar{\varphi}_a(x) - \varphi_a(x) = \frac{i\theta_{\mu\nu}}{2} \mathcal{L}^{\mu\nu} \varphi_a(x)$$

or to write explicitly

$$\delta \varphi_a(x) = (i\theta_{12} \mathcal{L}^{12} + i\theta_{13} \mathcal{L}^{13} + i\theta_{31} \mathcal{L}^{31} + i\theta_{10} \mathcal{L}^{10} + i\theta_{20} \mathcal{L}^{20} + i\theta_{30} \mathcal{L}^{30}) \varphi_a(x)$$

with  $\mathcal{L}^{12}, \mathcal{L}^{23}, \mathcal{L}^{31}$  related to the orbital angular momentum of particles.

Thus scalar particles are spinless particles.

## Free field theory

Consider single component scalar field  $\varphi(x)$ , with



$$E, \vec{p}: \quad E^2 - |\vec{p}|^2 = (\text{rest mass})^2$$

The matter wave

$$e^{-i\omega t + i\vec{k} \cdot \vec{x}}, \quad E = \hbar\omega, \vec{p} = \hbar\vec{k}$$

Using  $-E^2 + |\vec{p}|^2 + \text{Const} = 0$  we have

$$\left( \hbar^2 \frac{\partial^2}{\partial t^2} - \hbar^2 \nabla^2 + \text{Const} \right) e^{-i\omega t + i\vec{k} \cdot \vec{x}} = 0$$

Let  $\frac{\text{Const}}{\hbar^2} = \omega_0^2$  then

$$\left( \frac{\partial^2}{\partial t^2} - \nabla^2 + \omega_0^2 \right) \varphi(x) = 0$$

which is the **Klein-Gordon equation**. Let

$$\partial^2 = \partial_\mu \partial^\mu = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

the equation can be written as

$$(-\partial^2 + \omega_0^2) \varphi(x) = 0$$

Consider the quantization

$$\varphi(x)\varphi(y) \mp \varphi(y)\varphi(x) = C(x-y)$$

Or

$$\varphi(x)\varphi(y) - \sigma\varphi(y)\varphi(x) = C(x-y), \quad \sigma = \pm 1$$

For spacelike interval,

$$C(x-y) = 0, \quad \text{if } (x-y)^2 > 0$$

and

$$C(\Lambda x) = C(x), \quad \text{if } \Lambda \text{ is proper orthochronous}$$

Let  $y = 0$  we find

$$\begin{cases} \varphi(x)\varphi(0) - \sigma\varphi(0)\varphi(x) = C(x) \\ \varphi(0)\varphi(x) - \sigma\varphi(x)\varphi(0) = C(-x) \end{cases} \implies C(-x) = -\sigma C(x)$$

thus if  $\sigma = +1$  then  $C(x)$  is odd, and if  $\sigma = -1$  then  $C(x)$  is even.

Using the Fourier transform

$$C(x) = \int \frac{d^4 k}{(2\pi)^4} e^{ik \cdot x} \tilde{C}(k) = \int_{-\infty}^{+\infty} \frac{dk^0 dk^1 dk^2 dk^3}{(2\pi)^4} e^{-ik^0 \cdot x^0 + i\vec{k} \cdot \vec{x}} \tilde{C}(k)$$

using the equation we have

$$0 = (-\partial^2 + \omega_0^2) \int \frac{d^4 k}{(2\pi)^4} e^{ik \cdot x} \tilde{C}(k) = \int \frac{d^4 k}{(2\pi)^4} (k^2 + \omega_0^2) e^{ik \cdot x} \tilde{C}(k)$$

with  $k^2 = -k^{02} + |\vec{k}|^2$ . Thus

$$(k^2 + \omega_0^2) \tilde{C}(k) = 0, \quad (-k^{02} + |\vec{k}|^2 + \omega_0^2) \tilde{C}(k) = 0$$

Since  $C(\Lambda x) = C(x)$  for proper orthochronous LT, we have

$$\tilde{C}(\Lambda k) = \tilde{C}(k) \quad \text{for proper orthochronous LT}$$

therefore

$$\tilde{C}(k) = c_+ 2\pi \delta(k^2 + \omega_0^2) \theta(k^0) + c_- 2\pi \delta(k^2 + \omega_0^2) \theta(-k^0)$$

thus

$$C(x) = \int \frac{d^4 k}{(2\pi)^4} [c_+ 2\pi\delta(k^2 + \omega_0^2)\theta(k^0) + c_- 2\pi\delta(k^2 + \omega_0^2)\theta(-k^0)] = c_+ \int \frac{dk^1 dk^2 dk^3}{(2\pi)^3 2\omega_{\vec{k}}} e^{-i\omega_{\vec{k}}x^0 + i\vec{k}\cdot\vec{x}} + c_- \int \frac{dk^1 dk^2 dk^3}{(2\pi)^3 2\omega_{\vec{k}}} e^{+i\omega_{\vec{k}}x^0 + i\vec{k}\cdot\vec{x}}$$

which has used

$$\int a^b \delta(g(\xi)) d\xi = \sum_i \frac{1}{|g'(\xi_i)|}, \quad g(\xi_i) = 0, a < \xi_i < b$$

Define the **Lorentz invariant measure**

$$\widetilde{d}k = \frac{d^3 k}{(2\pi)^3 2\omega_{\vec{k}}}$$

or

$$\int \widetilde{d}k = \int \frac{d^4 k}{(2\pi)^4} 2\pi\delta(k^2 + \omega_0^2)\theta(k^0)$$

For  $t = 0$  we have

$$C(0, \vec{x}) = \int \widetilde{d}k (c_+ + c_-) e^{i\vec{k}\cdot\vec{x}} \propto (c_+ + c_-) \frac{K_1(\omega_0|\vec{x}|)}{|\vec{x}|}$$

Since  $C(x) = 0$  if  $x^2 > 0$ , we require  $C(0, \vec{x}) = 0$ , thus

$$c_+ + c_- = 0, \quad c_- = -c_+$$

therefore

$$\tilde{C}(k) = c_+ [2\pi\delta(k^2 + \omega_0^2)\theta(k^0) - 2\pi\delta(k^2 + \omega_0^2)\theta(-k^0)]$$

we can see that

$$\tilde{C}(-k) = -\tilde{C}(k) \implies C(-x) = -C(x) \implies \sigma = +1$$

and then

$$C(x) = c_+ \int \widetilde{d}k (e^{-i\omega_{\vec{k}}x^0 + i\vec{k}\cdot\vec{x}} - e^{+i\omega_{\vec{k}}x^0 + i\vec{k}\cdot\vec{x}})$$

Therefore the commutator,

$$[\varphi(x), \varphi(y)] = \varphi(x)\varphi(y) - \varphi(y)\varphi(x) = C(x - y)$$

we have

$$[\varphi(\vec{x}), \varphi(\vec{y})] = c_+ \operatorname{sgn}(x^0 - y^0) \left[ i\omega_0\theta(\tau^2) \frac{J_1(\omega_0\tau)}{4\pi\tau} - \frac{i}{2\pi} \delta(\tau^2) \right], \quad \tau = \sqrt{-(x - y)^2}$$

Since

$$[\varphi(x), \varphi(y)]^* = [\varphi(x), \varphi(y)]^\dagger = [\varphi(y)^\dagger, \varphi(x)^\dagger] = [\varphi(y), \varphi(x)] = -[\varphi(x), \varphi(y)]$$

we require that  $c_+$  is real.

Similarly,

$$[\varphi(t, \vec{x}), \frac{\varphi(t + \varepsilon, \vec{y}) - \varphi(t, \vec{y})}{\varepsilon}] = \frac{c_+}{\varepsilon} \operatorname{sgn}(-\varepsilon) \left[ i\omega_0\theta(\varepsilon^2 - |\vec{x} - \vec{y}|^2) \frac{J_1(\omega_0\sqrt{\varepsilon^2 - |\vec{x} - \vec{y}|^2})}{4\pi\sqrt{\varepsilon^2 - |\vec{x} - \vec{y}|^2}} - \frac{i}{2\pi} \delta(\varepsilon^2 - |\vec{x} - \vec{y}|^2) \right]$$

Integrate over volume,

$$\int d^3 y [\varphi(t, \vec{x}), \frac{\varphi(t + \varepsilon, \vec{y}) - \varphi(t, \vec{y})}{\varepsilon}] = ic_+, \quad \varepsilon \rightarrow 0$$

thus

$$[\varphi(t, \vec{x}), \dot{\varphi}(t, \vec{y})] = ic_+ \delta(\vec{x} - \vec{y})$$

Also,

$$[\varphi(t, \vec{x}), \varphi(t, \vec{y})] = 0$$

Choose amplitude of  $\varphi(x)$  such that  $|c_+| = 1$ , so  $c_+ = \pm 1$ .

Since

$$\begin{cases} i\dot{\varphi}(x) = [\varphi(x), H] \\ i\ddot{\varphi}(x) = [\dot{\varphi}(x), H] \end{cases}$$

and for Klein-Gordon field,

$$\ddot{\varphi} - \nabla^2 \varphi + \omega_0^2 \varphi = 0$$

thus

$$[\dot{\varphi}(x), H] = i(\nabla^2 \varphi - \omega_0^2 \varphi)$$

We can guess the Hamiltonian as

$$H(t) = \frac{1}{2c_+} \int d^3x [\dot{\varphi}^2 + (\nabla \varphi)^2 + \omega_0^2 \varphi^2] + (\text{arbitrary constant})$$

We may choose  $c_+ > 0$  such that  $\hat{H}$  has lower bound, thus we have the convention

$$c_+ = 1$$

therefore,

$$\begin{cases} [\varphi(t, \vec{x}), \varphi(t, \vec{y})] = 0 \\ [\varphi(t, \vec{x}), \dot{\varphi}(t, \vec{y})] = i\delta(\vec{x} - \vec{y}) \\ H = \frac{1}{2} \int d^3x [\dot{\varphi}^2 + (\nabla \varphi)^2 + \omega_0^2 \varphi^2] + \text{Const} \end{cases}$$

We can see dimension

$$[\varphi] = \frac{1}{\text{length}}, \quad [H] = \frac{1}{\text{length}}$$

Also we can guess  $\vec{p}$  such that

$$\begin{cases} \frac{\partial \varphi(x)}{\partial x^i} = -i[p^i, \varphi(x)] \\ \frac{\partial \dot{\varphi}(x)}{\partial x^i} = -i[p^i, \dot{\varphi}(x)] \end{cases}$$

the result is

$$\vec{p}(t) = - \int d^3x y \dot{\varphi} \nabla \varphi + \text{another arbitrary constant}$$

We do Fourier transform to  $\varphi(x)$ ,

$$\varphi(x) = \int \frac{d^4k}{(2\pi)^4} \tilde{\varphi}(k) e^{ik \cdot x}, \quad k \cdot x = -k^0 x^0 + \vec{k} \cdot \vec{x}$$

and

$$\tilde{\varphi}(k) = \int d^4x \varphi(x) e^{-ik \cdot x}$$

Since

$$(-\partial^2 + \omega_0^2) \varphi(x) = 0 \implies (k^2 + \omega_0^2) \tilde{\varphi}(k) = 0$$

therefore

$$\tilde{\varphi}(k) = 2\pi \delta(k^2 + \omega_0^2) [a(\vec{k}) \theta(k^0) + b(\vec{k}) \theta(-k^0)]$$

meaning that nonzero requires

$$\vec{k} = \omega_{\vec{k}}, \quad \omega_{\vec{k}} = \sqrt{\omega_0^2 + |\vec{k}|^2}$$

Also

$$\varphi^\dagger(x) = \varphi(x) \implies \tilde{\varphi}(k)^\dagger = \tilde{\varphi}(-k)$$

we get

$$a(-\vec{k}) = b^\dagger(\vec{k}), \quad b(\vec{k}) = a^\dagger(-\vec{k})$$

thus  $\tilde{\varphi}(k)$  could be rewritten as

$$\tilde{\varphi}(k) = 2\pi\delta(k^2 + \omega_0^2)[a(\vec{k})\theta(k^0) + a^\dagger(-\vec{k})\theta(-k^0)]$$

then

$$\varphi(x) = \int \widetilde{dk} (a(\vec{k})e^{ik \cdot x} + a^\dagger(\vec{k})e^{-ik \cdot x}), \quad \widetilde{dk} = \frac{dk^1 dk^2 dk^3}{(2\pi)^3 2\omega_{\vec{k}}}$$

we can obtain

$$a(\vec{k}) = \int d^3x [\omega_{\vec{k}} \varphi(t, \vec{x}) + i\dot{\varphi}(t, \vec{x})] e^{-ik \cdot x}, \quad t \text{ is arbitrary}$$

since

$$\int_0^\infty \tilde{\varphi}(k^0, \vec{k}) dk^0 = \frac{2\pi}{2\omega_{\vec{k}}} a(\vec{k})$$

from which we have

$$\begin{cases} a(\vec{k}) = \frac{\omega_{\vec{k}}}{\pi} \int_0^\infty dk^0 \tilde{\varphi}(k) \\ a^\dagger(\vec{k}) = \frac{\omega_{\vec{k}}}{\pi} \int_0^\infty dk^0 \tilde{\varphi}(-k) \end{cases}$$

thus

$$\begin{aligned} [\varphi(x), \varphi(y)] &= \int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2 + \omega_0^2) [\theta(k^0) - \theta(-k^0)] e^{ik \cdot (x-y)} \\ \implies [\tilde{\varphi}(k), \tilde{\varphi}(l)] &= \int d^4x d^4y [\varphi(x) e^{-ik \cdot x}, \varphi(y) e^{-il \cdot y}] = 2\pi\delta(k^2 + \omega_0^2) \text{sgn}(k^0) (2\pi)^4 \delta(k+l) \end{aligned}$$

Therefore,

$$[a(k), a(l)] = \frac{\omega_{\vec{k}} \omega_{\vec{l}}}{\pi^2} \int_0^\infty dk^0 \int_0^\infty dl^0 [\tilde{\varphi}(k), \tilde{\varphi}(l)] = 0$$

Similarly

$$[a^\dagger(k), a^\dagger(l)] = 0$$

However,

$$[a(k), a^\dagger(l)] = (2\pi)^3 2\omega_{\vec{k}} \delta(\vec{k} - \vec{l})$$

which means  $a(k)$  is boson annihilation operator.

To summarize,

$$\begin{cases} [a(\vec{k}), a(\vec{l})] = [a^\dagger(\vec{k}), a^\dagger(\vec{l})] = 0 \\ [a(\vec{k}), a^\dagger(\vec{l})] = (2\pi)^3 2\omega_{\vec{k}} \delta(\vec{k} - \vec{l}) \\ \varphi(x) = \int \widetilde{dk} [a(\vec{k}) e^{ik \cdot x} + a^\dagger(\vec{k}) e^{-ik \cdot x}] \quad (k^0 = \omega_{\vec{k}}) \end{cases}$$

Therefore the Hamiltonian,

$$H = \int \widetilde{dk} \frac{\omega_{\vec{k}}}{2} [a(\vec{k}) a^\dagger(\vec{k}) + a^\dagger(\vec{k}) a(\vec{k})] + \text{Const}$$

which could be written as

$$H = \int \widetilde{dk} \omega_{\vec{k}} a^\dagger(\vec{k}) a(\vec{k}) + \sum_{\vec{k}} \frac{\omega_{\vec{k}}}{2} + \text{Const}$$

And the momentum

$$\vec{p} = - \int d^3x \dot{\varphi}(x) \nabla \varphi(x) + \text{Const} = \int d\vec{k} \vec{k} a^\dagger(\vec{k}) a(\vec{k}) + \text{Const}$$

thus  $a^\dagger(\vec{k})$  creates a boson with energy  $\omega_{\vec{k}}$  and momentum  $\vec{k}$ , and

$$\omega_{\vec{k}}^2 - |\vec{k}|^2 = \omega_0^2$$

where  $\omega_0$  can be regarded as the rest mass of the boson (in natural units  $\hbar = c = 1$ ).

Ground state of quantum field is commonly denoted as  $|0\rangle$ , also called vacuum state. Thus

$$a(\vec{k})|0\rangle \text{ for all } \vec{k}$$

We also assume that

$$p^\mu |0\rangle = 0$$

thus  $|0\rangle$  is invariant under spacetime translations.

What is the spin of the Klein-Gordon particle?

Solution: Consider a single Klein-Gordon particle with zero momentum,

$$a^\dagger(\vec{0})|0\rangle = \int d^3x [\omega_0 \varphi(t, \vec{x}) - i \frac{\partial}{\partial t} \varphi(t, \vec{x})] e^{-i\omega_0 t} |0\rangle$$

Rotate the field using the Lorentz transformation

$$\varphi(t, \vec{x}) \rightarrow \varphi(t, \vec{y}), \quad \vec{y} = R^{-1} \vec{x}$$

thus

$$a^\dagger(\vec{0})|0\rangle \rightarrow \int \underbrace{d^3(R\vec{y})}_{=d^3\vec{y}} [\omega_0 \varphi(t, \vec{y}) - i \frac{\partial}{\partial t} \varphi(t, \vec{y})] e^{-i\omega_0 t} |0\rangle = a^\dagger(\vec{0})|0\rangle \implies 0$$

We conclude that Klein-Gordon bosons are **spinless** bosons.

Nonlinear scalar field theory: single-particle states and scattering amplitudes

First study KG (which is linear scalar field theory)

$$|\vec{k}\rangle = a^\dagger(\vec{k})|0\rangle, \quad [a(\vec{k}), a^\dagger(\vec{l})] = (2\pi)^3 2\omega_{\vec{k}} \delta(\vec{k} - \vec{l})$$

while

$$\langle \vec{k} | \vec{l} \rangle = \langle 0 | a(\vec{k}) a^\dagger(\vec{l}) | 0 \rangle = \langle 0 | [a(\vec{k}) a^\dagger(\vec{l}) - a^\dagger(\vec{l}) a(\vec{k})] | 0 \rangle = (2\pi)^3 2\omega_{\vec{k}} \delta(\vec{k} - \vec{l}) \langle 0 | 0 \rangle$$

Since  $\langle 0 | 0 \rangle = 1$ , then

$$\langle \vec{k} | \vec{l} \rangle = (2\pi)^3 2\omega_{\vec{k}} \delta(\vec{k} - \vec{l})$$

which is a Lorentz invariant normalization.

Consider

$$\langle \vec{k} | \varphi(x) | 0 \rangle = \langle 0 | a(\vec{k}) \int d\vec{l} (a(\vec{l}) e^{il \cdot x} + a^\dagger(\vec{l}) e^{-il \cdot x}) | 0 \rangle = e^{-ik \cdot x}$$

In particular,

$$\langle \vec{k} | \varphi(0) | 0 \rangle = 1$$

Also,

$$\tilde{\varphi}(-k)|0\rangle = 2\pi\delta(k^2 + \omega_0^2)(a(-\vec{k})\theta(-k^0) + a^\dagger(\vec{k})\theta(k^0))|0\rangle = 2\pi\delta(k^2 + \omega_0^2)\theta(k^0)|\vec{k}\rangle$$

thus

$$\varphi(x)|0\rangle = \int d\vec{k} e^{-ik \cdot x} |\vec{k}\rangle$$

is a single-particle state.

If we switch to nonlinear scalar field theory, for example

$$(-\partial^2 + \omega_0^2)\varphi(x) = c_0 + c_1\varphi + \underbrace{c_2\varphi^2 + c_3\varphi^3 + c_4\varphi^4 + \dots}_{\text{nonlinear terms}}$$

Many of our prior assumptions should be abandoned.

But in many situations, we may still assume

1.  $H, \vec{p}, \quad p^\mu = (H, \vec{p})$
2. There is unique ground state  $|0\rangle$ , also called vacuum state, with  $p^\mu|0\rangle = 0, \langle 0|0\rangle = 1$ .
3. There are still one-particle states which form a Lorentz invariant set in the  $(k^0, \vec{k})$  space, i.e.

$$\begin{cases} k^{0^2} - |\vec{k}|^2 = m^2 \\ k^0 \geq m \end{cases}$$

where  $m$  is the rest mass of a single particle. But now  $m \neq \omega_0$  in general. And

$$\langle \vec{k} | \vec{l} \rangle = (2\pi)^3 2\omega_{\vec{k}} \delta(\vec{k} - \vec{l}), \quad \omega_{\vec{k}} = \sqrt{m^2 + |\vec{k}|^2}$$

4. There are 2-particle states, 3-particle states,  $\dots$  and perhaps more exotic states such as bound states with energy  $P^0$  and momentum  $\vec{P}$  satisfying

$$\begin{cases} p^{0^2} - |\vec{p}|^2 \geq \Delta^2, \\ p^0 \geq \Delta \end{cases}, \quad m < \Delta \leq 2m$$

$\Delta = 2m$  if there is no bound state, and  $\Delta < 2m$  if there are two-particle bound states.

Question: How to construct single-particle state using  $\varphi(x)$  and  $|0\rangle$ ?

Now  $\varphi(x)|0\rangle$  is usually no longer a pure single-particle state.

For harmonic oscillator,

$$H = -\frac{1}{2} \frac{\partial^2}{\partial \varphi^2} + \frac{\omega_0^2}{2} \varphi^2$$

while the ground state  $\psi_0(\varphi) \propto e^{-\omega_0 \frac{\varphi^2}{2}}$  and the first excited state  $\psi_1(\varphi) \propto \varphi|0\rangle$ .

But for anharmonic oscillator,

$$H = -\frac{1}{2} \frac{\partial^2}{\partial \varphi^2} + \frac{\omega_0^2}{2} \varphi^2 + (H_1\varphi + H_2\varphi^2 + H_3\varphi^3 + \dots)$$

$\psi_1$  is no longer proportional to  $\varphi\psi_0$ .

Let

$$\tilde{\varphi}(k) = \int d^4x \varphi(x) e^{-ik \cdot x}$$

then using 4 Heisenberg equations we have

$$[p^\mu, \tilde{\varphi}(k)] = -k^\mu \tilde{\varphi}(k)$$

So if  $|\psi\rangle$  is an energy-momentum eigenstate with 4-momentum  $p^\mu$ , then  $\tilde{\varphi}(k)|\psi\rangle$  is either zero or another energy-momentum eigenstate with 4-momentum  $p^\mu - k^\mu$ , i.e.

$$\tilde{\varphi}(-k)|0\rangle = \int d^4x e^{ik \cdot x} \varphi(x) |0\rangle$$

is either zero or another energy-momentum eigenstate with 4-momentum  $k^\mu$ .

If

$$k^0 < 0 \text{ or } \begin{cases} k^0 > 0 \\ k^{0^2} - |\vec{k}|^2 < m^2 \text{ or } m^2 < k^{0^2} - |\vec{k}|^2 < \Delta^2 \end{cases}$$

there is no state with 4-momentum  $k^\mu$ , and so  $\tilde{\varphi}(-k)|0\rangle = 0$ .

This suggests that when  $k^\mu$  is near the "mass shell", namely when

$$\begin{cases} k^0 > 0 \\ k^{0^2} - |\vec{k}|^2 \approx m^2 \end{cases}$$

then

$$\tilde{\varphi}(-k)|0\rangle \propto 2\pi\delta(k^2 + m^2)|\vec{k}\rangle$$

we can choose amplitude of  $\varphi(x)$  such that

$$\begin{cases} \tilde{\varphi}(-k)|0\rangle = 2\pi\delta(k^2 + m^2)|\vec{k}\rangle \\ \tilde{\varphi}(k)|0\rangle = 0 \end{cases} \implies \langle 0|\tilde{\varphi}(-k) = 0, \quad \text{if } k^0 > 0 \text{ and } k^{0^2} - |\vec{k}|^2 < \Delta^2$$

where  $|\vec{k}\rangle$  is normalized as

$$\langle \vec{k}|\vec{l}\rangle = (2\pi)^3 2\omega_{\vec{k}} \delta(\vec{k} - \vec{l})$$

which is "a-shell" scheme for the amplitude of  $\varphi(x)$ .

Therefore, when  $l^0 = \omega_{\vec{l}}$ ,

$$\langle \vec{l}|\tilde{\varphi}(-k)|0\rangle = \begin{cases} (2\pi)\delta(k^2 + m^2) \langle \vec{l}|\vec{k}\rangle = (2\pi)^4 \delta(k - l), & \text{if } k^0 > 0 \text{ and } k^{0^2} - |\vec{k}|^2 < \Delta^2 \\ 0 = (2\pi)^4 \delta(k - l), & \text{otherwise} \end{cases}$$

thus

$$\begin{aligned} \langle \vec{l}|\tilde{\varphi}(-k)|0\rangle &= (2\pi)^4 \delta(k - l), \quad \text{for any } k \\ \langle \vec{l}|\varphi(x)|0\rangle &= \langle \vec{l}|\int \frac{d^4 k}{(2\pi)^4} \tilde{\varphi}(k) e^{ik \cdot x} |0\rangle = \int \frac{d^4 k}{(2\pi)^4} (2\pi)^4 \delta(-k - l) e^{ik \cdot x} = e^{-il \cdot x} \end{aligned}$$

In particular,

$$\langle \vec{l}|\varphi(0)|0\rangle = 1$$

We consider two functions  $\psi_1(x)$  and  $\psi_2(x)$ , let

$$\tilde{\psi}_i(k) = \int d^4 x \psi_i(x) e^{-ik \cdot x}$$

If  $\tilde{\psi}_1(l) = \tilde{\psi}_2(l)$  whenever  $l^2 + m^2 = 0$ , we say that  $\psi_1(x), \psi_2(x)$  are **on-shell equivalent**.

Moreover, if  $\tilde{\psi}_1(l), \tilde{\psi}_2(l)$  are narrow peaks, then

$$\int d^4 x \psi_1(x) \varphi(x)|0\rangle = \int d^4 x \psi_2(x) \varphi(x)|0\rangle$$

which both represent the same single-particle state.

It's easy to prove it,

$$\int d^4 x \psi_i(x) \varphi(x)|0\rangle = \int \frac{d^4 l}{(2\pi)^4} \tilde{\psi}_i(l) \tilde{\varphi}(-l)|0\rangle$$

Since  $\tilde{\psi}_1(l), \tilde{\psi}_2(l)$  are narrow peaks, the both expressions become the same.

Consider

$$F(x) = \int \frac{d^3 l}{(2\pi)^3} f(\vec{l}) e^{-i\omega_{\vec{l}} t + i\vec{l} \cdot \vec{x}}$$

which satisfies

$$\left( \frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 \right) F(x) = 0$$

Next add  $g(t - T)$  which produces

$$F(x)g(t - T)$$

which is on-shell equivalent for different  $T$ 's. To prove this, we consider its Fourier transform,

$$\int F(x)g(t - T)e^{-il \cdot x} d^4 x = f(\vec{l})\tilde{g}(l^0 - \omega_{\vec{l}})e^{i(l^0 - \omega_{\vec{l}})T}, \quad \tilde{g}(\nu) = \int_{-\infty}^{+\infty} g(t)e^{i\nu t} dt$$

Assume  $\tilde{g}(\nu)$  has narrow peaks at  $\nu = 0$ , then the expression above is zero except

$$l^0 = \omega_{\vec{l}} \implies l^2 + m^2 = 0 \implies \text{on shell equivalent}$$

We define two states:

"in" state:  $n$  particles coming-in in the distant past;

"out" state:  $n'$  particles coming-out in the distant future.

The "in" state creation operator could be described as

$$C_\alpha = \int d^4x_\alpha u_\alpha(x_\alpha) \varphi(x_\alpha)$$

with

$$u_\alpha(x_\alpha) = g(t_\alpha - T_-) \int \frac{d^3p}{(2\pi)^3} f_\alpha(\vec{p}) e^{-i\omega_{\vec{p}} t_\alpha + i\vec{p} \cdot \vec{x}_\alpha}, \quad f_\alpha(\vec{p}) : \text{narrow peak centered around } \vec{k}_\alpha, \quad T_- \rightarrow -\infty$$

we have the "in" state

$$|in\rangle = C_1 \cdots C_n |0\rangle$$

while for the "out" state,

$$D_\alpha = \int d^4x'_\alpha u'_\alpha(x'_\alpha) \varphi(x'_\alpha)$$

with

$$u'_\alpha(x'_\alpha) = g(t'_\alpha - T_+) \int \frac{d^3p}{(2\pi)^3} f'_\alpha(\vec{p}) e^{-i\omega_{\vec{p}} t'_\alpha + i\vec{p} \cdot \vec{x}'_\alpha}, \quad f'_\alpha(\vec{p}) : \text{narrow peak centered around } \vec{k}'_\alpha, \quad T_+ \rightarrow +\infty$$

we have the "out" state

$$|out\rangle = D_1 \cdots D_{n'} |0\rangle$$

The transition amplitude

$$\langle out|in\rangle = \langle 0|D_{n'}^\dagger D_{n'-1}^\dagger \cdots D_2^\dagger D_1^\dagger C_1 C_2 \cdots C_n |0\rangle$$

We introduce the **time-ordering symbol** : if  $\varphi(x)$  is scalar field then

$$T\varphi(x_1)\varphi(x_2) = \begin{cases} \varphi(x_1)\varphi(x_2), & x_1^0 > x_2^0 \\ \varphi(x_2)\varphi(x_1), & x_1^0 < x_2^0 \end{cases}$$

Since  $D_n$  and  $C_n$  commute with each other,

$$\langle out|in\rangle = \langle 0|T D_{n'}^\dagger D_{n'-1}^\dagger \cdots D_2^\dagger D_1^\dagger C_1 C_2 \cdots C_n |0\rangle$$

Define

$$\bar{C}_\alpha = \int d^4x_\alpha g(t_\alpha - T_+) \int \frac{d^3p}{(2\pi)^3} f_\alpha(\vec{p}) e^{-i\omega_{\vec{p}} t_\alpha + i\vec{p} \cdot \vec{x}_\alpha} \varphi(x_\alpha)$$

We have

$$C_\alpha |0\rangle = \bar{C}_\alpha |0\rangle$$

Similarly,

$$\bar{D}_\alpha = \int d^4x'_\alpha g(t'_\alpha - T_-) \int \frac{d^3p}{(2\pi)^3} f'_\alpha(\vec{p}) e^{-i\omega_{\vec{p}} t'_\alpha + i\vec{p} \cdot \vec{x}'_\alpha} \varphi(x'_\alpha)$$

with

$$D_\alpha |0\rangle = \bar{D}_\alpha |0\rangle$$

We consider replacing one  $C_\alpha$  with  $\bar{C}_\alpha$ , i.e.

$$\langle 0|T D_{n'}^\dagger D_{n'-1}^\dagger \cdots D_2^\dagger D_1^\dagger C_1 C_2 \cdots C_{\alpha-1} \bar{C}_\alpha C_{\alpha+1} \cdots C_n |0\rangle = \langle 0|\bar{C}_\alpha T D_{n'}^\dagger D_{n'-1}^\dagger \cdots D_2^\dagger D_1^\dagger C_1 C_2 \cdots C_{\alpha-1} C_{\alpha+1} \cdots C_n |0\rangle = 0$$

since



$$\bar{C}_\alpha = \int d^4 l \tilde{u}_\alpha(l) \tilde{\varphi}(-l), \quad \bar{C}_\alpha^\dagger = \int d^4 l \tilde{u}_\alpha^*(l) \tilde{\varphi}(l) \implies \bar{C}_\alpha^\dagger |0\rangle = 0 \implies \langle 0 | \bar{C}_\alpha = 0$$

Similarly,

$$\langle 0 | T D_{n'}^\dagger D_{n'-1}^\dagger \cdots D_{\alpha+1}^\dagger \bar{D}_\alpha^\dagger D_{\alpha-1}^\dagger \cdots D_2^\dagger D_1^\dagger C_1 C_2 \cdots C_n | 0 \rangle = 0$$

Therefore the transition amplitude could be written as

$$\langle out | in \rangle = \langle 0 | T (D_{n'}^\dagger - \bar{D}_{n'}^\dagger) (D_{n'-1}^\dagger - \bar{D}_{n'-1}^\dagger) \cdots (D_1^\dagger - \bar{D}_1^\dagger) (C_1 - \bar{C}_1) \cdots (C_n - \bar{C}_n) | 0 \rangle$$

While the difference between  $C_\alpha, \bar{C}_\alpha$ ,

$$C_\alpha - \bar{C}_\alpha = \int d^4 x [g(t - T_-) - g(t - T_+)] \int \frac{d^3 p}{(2\pi)^3} f_\alpha(\vec{p}) e^{-i\omega_{\vec{p}} t + i\vec{p} \cdot \vec{x}} = i \int \widetilde{dp} f_\alpha(\vec{p}) \int d^4 x e^{ipx} (m^2 - \partial^2) \varphi(x)$$

(motherfucker calculation)

Similarly,

$$\begin{aligned} \bar{D}_\alpha - D_\alpha &= i \int \widetilde{dp} f'_\alpha(\vec{p}) \int d^4 x' e^{ipx'} (m^2 - \partial^2) \varphi(x') \\ D_\alpha^\dagger - \bar{D}_\alpha^\dagger &= i \int \widetilde{dp} f_\alpha'^*(\vec{p}) \int d^4 x' e^{-ipx'} (m^2 - \partial^2) \varphi(x') \end{aligned}$$

Therefore

$$\begin{aligned} \langle out | in \rangle &= i^{n+n'} \int \widetilde{dp}_1 f_1(\vec{p}_1) \cdots \int \widetilde{dp}_n f_n(\vec{p}_n) \int \widetilde{dp}_1' f_1'^*(\vec{p}_1') \cdots \int \widetilde{dp}_n' f_n'^*(\vec{p}_n') \int d^4 x_1 e^{ip_1 x_1} (m^2 - \partial_1^2) \cdots \int d^4 x_n e^{ip_n x_n} (m^2 - \partial_n^2) \\ &\quad \int d^4 x_1' e^{ip_1' x_1'} (m^2 - \partial_1'^2) \cdots \int d^4 x_n' e^{ip_n' x_n'} (m^2 - \partial_n'^2) \langle 0 | T \varphi(x_1) \cdots \varphi(x_n) \varphi(x_1') \cdots \varphi(x_n') | 0 \rangle \end{aligned}$$

Consider the general normalization

$$\tilde{\varphi}(-k) | 0 \rangle = 2\pi \sqrt{\zeta} \delta(k^2 + m^2) \theta(k^0) |\vec{k}\rangle + \eta (2\pi)^4 \delta(k) | 0 \rangle, \quad -k^2 < \Delta^2$$

By using

$$\langle \vec{k} | \vec{k}' \rangle = (2\pi)^3 2\omega_{\vec{k}} \delta(\vec{k} - \vec{k}')$$

we obtain

$$\langle \vec{k} | \varphi(0) | 0 \rangle = \sqrt{\zeta}$$

Since

$$C_\alpha | 0 \rangle = \int \frac{d^4 l}{(2\pi)^4} \tilde{u}_\alpha(l) \tilde{\varphi}(-l) | 0 \rangle = \sqrt{\zeta} \int \widetilde{dp} f_\alpha(\vec{p}) |\vec{p}\rangle$$

thus

$$|C_\alpha | 0 \rangle|^2 = \zeta \int \widetilde{dp} |f_\alpha(\vec{p})|^2$$

consequently

$$\langle in | in \rangle = |C_1 \cdots C_n | 0 \rangle|^2 = \zeta^n \prod_{\alpha=1}^n \widetilde{dp}_\alpha |f_\alpha(\vec{p}_\alpha)|^2$$

Similarly,

$$\langle out | out \rangle = \zeta^n \prod_{\alpha=1}^{n'} \widetilde{dp}_\alpha' |f_\alpha'(\vec{p}_\alpha')|^2$$

The Born's rule requires that

$$P_{|in\rangle \rightarrow |out\rangle} = \frac{|\langle out | in \rangle|^2}{\langle in | in \rangle \langle out | out \rangle}$$

Consider the plane wave form

$$f_\alpha(\vec{p}) = \frac{(2\pi)^3}{\sqrt{\zeta}} 2\omega_{\vec{p}} \delta(\vec{p} - \vec{k}_\alpha), \quad f'_\alpha(\vec{p}) = \frac{(2\pi)^3}{\sqrt{\zeta}} 2\omega_{\vec{p}} \delta(\vec{p} - \vec{k}'_\alpha)$$

thus

$$C_\alpha|0\rangle = |\vec{k}_\alpha\rangle, \quad D_\alpha|0\rangle = |\vec{k}'_\alpha\rangle$$

then

$$|in\rangle = C_1 \cdots C_n |0\rangle = |\vec{k}_1, \cdots, \vec{k}_n\rangle_{in}, \quad |out\rangle = D_1 \cdots D_n |0\rangle = |\vec{k}'_1, \cdots, \vec{k}'_n\rangle_{out}$$

## Calculating ground state expectation values using path integral

Let

$$|0\rangle : \text{ground state} \quad H : \text{Hamiltonian} \quad Q(t) : \text{generalized coordinate}, \quad Q(t)^\dagger = Q(t), \quad Q(t) = e^{iHt} Q(0) e^{-iHt}$$

How to calculate  $\langle 0 | Q(t_1) \cdots Q(t_n) | 0 \rangle$ ?

If  $\text{Im}(\Delta t) \rightarrow -\infty$ , then

$$e^{-iH\Delta t} = \sum_n e^{-iE_n\Delta t} |\psi_n\rangle \langle \psi_n|$$

Let  $\Delta t = a + ib$ ,  $b = -i\infty$ , then

$$-iE_n\Delta t = -iE_n(a + ib) = +E_nb - iE_na$$

thus

$$e^{-iH\Delta t} \rightarrow \text{projection onto ground state up to some coefficient}$$

Let  $|1\rangle, |2\rangle$  be two generic states having nonzero overlap with  $|0\rangle$ , then if  $\text{Im} \Delta t \rightarrow -\infty$ ,

$$\begin{cases} e^{-iH\Delta t} |2\rangle \rightarrow c_2 e^{-iE_0\Delta t} |0\rangle \\ \langle 1 | e^{-iH\Delta t} \rightarrow c_1^* e^{-iE_0\Delta t} \langle 0 | \end{cases}$$

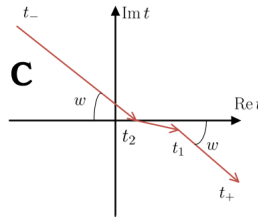
Therefore

$$\langle 0 | Q(t_1) Q(t_2) | 0 \rangle = \frac{\langle 1 | e^{-iHt_+} Q(t_1) Q(t_2) e^{+iHt_-} | 2 \rangle}{\langle 1 | e^{-iHt_+} e^{+iHt_-} | 2 \rangle} = \frac{\langle 1 | e^{-iHt_+} e^{iHt_1} Q(0) e^{-iHt_1} e^{iHt_2} Q(0) e^{-iHt_2} e^{+iHt_-} | 2 \rangle}{\langle 1 | e^{-iHt_+} e^{+iHt_-} | 2 \rangle}, \quad \text{Im } t_+ \rightarrow -\infty, \text{Im } t_- \rightarrow +\infty$$

consequently

$$\langle 0 | Q(t_1) Q(t_2) | 0 \rangle = \frac{\langle 1 | e^{-iH(t_+-t_1)} Q(0) e^{-iH(t_1-t_2)} Q(0) e^{-iH(t_2-t_-)} | 2 \rangle}{\langle 1 | e^{-iH(t_+-t_-)} | 2 \rangle}$$

This procedure could be portrayed as



where  $w$  is the **wick angle**.

Consider the interval

$$\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$$

Introducing  $\tau = it$  we have

$$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 + \Delta \tau^2$$

which is Euclidean distance and has  $SO(4)$  symmetry.

We need to calculate

$$\langle q'' | e^{-iHT} | q' \rangle$$

Let  $\delta t = \frac{T}{N+1}$ , then  $e^{-iHT} = (e^{-iH\delta t})^{N+1}$ .

If

$$H = K + V, \quad [K, V] \neq 0$$

then we have the **Suzuki-Trotter decomposition**

$$e^{-iH\delta t} = e^{-iK\delta t} e^{-iV\delta t} + O((\delta t)^2)$$

If  $V$  is diagonal in  $Q(0)$  representation,

$$\langle q_{j+1} | e^{-iH\delta t} | q_j \rangle = \langle q_{j+1} | e^{-iK\delta t} e^{-iV\delta t} | q_j \rangle + O(\delta t^2) = \langle q_{j+1} | e^{-iK\delta t} | q_j \rangle e^{-iV(q_j)\delta t} + O(\delta t^2)$$

For example,

$$H = \frac{P^2}{2Z} + V(Q)$$

If we define

$$|q\rangle = e^{-i(q-q_0)P} |q_0\rangle$$

then

$$\langle q_{j+1} | e^{-iH\delta t} | q_j \rangle = \langle q_{j+1} | e^{-\frac{i}{2Z} P^2 \delta t} | q_j \rangle e^{-iV(q_j)\delta t} + O(\delta t^2) = \frac{\sqrt{Z}}{\sqrt{2\pi i \delta t}} e^{\frac{iZ\delta t}{2} \left( \frac{q_{j+1}-q_j}{\delta t} \right)^2 - iV(q_j)\delta t} + O(\delta t^2)$$

Since the Lagrangian

$$L = \frac{Z}{2} \dot{q}^2 - V(q)$$

then

$$\langle q'' | e^{-iHT} | q' \rangle = \int_{q(0)=q', q(T)=q''} \mathcal{D}q e^{i \int_0^T L dt} = \lim_{N \rightarrow \infty} \left( \frac{\sqrt{Z}}{\sqrt{2\pi i \delta t}} \right)^{N+1} \int_{-\infty}^{+\infty} dq_1 \cdots dq_N e^{i \delta t \left[ \frac{Z}{2} \left( \frac{q''-q_N}{\delta t} \right)^2 - V(q_N) + \frac{Z}{2} \left( \frac{q_N-q_{N-1}}{\delta t} \right)^2 - V(q_{N-1}) + \cdots + \frac{Z}{2} \left( \frac{q_1-q'}{\delta t} \right)^2 - V(q_1) \right]}$$

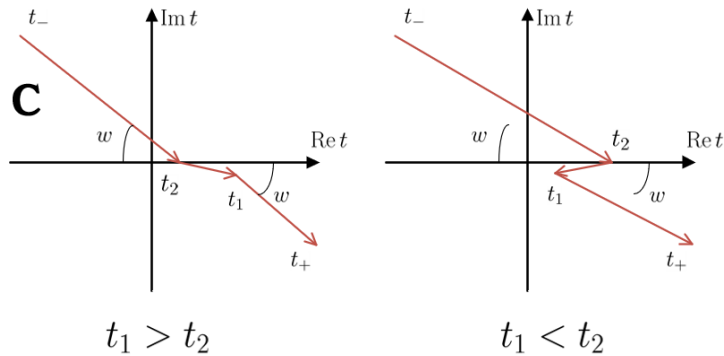
thus

$$\langle 0 | Q(t_1) Q(t_2) | 0 \rangle = \frac{\int \mathcal{D}q q(t_1) q(t_2) e^{i \int_C L dt}}{\int \mathcal{D}q e^{i \int_C L dt}}$$

where  $C$  is a path:

$$t_- \rightarrow t_2 \rightarrow t_1 \rightarrow t_+$$

with  $\text{Im } t$  decreasing.



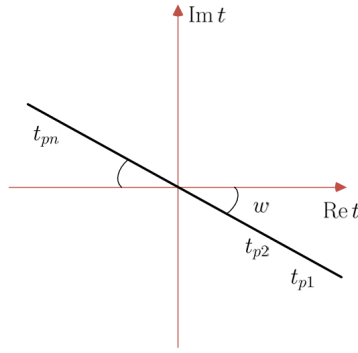
$\langle 0 | TQ(t_1) \cdots Q(t_n) | 0 \rangle$  is associated with a straight-line path without U-turns if  $w \rightarrow 0^+$ . Thus

$$\langle 0 | TQ(t_1) \cdots Q(t_n) | 0 \rangle = \frac{\int \mathcal{D}q q(t_1) \cdots q(t_n) e^{iS}}{\int \mathcal{D}q e^{iS}}, \quad S = \int_C L dt$$

When  $t$  is purely imaginary,  $iS$  is often real.

In path integral, it is beneficial to let time flow from  $(+i)$ future to  $(-i)$ future, rather than from past to future.

For example, if  $t_i = \text{real} \cdot e^{-iw}$ ,



we have

$$\text{Re } t_{p1} > \text{Re } t_{p2} > \dots > \text{Re } t_{pn} \Leftrightarrow \text{Im } t_{p1} < \text{Im } t_{p2} < \dots < \text{Im } t_{pn}$$

We consider the **Wick rotation** as follows:

- First step:  $w = \frac{\pi}{2}$
- Then: rotate things on the complex time plane freely, so long as  $\text{Im } \delta t < 0$  along  $C$ .

In this way we have

$$\begin{array}{ccc} \text{relativistic QFT} & \xrightarrow{90^\circ \text{ Wick rotation}} & \text{Classical statistical mechanics} \\ \text{in } d \text{ spacetime dimensions} & & \text{for a fluctuating classical field in } d\text{-dimensional Euclidean space with } SO(d) \text{ symmetry} \end{array}$$

Example: K-G in 1 dimensional spacetime

$$\begin{cases} -\ddot{\varphi} = \omega_0^2 \varphi \\ [\varphi, \dot{\varphi}] = i \end{cases} \xrightarrow{\varphi \rightarrow \Pi_\varphi} \begin{cases} H = \frac{\Pi_\varphi^2}{2} + \frac{\omega_0^2}{2} \varphi^2 \\ [\varphi, \Pi_\varphi] = i \end{cases}$$

thus

$$L = \frac{\dot{\varphi}^2}{2} - \frac{\omega_0^2}{2} \varphi^2, \quad iS = \int \left( \frac{\dot{\varphi}^2}{2} - \frac{\omega_0^2}{2} \varphi^2 \right) d\tau, \quad \tau = it$$

Since  $\tau = it, t = -i\tau$ , we can introduce

$$\phi(\tau) = \varphi(t) = \varphi(-i\tau)$$

then

$$iS = - \int d\tau \left[ \frac{1}{2} \left( \frac{d\phi}{d\tau} \right)^2 + \frac{\omega_0^2}{2} \phi^2 \right] = -S_E$$

Then we define

$$\Delta_E(\tau_1 - \tau_2) = \langle 0 | T \varphi(-i\tau_1) \varphi(-i\tau_2) | 0 \rangle = \langle 0 | \phi(\tau_1) \phi(\tau_2) | 0 \rangle = \frac{\int D\phi \phi(\tau_1) \phi(\tau_2) e^{-S_E}}{\int D\phi e^{-S_E}}$$

If we use the Fourier transform of  $\phi(\tau)$ ,

$$\phi(\tau) = \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi} e^{-i\nu\tau} \tilde{\phi}(\nu), \quad \tilde{\phi}(\nu) = \int_{-\infty}^{+\infty} d\tau e^{i\nu\tau} \phi(\tau)$$

then

$$S_E = \int_0^\infty \frac{d\nu}{2\pi} (\nu^2 + \omega_0^2) |\tilde{\phi}(\nu)|^2$$

Or to discretize it,

$$S_E = \sum_{\nu>0} \frac{\Delta\nu}{2\pi} (\nu^2 + \omega_0^2) |\tilde{\phi}(\nu)|^2$$

thus

$$e^{-S_E} = e^{-\sum_{\nu>0} \frac{\Delta\nu}{2\pi} (\nu^2 + \omega_0^2) |\tilde{\phi}(\nu)|^2} = \prod_{\nu>0} e^{-\frac{\Delta\nu}{2\pi} (\nu^2 + \omega_0^2) |\tilde{\phi}(\nu)|^2}$$

then

$$\langle 0 | |\tilde{\phi}(\nu)|^2 | 0 \rangle = \frac{\int d\tilde{\phi}^* d\tilde{\phi} \tilde{\phi}^* \tilde{\phi} e^{-\frac{\Delta\nu}{2\pi} (\nu^2 + \omega_0^2) \tilde{\phi}^* \tilde{\phi}}}{\int d\tilde{\phi}^* d\tilde{\phi} e^{-\frac{\Delta\nu}{2\pi} (\nu^2 + \omega_0^2) \tilde{\phi}^* \tilde{\phi}}} = \frac{2\pi}{\Delta\nu (\nu^2 + \omega_0^2)}$$

Back to continuum  $\tau$ :

$$\langle 0 | \tilde{\phi}(\nu) \tilde{\phi}(\nu') | 0 \rangle = \frac{2\pi}{\nu^2 + \omega_0^2} \delta(\nu + \nu')$$

$$\langle 0 | \phi(\tau) \phi(\tau') | 0 \rangle = \langle 0 | \int \frac{d\nu}{2\pi} \tilde{\phi}(\nu) e^{-i\nu\tau} \int \frac{d\nu'}{2\pi} \tilde{\phi}(\nu') e^{-i\nu'\tau'} | 0 \rangle = \int \frac{d\nu}{2\pi} \frac{d\nu'}{2\pi} \frac{2\pi}{\nu^2 + \omega_0^2} \delta(\nu + \nu') e^{-i\nu\tau - i\nu'\tau'} = \frac{1}{2\omega_0} e^{-\omega_0|\tau - \tau'|}$$

Since

$$\langle 0 | \phi(\tau) \phi(\tau') | 0 \rangle = \langle 0 | T \phi(-i\tau) \phi(-i\tau') | 0 \rangle$$

If  $\text{Im } t < 0$  which means  $t$  is "later" than 0, then

$$\langle 0 | \varphi(t) \varphi(0) | 0 \rangle = \frac{1}{2\omega_0} e^{-\omega_0\tau} = \frac{1}{2\omega_0} e^{-i\omega_0 t}$$

If  $t$  is real,

$$\langle 0 | T \varphi(t) \varphi(0) | 0 \rangle = \frac{1}{2\omega_0} e^{-i\omega_0|t|}$$

which satisfies for KG in 1-dimensional spacetime.

And

$$\langle 0 | T \varphi(t_1) \varphi(t_2) \varphi(t_3) | 0 \rangle = \frac{\int D\varphi \varphi(t_1) \varphi(t_2) \varphi(t_3) e^{iS}}{\int D\varphi e^{iS}} = 0$$

since when  $\varphi(t) \rightarrow \varphi(t)$ ,  $S$  is invariant.

And

$$\langle 0 | T \varphi(t_1) \varphi(t_2) \varphi(t_3) \varphi(t_4) | 0 \rangle = \Delta(t_1 - t_2) \Delta(t_3 - t_4) + \Delta(t_1 - t_3) \Delta(t_2 - t_4) + \Delta(t_1 - t_4) \Delta(t_2 - t_3)$$

Moreover,

$$\langle 0 | T \varphi(t_1) \cdots \varphi(t_{2n}) | 0 \rangle = \sum_{\text{pairings}} \Delta(t_{i_1} - t_{i_2}) \cdots \Delta(t_{i_{2n-1}} - t_{i_{2n}})$$

which has sum of  $(2n-1)(2n-3) \cdots 1 = (2n-1)!!$  terms.

We can define the FT of  $\Delta(t)$ ,

$$\hat{\Delta}(\omega) = i \int_{-\infty}^{+\infty} dt e^{i\omega t} \Delta(t) = \int_{-\infty}^{+\infty} d\tau e^{i\nu\tau} \langle 0 | \phi(\tau) \phi(0) | 0 \rangle = \frac{1}{\nu^2 + \omega_0^2} = \frac{1}{-\omega^2 + \omega_0^2}$$

which has introduced

$$\tau = it, \quad \nu = -i\omega$$

which maintains  $\omega t$  real.

$\hat{\Delta}(\omega)$  has poles at  $\omega = \pm\omega_0$ . To get away from the poles we can make a shift,

$$\hat{\Delta}(\omega) = \frac{1}{-\omega^2 + (\omega_0 - i\varepsilon)^2} = \frac{1}{-\omega^2 + \omega_0^2 - 2i\varepsilon}$$

In relativistic QFT,  $k^0, x^0$  rotate in opposite directions:

$$-k^0 x^0 + \vec{k} \cdot \vec{x} = (ik^0)(ix^0) + \vec{k} \cdot \vec{x} = k_1 x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4$$

with

$$x_4 = x^4 = \tau = ix_0 = it, \quad k_4 = k^4 = ik^0$$

For KG in 4-dimensional spacetime:

$$H = \int d^3x \left( \frac{1}{2} \Pi_\varphi^2 + \frac{1}{2} (\nabla \varphi)^2 + \frac{1}{2} m^2 \varphi^2 \right), \quad \Pi_\varphi = \dot{\varphi}, \quad [\varphi(t, \vec{x}), \Pi_\varphi(t, \vec{y})] = i\delta(\vec{x} - \vec{y})$$

Discretize:

$$\vec{x} = (\lambda n_1, \lambda n_2, \lambda n_3) = \lambda \vec{n}, \quad \varphi_{\vec{n}} = \varphi(\lambda \vec{n}), \quad \Pi_{\vec{n}} = \Pi_\varphi(\lambda \vec{n})$$

then

$$H_{dis} = \lambda^3 \sum_{\vec{n}} \left[ \frac{1}{2} \Pi_{\vec{n}}^2 + \frac{1}{2} \sum_{i=1}^3 \left( \frac{\varphi_{\vec{n}+\vec{e}_i} - \varphi_{\vec{n}-\vec{e}_i}}{2\lambda} \right)^2 + \frac{1}{2} m^2 \varphi_{\vec{n}}^2 \right]$$

with

$$[\varphi_{\vec{n}}(t), \Pi_{\vec{n}'}(t)] = i \frac{1}{\lambda^3} \delta_{\vec{n}, \vec{n}'}$$

If we introduce

$$P_{\vec{n}} = \lambda^3 \Pi_{\vec{n}}$$

we obtain

$$[\varphi_{\vec{n}}, P_{\vec{n}'}] = i \delta_{\vec{n}, \vec{n}'}$$

thus the Hamiltonian becomes

$$H_{dis} = \left( \sum_{\vec{n}} \frac{P_{\vec{n}}^2}{2\lambda^3} \right) + V, \quad V = \lambda^3 \sum_{\vec{n}} \left[ \frac{1}{2} \sum_{i=1}^3 \left( \frac{\varphi_{\vec{n}+\vec{e}_i} - \varphi_{\vec{n}-\vec{e}_i}}{2\lambda} \right)^2 + \frac{m^2}{2} \varphi_{\vec{n}}^2 \right]$$

To generalize to QM with  $M$  degrees of freedom, we introduce coordinates

$$Q_1, \dots, Q_M, P_1, \dots, P_M, \quad [Q_j, Q_k] = 0, [P_j, P_k] = 0, [Q_j, P_k] = i\delta_{jk}$$

To calculate

$$X = \langle 0 | F_1(t_1) \dots F_N(t_N) | 0 \rangle$$

with

$$F_a(t) = F_a(Q_1(t), Q_2(t), \dots, Q_N(t)) = F_a(Q(t))$$

The similar procedure as before

$$X = \frac{\langle 1 | e^{-iH(t_+-t_1)} F_1(t=0) e^{-iH(t_1-t_2)} F_2(t=0) \dots e^{-iH(t_{N-1}-t_N)} F_N(t=0) e^{-iH(t_N-t_-)} | 2 \rangle}{\langle 1 | e^{-iH(t_+-t_-)} | 2 \rangle}$$

We introduce

$$Q_j(0) |q\rangle = q_j |q\rangle, \quad |q\rangle = |q_1, \dots, q_M\rangle, \quad \int dq = \int_{-\infty}^{+\infty} dq_1 \dots \int_{-\infty}^{+\infty} dq_M, \quad \int dq |q\rangle \langle q| = 1$$

and expanding  $F_a(t=0)$

$$F_a(t=0) = \int dq F_a(q) |q\rangle \langle q|$$

Again we have the function

$$\mathcal{U}(q, t, q', t') = \langle q | e^{-iH(t-t')} | q' \rangle = \left[ \frac{\sqrt{2}}{\sqrt{2\pi i(t-t')}} \right]^M e^{i(t-t') \left[ \frac{1}{2} \sum_{j=1}^M \left( \frac{q_j - q'_j}{t-t'} \right)^2 - V(q') \right]}$$

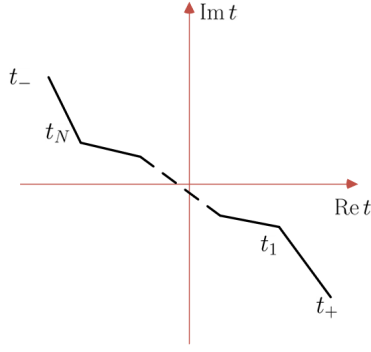
then we have

$$X = \frac{\int Dq F_1(q(t_1)) \dots F_N(q(t_N)) e^{i \int_C L dt}}{\int Dq e^{i \int_C L dt}}$$

where

$$L = \frac{Z}{2} \sum_{j=1}^M \dot{q}_j^2 - V(q), \quad Dq = \prod_{t \in C} dq_1(t) dq_2(t) \cdots dq_M(t)$$

with the path  $C$  as follows:



Using the Hamiltonian before, we have the discrete Lagrangian

$$L = \left( \sum_{\vec{n}} \frac{\lambda^3}{2} \dot{\varphi}_{\vec{n}}^2 \right) - V = \sum_{\vec{n}} \lambda^3 \left[ \frac{\dot{\varphi}_{\vec{n}}^2}{2} - \frac{m^2}{2} \varphi_{\vec{n}}^2 - \frac{1}{2} \sum_{i=1}^3 \left( \frac{\varphi_{\vec{n}+\vec{e}_i} - \varphi_{\vec{n}-\vec{e}_i}}{2\lambda} \right)^2 \right]$$

while in continuum

$$L_{KG} = \int d^3x \left( \frac{\dot{\varphi}^2}{2} - \frac{m^2}{2} \varphi^2 - \frac{1}{2} (\nabla \varphi)^2 \right) = \int d^3x \mathcal{L}_{KG}, \quad \mathcal{L}_{KG} = \frac{\dot{\varphi}^2}{2} - \frac{1}{2} (\nabla \varphi)^2 - \frac{m^2}{2} \varphi^2 = -\frac{1}{2} (\partial_\mu \varphi) (\partial^\mu \varphi) - \frac{m^2}{2} \varphi^2$$

And the action

$$S = \int dt d^3x \mathcal{L} = \int d^4x \mathcal{L}, \quad d^4x = dx^0 dx^1 dx^2 dx^3, \quad t = x^0$$

and  $e^{iS} = e^{-S_E}$  with

$$S_E = \int d\tau d^3x \left[ \frac{1}{2} \left( \frac{\partial \varphi}{\partial t} \right)^2 + \frac{1}{2} (\nabla \varphi)^2 + \frac{m^2}{2} \varphi^2 \right]$$

with

$$x_E = (x^1, x^2, x^3, ix^0), \quad \tau = it = ix^0 = x^4 = x_4$$

thus

$$S_E = \int d^4x_E \left[ \frac{1}{2} \sum_{i=1}^4 \left( \frac{\partial \phi}{\partial x_i} \right)^2 + \frac{m^2}{2} \phi^2 \right], \quad d^4x_E = dx^1 dx^2 dx^3 dx^4, \quad \phi(\vec{x}, \tau) = \varphi(-i\tau, \vec{x}) = \varphi(t, \vec{x})$$

Define the Fourier transform

$$\phi(x) = \int \frac{d^4k_E}{(2\pi)^4} \tilde{\phi}(k) e^{ik_E \cdot x_E}, \quad d^4k_E = dk^1 dk^2 dk^3 dk^4$$

with

$$k^4 = ik^0, x^4 = ix^0, \quad k_E \cdot x_E = k^1 x^1 + k^2 x^2 + k^3 x^3 + k^4 x^4 = -k^0 x^0 + k^1 x^1 + k^2 x^2 + k^3 x^3 = k \cdot x$$

thus we obtain

$$S_E = \int \frac{d^4k_E}{(2\pi)^4} \left( \frac{k^2}{2} + \frac{m^2}{2} \right) |\tilde{\phi}(k)|^2, \quad k^2 = k \cdot k = k_E \cdot k_E = k^{1^2} + k^{2^2} + k^{3^2} - k^{0^2}$$

therefore

$$\langle 0 | \tilde{\phi}(k) \tilde{\phi}(l) | 0 \rangle = \frac{(2\pi)^4 \delta(k_E + l_E)}{k^2 + m^2}$$

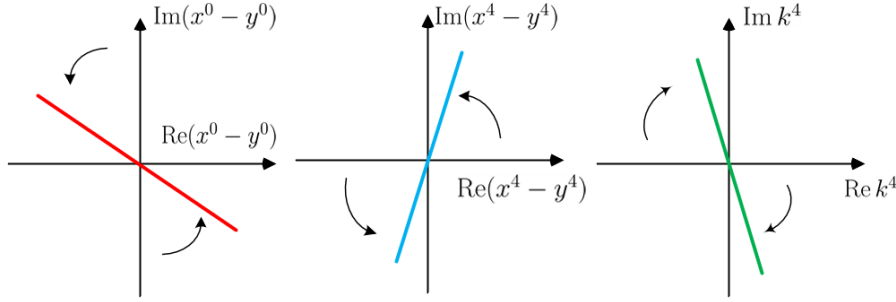
finally the propagator of KG

$$\langle 0 | T \varphi(x) \varphi(y) | 0 \rangle = \langle 0 | \phi(x) \phi(y) | 0 \rangle = \int \frac{d^4k_E}{(2\pi)^4} \frac{e^{ik_E \cdot (x_E - y_E)}}{k_E^2 + m^2} \approx \frac{m K_1(m|x_E - y_E|)}{|x_E - y_E|}$$

while it can also be expressed as

$$\langle 0 | T \varphi(x) \varphi(y) | 0 \rangle = \int \frac{d^3 k}{(2\pi)^3} e^{i\vec{k} \cdot (\vec{x} - \vec{y})} \int_{-\infty}^{+\infty} \frac{dk_4}{2\pi} \frac{e^{ik_4(x_4 - y_4)}}{k_4^2 + \omega_k^2}$$

We can do wick rotation as follows:



thus we can let

$$x^4 - y^4 = ie^{-i\varepsilon}(x^0 - y^0), \quad \varepsilon = 0^+$$

and

$$k^4 = ie^{i\varepsilon}k^0, \quad k^0 : +\infty \rightarrow -\infty$$

When  $x^0 - y^0$  is real,

$$\langle 0 | T \varphi(x) \varphi(y) | 0 \rangle = \int \frac{d^3 k}{(2\pi)^3} e^{i\vec{k} \cdot (\vec{x} - \vec{y})} \int_{k^0=+\infty}^{-\infty} \frac{ik^0}{2\pi} \frac{e^{-ik^0(x^0 - y^0)}}{-k^{02}e^{2i\varepsilon} + |\vec{k}|^2 + m^2}$$

After some trivial manipulation,

$$\langle 0 | T \varphi(x) \varphi(y) | 0 \rangle = \int \frac{-id^4 k}{(2\pi)^4} \frac{e^{ik \cdot (x - y)}}{k^2 + m^2 - i\varepsilon}$$

## Path integral for interacting field theory: perturbative calculations

$$L_{QFT} = \int d^3 x \mathcal{L}(\varphi, \partial \varphi, \dots)$$

$\mathcal{L}$ : invariant under proper orthochronous Lorentz transformation

Consider  $\varphi^3$  theory:

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{m^2}{2}\varphi^2 + \frac{g}{6}\varphi^3 + Y\varphi$$

$g, Y$ : infinitesimal constants

$\mathcal{L}$  can also be written as

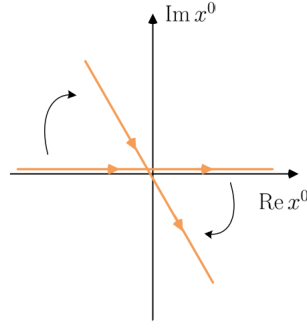
$$\mathcal{L} = -\mathcal{L}_E, \quad \mathcal{L}_E = \underbrace{\frac{1}{2} \sum_{i=1}^4 (\partial_i \varphi)^2 + \frac{1}{2} m^2 \varphi^2}_{\mathcal{L}_E^0} - \underbrace{\frac{g}{6} \varphi^3 + Y \varphi}_{-\mathcal{L}_E^1}$$

thus

$$e^{iS} = e^{i \int d^4 x \mathcal{L}} = e^{\int d^4 x_E \mathcal{L}} = e^{\int d^4 x_E (-\mathcal{L}_E)} = e^{-S_E}$$

which we did wick rotation as





and

$$S_E = \int d^4x_E (\mathcal{L}_E^0 - \mathcal{L}_E^1)$$

thus

$$\langle 0 | T \varphi(x_1) \cdots \varphi(x_N) | 0 \rangle = \frac{\int D\varphi \varphi(x_1) \cdots \varphi(x_N) e^{-S_E}}{\int D\varphi e^{-S_E}} = \frac{\frac{\int D\varphi (A e^{+\int d^4x_E \mathcal{L}_E^1}) e^{-\int d^4x_E \mathcal{L}_E^0}}{\int D\varphi e^{-\int d^4x_E \mathcal{L}_E^0}}}{\frac{\int D\varphi (e^{+\int d^4x_E \mathcal{L}_E^1}) e^{-\int d^4x_E \mathcal{L}_E^0}}{\int D\varphi e^{-\int d^4x_E \mathcal{L}_E^0}}}$$

with  $A = T\varphi(x_1) \cdots \varphi(x_N)$ . Let

$$B = A e^{+\int d^4x_E \mathcal{L}_E^1}$$

we have

$$\langle B \rangle_0 = \frac{\int D\varphi B e^{-\int d^4x_E \mathcal{L}_E^0}}{\int D\varphi e^{-\int d^4x_E \mathcal{L}_E^0}}, \quad \langle A \rangle = \frac{\langle A e^{\int d^4x_E \mathcal{L}_E^1} \rangle_0}{\langle e^{\int d^4x_E \mathcal{L}_E^1} \rangle_0}$$

where

$$\langle \cdot \rangle_0 = \langle 0 | \cdot | 0 \rangle_{KG}$$

Our objective is to calculate  $A$ . First consider the **denominator**,

$$G = \langle e^{\int d^4x_E \mathcal{L}_E^1} \rangle_0 = \langle e^{\int d^4x_E (\frac{g}{6} \varphi^3 + Y \varphi)} \rangle_0 = \sum_{n_g=0}^{\infty} \sum_{n_Y=0}^{\infty} G_{n_g, n_Y}, \quad G_{n_g, n_Y} = \left\langle \frac{1}{n_g! n_Y!} \left( \frac{g}{6} \int d^4x_E \varphi^3(x) \right)^{n_g} \left( Y \int d^4y_E \varphi(y) \right)^{n_Y} \right\rangle_0 \propto g^{n_g} Y^{n_Y}$$

If  $n_g + n_Y$  is odd, then

$$G_{n_g, n_Y} = 0$$

Also we can see that

$$G_{00} = 1 \text{ (empty diagram)}, \quad G_{10} = G_{01} = 0$$

Moreover,

$$\begin{aligned} G_{20} &= \frac{1}{2!0!} \left( \frac{g}{6} \right)^2 \int d^4x_E d^4y_E \langle \varphi(x) \varphi(x) \varphi(x) \varphi(y) \varphi(y) \varphi(y) \rangle_0 \\ &= \frac{1}{2!} \left( \frac{g}{3!} \right)^2 \int d^4x_E d^4y_E [3! \Delta(x-y)^3 + 3 \cdot 3 \Delta(x-x) \Delta(y-y) \Delta(x-y)] \\ &= \frac{g^2}{2!3!} \int d^4x_E d^4y_E \Delta(x-y)^3 + \frac{g^2}{2!2!2!} \int d^4x_E d^4y_E \Delta(x-x) \Delta(y-y) \Delta(x-y) \end{aligned}$$

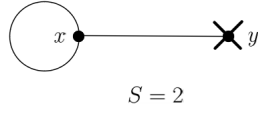
The last expression could be re-expressed as

$$\begin{array}{c} \text{Diagram 1: Two vertices } x \text{ and } y \text{ connected by two curved lines.} \\ S = 2!3! \end{array} \quad + \quad \begin{array}{c} \text{Diagram 2: Two vertices } x \text{ and } y \text{ connected by a straight line, each with a self-loop.} \\ S = 2!2!2! \end{array}$$

And

$$G_{11} = \frac{gY}{6} \int d^4x_E d^4y_E \langle \varphi(x) \varphi(x) \varphi(x) \varphi(y) \rangle_0 = \frac{gY}{2} \int d^4x_E d^4y_E \Delta(x-x) \Delta(x-y)$$

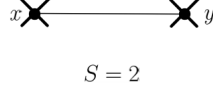
which could be re-expressed as



And

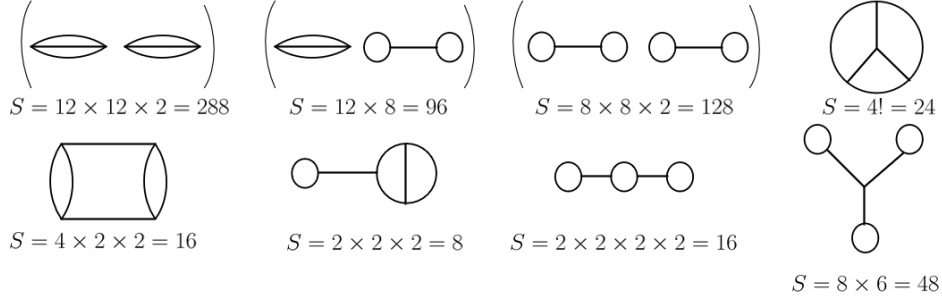
$$G_{02} = \frac{Y^2}{2!} \int d^4x_E d^4y_E \Delta(x-y)$$

which could be re-expressed as

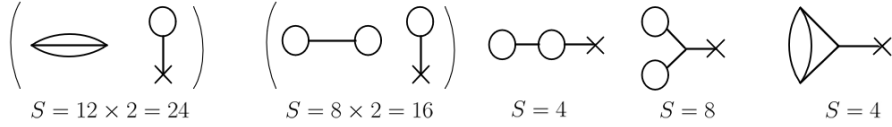


Similarly, we have

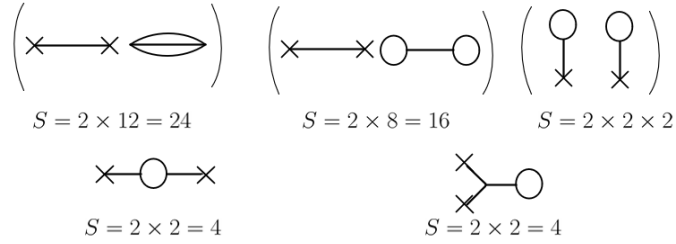
$G_{40}$ :



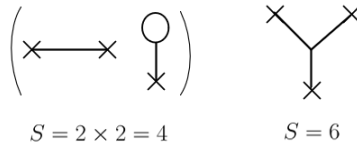
$G_{31}$  :



$G_{22}$  :



$G_{13}$ :



$G_{04}$ :

$$\left( \begin{array}{cc} \times & \times \\ \hline \times & \times \end{array} \right)$$

$$S = 2 \times 2 \times 2 = 8$$

thus we have

$$G = \langle e^{\int d^4 x_E (\frac{g}{6} \varphi^3 + Y \varphi)} \rangle_0 = \sum (\text{all possible vacuum diagrams})$$

Connected vacuum diagrams:

$$C_1 = \times \text{---} \times \quad C_2 = \text{---} \text{---} \text{---} \quad C_3 = \bigcirc \text{---} \bigcirc \quad C_4 = \times \text{---} \bigcirc \quad \dots\dots$$

then

$$G = \sum_{n_1, n_2, n_3, \dots} \frac{1}{n_1! n_2! n_3! \dots} C_1^{n_1} C_2^{n_2} C_3^{n_3} \dots = e^{C_1 + C_2 + C_3 + C_4} = e^{\sum \text{Connected vacuum diagrams}}$$

Then consider the **numerator**,

$$G^{(N)}(x_1, \dots, x_N) = \langle \varphi(x_1) \dots \varphi(x_N) e^{\int d^4 y_E [\frac{g}{6} \varphi(y)^3 + Y \varphi(y)]} \rangle_0 = \sum_{n_g, n_Y=0}^{\infty} G_{n_g, n_Y}^{(N)}(x_1, \dots, x_N)$$

For example, consider  $N = 1$ . It's easy to see

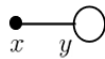
$$G_{n_g, n_Y}^{(1)}(x) = 0 \quad \text{if} \quad n_g + n_Y + 1 \text{ is odd}$$

thus

$$G_{00}^{(1)}(x) = 0$$

and

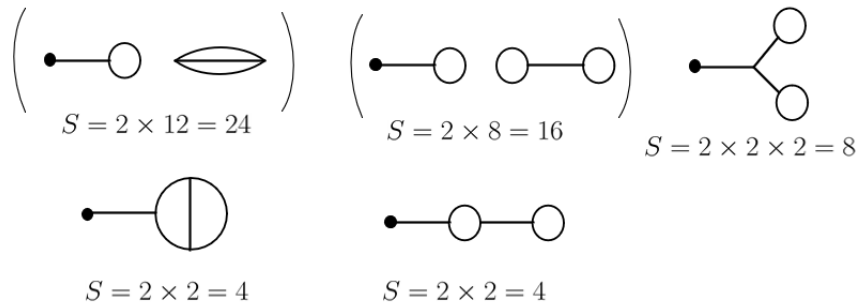
$$G_{10}^{(1)}(x): (S = 2)$$



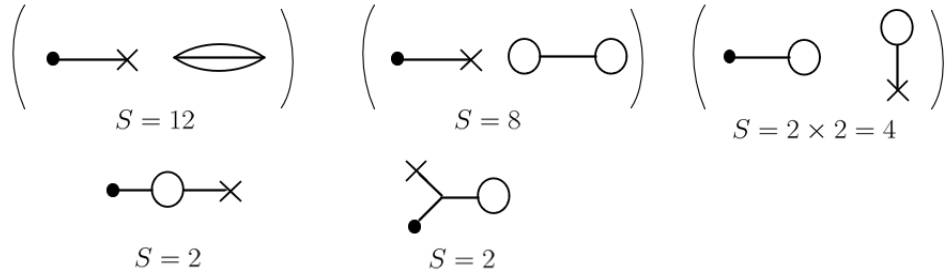
$$G_{01}^{(1)}(x): (S = 1)$$



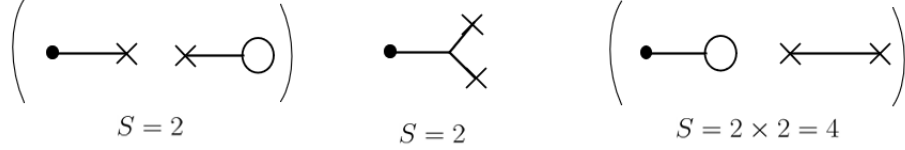
$$G_{30}^{(1)}(x):$$



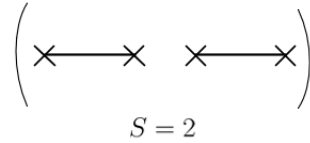
$$G_{21}^{(1)}(x):$$



$G_{12}^{(1)}(x)$ :



$G_{03}^{(1)}(x)$ :



Or considering the general part:

$$\langle \varphi(x_1) \cdots \varphi(x_N) e^{\int d^4 x_E \mathcal{L}_E^1} \rangle_0 = \sum_{J, n_1, n_2, \dots} C_J^{(N)} \frac{C_1^{n_1} C_2^{n_2} C_3^{n_3} \cdots}{n_1! n_2! n_3! \cdots} = \sum_J C_J^{(N)} e^{C_1 + C_2 + C_3 + \cdots}$$

where  $C_J^{(N)}$  is a diagram containing **NO** disconnected vacuum pieces.

Finally we reach that

$$\langle 0 | T \varphi(x_1) \cdots \varphi(x_N) | 0 \rangle = \frac{\langle \varphi(x_1) \cdots \varphi(x_N) e^{\int d^4 x_E \mathcal{L}_E^1} \rangle_0}{\langle e^{\int d^4 x_E \mathcal{L}_E^1} \rangle_0} = \sum_J C_J^{(N)}(x_1, \dots, x_N)$$

We consider the Lagrangian density with the form

$$\mathcal{L} = \underbrace{-\frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2}\omega_0^2 \varphi^2}_{\mathcal{L}^0} + \underbrace{\frac{g_1}{1!}\varphi + \frac{g_2}{2!}\varphi^2 + \cdots + \frac{g_M}{M!}\varphi^M}_{\mathcal{L}^1}$$

what we're interested in is

$$\langle \varphi(x_1) \cdots \varphi(x_E) e^{i \int d^4 x \mathcal{L}^1} \rangle_0 = \left\langle \varphi(x_1) \cdots \varphi(x_E) \sum_{V_1, \dots, V_M} \frac{\left( i \int d^4 x \frac{g_1}{1!} \varphi(x) \right)^{V_1}}{V_1!} \frac{\left( i \int d^4 x \frac{g_2}{2!} \varphi(x) \varphi(x) \right)^{V_2}}{V_2!} \cdots \frac{\left( i \int d^4 x \frac{g_M}{M!} \varphi(x)^M \right)^{V_M}}{V_M!} \right\rangle$$

The external points:

$$i = 1 : \varphi(x_1) \quad i = 2 : \varphi(x_2) \quad \cdots \quad i = E : \varphi(x_E)$$

The vertices:

$$\begin{cases} E + 1 \leq i \leq E + V_1 : \text{1-point vertices} \\ E + V_1 \leq i \leq E + V_1 + V_2 : \text{2-point vertices} \\ \dots\dots\dots \\ E + V_1 + \cdots + V_{M-1} \leq i \leq E + V_1 + \cdots + V_M : \text{M-point vertices} \end{cases}$$

The total number of points in the diagram

$$N_0 = E + V_1 + \cdots + V_M$$

The  $i$ th point and  $j$ th point are connected by  $C_{ij} \geq 0$  lines. ( $i \neq j$ )

If  $i = j$ , then

$$C_{ii} = \text{twice the number of lines connecting } i \text{ to itself}$$

Taking the sum

$$C_i = \sum_{j=1}^{N_0} C_{ij} = \begin{cases} 1, & \text{external point } (1 \leq i \leq E) \\ n, & n\text{-point vertex } (1 \leq n \leq M) \end{cases}$$

Let  $D(C)$  be number of different  $C$  matrices that are similar to  $C$ .

The overall factor

$$\eta(C) = \left[ \prod_{n=1}^M \frac{\left(\frac{1}{n!}\right)^{V_n}}{V_n!} \right] \left[ \prod_{i=1}^{N_0} \frac{C_i!}{C_{i1}! \cdots C_{iN_0}!} \right] \left( \prod_{i < j} C_{ij}! \right) \left( \prod_{i=1}^M (C_{ii} - 1)!! \right) D(C)$$

Noting that the two factor cancels:

$$\prod_{n=1}^M \left(\frac{1}{n!}\right)^{V_n} \prod_{i=1}^{N_0} C_i! = 1$$

thus

$$\eta(C) = \frac{1}{\left(\prod_{n=1}^M V_n!\right)} \left(\prod_{i,j=1}^{N_0} \frac{1}{C_{ij}!}\right) \left(\prod_{i < j} C_{ij}!\right) \left(\prod_{i=1}^M (C_{ii} - 1)!!\right) D(C) = \frac{1}{\prod_{n=1}^M V_n!} \frac{1}{\prod_{i > j} C_{ij}!} \frac{1}{\prod_i C_{ii}!!} D(C)$$

Let  $F(C)$  be vertex renaming freedom, then

$$F(C)D(C) = V_1! \cdots V_M! \implies \frac{D(C)}{V_1! \cdots V_M!} = \frac{1}{F(C)}$$

Hence

$$\eta(C) = \frac{1}{F(C) \left(\prod_{i > j} C_{ij}!\right) \left(\prod_i C_{ii}!!\right)} = \frac{1}{S(C)}$$

Consider the case  $g_1 = Y, g_3 = g, g \sim Y$  and all other  $g$ 's are zero, then

$$\langle 0 | \varphi(x) | 0 \rangle = \frac{g}{2} \int d^4 y_E \Delta(x - y) \Delta(0) + Y \int d^4 y_E \Delta(x - y) + O(g^3) = \int d^4 y_E \Delta(x - y) \left( Y + \frac{g}{2} \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{k_E^2 + m^2} \right) + O(g^3)$$

However, noting that

$$\lim_{\Lambda \rightarrow \infty} \begin{cases} \int_{k_E < \Lambda} d^2 k_E \frac{1}{k_E^2 + m^2} : & \text{logarithmically divergent} \\ \int_{k_E < \Lambda} d^3 k_E \frac{1}{k_E^2 + m^2} : & \text{linearly divergent} \\ \int_{k_E < \Lambda} d^4 k_E \frac{1}{k_E^2 + m^2} : & \text{quadratically divergent} \end{cases}$$

A way to handle this divergence is

$$\int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{k_E^2 + m^2} = \left( \int_{k_E < \Lambda} \frac{d^4 k_E}{(2\pi)^4} \frac{1}{k_E^2 + m^2} \right) + \left( \int_{k_E > \Lambda} \frac{d^4 k_E}{(2\pi)^4} \frac{1}{k_E^2 + m^2} \right) = \text{any value independent of } \Lambda$$

thus we have the freedom of integration. Then the part

$$Y + \frac{g}{2} \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{k_E^2 + m^2}$$

could be set to 0 by choosing the zero of  $\varphi$ .

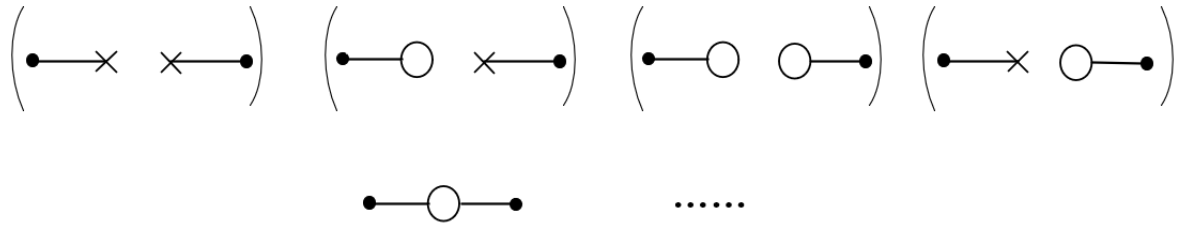
One particular scheme is:

- Each tadpole is set to zero.
- $Y$  is set to zero.

then

$$\langle 0 | \varphi(x) | 0 \rangle = 0$$

Since  $\langle 0 | T \varphi(x) \varphi(y) | 0 \rangle$  is



The first line part contains disconnected pieces, which equals

$$\langle 0 | \varphi(x) | 0 \rangle \langle 0 | \varphi(y) | 0 \rangle$$

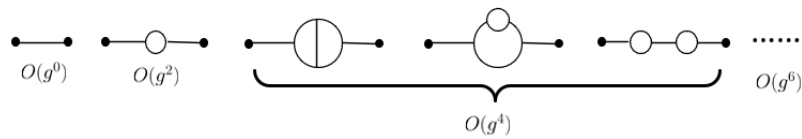
which is set to zero by our scheme.

The Lagrangian density is now

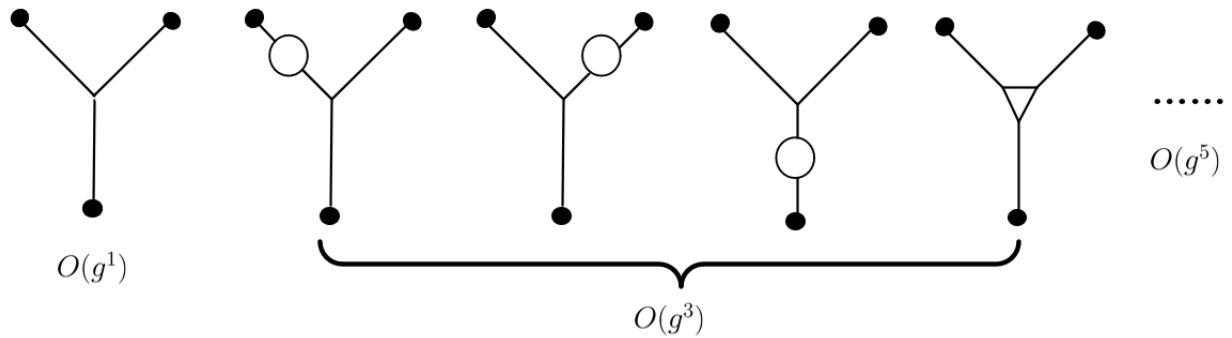
$$\mathcal{L}_E = \frac{1}{2}(\partial\varphi)^2 + \frac{m^2}{2}\varphi^2 - \frac{g}{6}\varphi^3$$

then for  $\langle 0 | T \varphi(x_1) \cdots \varphi(x_N) | 0 \rangle$ , we only need to care about other connected parts. For example,

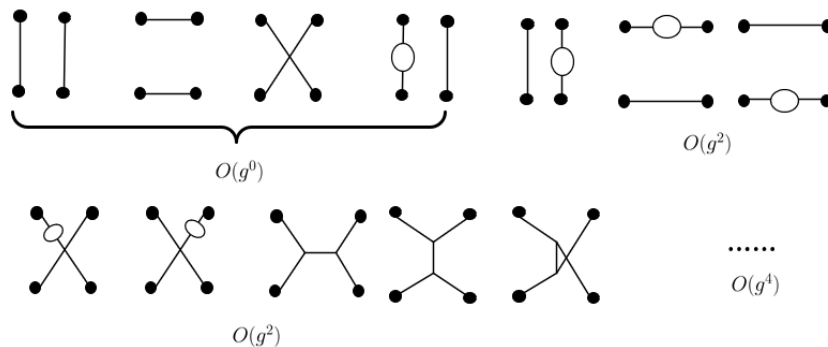
$\langle 0 | T \varphi(x) \varphi(y) | 0 \rangle$ :



$\langle 0 | T \varphi(x_1) \varphi(x_2) \varphi(x_3) | 0 \rangle$  :



$\langle 0 | T \varphi(x_1) \varphi(x_2) \varphi(x_3) \varphi(x_4) | 0 \rangle$  :



For the last part we can see that

.....

$$\begin{aligned}\langle 0|\mathcal{T}\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|0\rangle &= \langle 0|\mathcal{T}\varphi(x_1)\varphi(x_2)|0\rangle\langle 0|\mathcal{T}\varphi(x_3)\varphi(x_4)|0\rangle + \langle 0|\mathcal{T}\varphi(x_1)\varphi(x_4)|0\rangle\langle 0|\mathcal{T}\varphi(x_2)\varphi(x_3)|0\rangle \\ &+ \langle 0|\mathcal{T}\varphi(x_1)\varphi(x_3)|0\rangle\langle 0|\mathcal{T}\varphi(x_2)\varphi(x_4)|0\rangle + \langle 0|\mathcal{T}\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|0\rangle_C\end{aligned}$$

Similarly,

$$\langle 0|\mathcal{T}\varphi(x_1)\cdots\varphi(x_5)|0\rangle = \sum_{10 \text{ terms}} \langle 0|\mathcal{T}\varphi\varphi\varphi|0\rangle\langle 0|\mathcal{T}\varphi\varphi|0\rangle + \langle 0|\mathcal{T}\varphi\varphi\varphi\varphi|0\rangle_C$$

$$\langle 0|\mathcal{T}\varphi(x_1)\cdots\varphi(x_6)|0\rangle = \sum \langle 0|\mathcal{T}\varphi\varphi|0\rangle\langle 0|\mathcal{T}\varphi\varphi|0\rangle\langle 0|\mathcal{T}\varphi\varphi|0\rangle + \langle 0|\mathcal{T}\varphi\varphi\varphi\varphi|0\rangle_C\langle 0|\mathcal{T}\varphi\varphi|0\rangle + \langle 0|\mathcal{T}\varphi\varphi\varphi|0\rangle\langle 0|\mathcal{T}\varphi\varphi\varphi|0\rangle + \langle 0|\mathcal{T}\varphi\varphi\varphi\varphi\varphi|0\rangle_C$$

We consider a general condition. Suppose the diagram has:

- $N_0$  points
- $N_1$  lines
- $N_2$  independent loops
- $N_3$  disconnected pieces ( $N_3 = 1$  for connected diagram)

A formula for any diagram is

$$N_0 - N_1 + N_2 - N_3 = 0$$

Consider  $\langle 0|\mathcal{T}\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|0\rangle_C$ . According to the diagrams we have

$$\begin{aligned}\langle 0|\mathcal{T}\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|0\rangle_C &= g^2 \int d^4y_E d^4z_E \Delta(y-z) [\Delta(x_1-y)\Delta(x_2-y)\Delta(z-x_3)\Delta(z-x_4) \\ &+ \Delta(x_1-y)\Delta(x_3-y)\Delta(z-x_2)\Delta(z-x_4) + \Delta(x_1-y)\Delta(x_4-y)\Delta(z-x_2)\Delta(z-x_3)] + O(g^4)\end{aligned}$$

Consider particles incoming and outgoing as:

$$\vec{k}_1, \vec{k}_2 \rightarrow \vec{k}'_1, \vec{k}'_2, \quad \vec{k}_1 \neq \vec{k}'_1, \vec{k}_1 \neq \vec{k}'_2, \vec{k}_2 \neq \vec{k}'_1, \vec{k}_2 \neq \vec{k}'_2$$

According to the LSZ formula,

$$\begin{aligned}\langle \text{final} : k'_1 k'_2 | \text{initial} : k_1 k_2 \rangle &= \int (id^4 x_1)(id^4 x_2)(id^4 x'_1)(id^4 x'_2) e^{i(k_1 x_1 + k_2 x_2 - k'_1 x'_1 - k'_2 x'_2)} \\ &(m^2 - \partial_1^2)(m^2 - \partial_2^2)(m^2 - \partial_1'^2)(m^2 - \partial_2'^2) \langle 0|\mathcal{T}\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|0\rangle_C\end{aligned}$$

By using

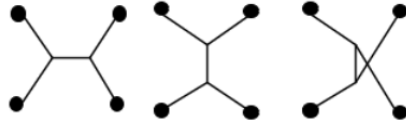
$$(m^2 - \partial_x^2)\Delta(x-y) = -i\delta(x-y)$$

we have

$$\int id^4 x e^{ikx} (m^2 - \partial_x^2)\Delta(x-y) = e^{iky}, \quad \int id^4 x' e^{-ik'x'} (m^2 - \partial_{x'}^2)\Delta(x'-y) = e^{-ik'y}$$

We have the **rule** : replace each external free propagator by  $e^{iky}$  (for incoming particle with 4-momentum  $k$ ), or  $e^{-ik'y}$  (for outgoing particle with 4-momentum  $k'$ ).

Therefore  $\langle \text{out} : k'_1 k'_2 | \text{in} : k_1 k_2 \rangle$  equals



plus  $O(g^4)$ .

Now consider its mathematical expression. Since

$$\Delta(y-z) = \int \frac{-id^4 k}{(2\pi)^4} \frac{e^{ik \cdot (z-y)}}{k^2 + m^2 - i\varepsilon}$$

we have

$$\langle \text{out} : k'_1 k'_2 | \text{in} : k_1 k_2 \rangle = g^2 \int id^4 y id^4 z \frac{-id^4 k}{(2\pi)^4} \frac{e^{ik \cdot (z-y)}}{k^2 + m^2 - i\varepsilon} [e^{i(k_1 + k_2) \cdot y - i(k'_1 + k'_2) \cdot z} + e^{i(k_1 - k'_1) \cdot y + i(k_2 - k'_2) \cdot z} + e^{i(k_1 - k'_2) \cdot y + i(k_2 - k'_1) \cdot z}] + O(g^4)$$

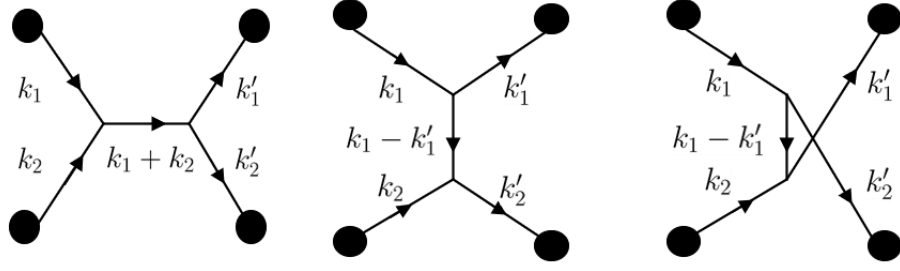
By calculating the integral over  $y, z$  we now have

$$\langle \text{out} : k'_1 k'_2 | \text{in} : k_1 k_2 \rangle = i(2\pi)^4 \delta(k_1 + k_2 - k'_1 - k'_2) \mathcal{J}_{k'_1 k'_2 \leftarrow k_1 k_2}$$

with

$$\mathcal{J}_{k'_1 k'_2 \leftarrow k_1 k_2} = g^2 \left[ \frac{1}{(k_1 + k_2)^2 + m^2 - i\varepsilon} + \frac{1}{(k_1 - k'_1)^2 + m^2 - i\varepsilon} + \frac{1}{(k_1 - k'_2)^2 + m^2 - i\varepsilon} \right] + O(g^4)$$

which could be portrayed as (the  $g^2$  part)



Or to generalize,

$$\langle \text{out} : k'_1 k'_2 \dots | \text{in} : k_1 k_2 \dots \rangle = i(2\pi)^4 \delta(k_{in} - k_{out}) \mathcal{J}_{k'_1 k'_2 \dots \leftarrow k_1 k_2 \dots}$$

and generalizing to  $d$ -dimensional spacetime,

$$\langle \text{out} : k'_1 k'_2 \dots | \text{in} : k_1 k_2 \dots \rangle = i(2\pi)^d \delta(k_{in} - k_{out}) \mathcal{J}_{k'_1 k'_2 \dots \leftarrow k_1 k_2 \dots}$$

Consider its dimension,

$$\langle \text{out} | \text{in} \rangle = \int i d^d x_1 \dots i d^d x_n i d^d x_{1'} \dots i d^d x_{n'} \cdot e^{\dots} (m^2 - \partial_1)^2 \dots \langle 0 | T \varphi \dots \varphi | 0 \rangle$$

Since

$$S = \int d^d x \left( -\frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi + \dots \right)$$

is dimensionless, then

$$[\varphi] = [x]^{\frac{2-d}{2}}$$

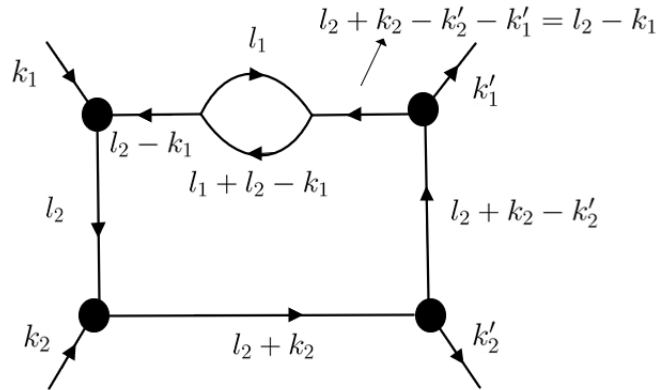
Therefore,

$$[\langle \text{out} | \text{in} \rangle] = [x]^{\frac{d-2}{2}(n+n')}$$

then

$$[\mathcal{J}] = [x]^{(n+n')\frac{d-2}{2} - d}$$

Consider this diagram:



We claim that this diagram equals

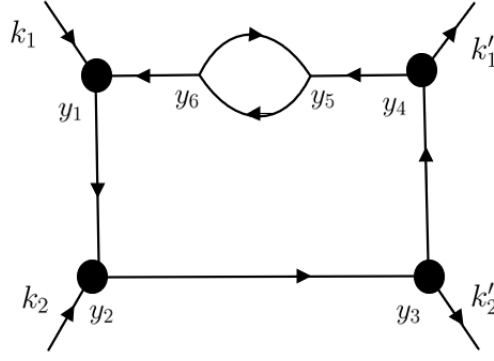
$$\frac{g^6}{2} \int \frac{(-i) d^4 l_1}{(2\pi)^4} \frac{(-i) d^4 l_2}{(2\pi)^4} \hat{\Delta}(l_2) \hat{\Delta}(l_2 + k_2) \hat{\Delta}(l_2 + k_2 - k'_2) [\hat{\Delta}(l_2 - k_1)]^2 \hat{\Delta}(l_1) \hat{\Delta}(l_1 + l_2 - k_1) = \mathcal{J}_{k'_1 k'_2 \leftarrow k_1 k_2}^{(\text{one term})}$$

where

$$\hat{\Delta}(l) = \frac{1}{l^2 + m^2 - i\varepsilon}$$



To derive this, we start from LSZ formula:



This term equals

$$\frac{g^6}{2} \int d^4 y_1 \dots d^4 y_6 e^{ik_1 y_1 + ik_2 y_2 - ik'_1 y_4 - ik'_2 y_3} \Delta_{12} \Delta_{23} \Delta_{34} \Delta_{45} \Delta_{56}^2 \Delta_{61}$$

where

$$\Delta_{ab} = \Delta(y_a - y_b) = \int \frac{-i d^4 q_j}{(2\pi)^4} \hat{\Delta}(q_j) e^{iq_j \cdot (y_a - y_b)}$$

then the term above equals

$$i(2\pi)^4 \delta(k_1 + k_2 - k'_1 - k'_2) \mathcal{J}_{k'_1 k'_2 \leftarrow k_1 k_2}^{(\text{one term})}$$

internal line in 4-momentum space: virtual particle

## Cross sections and decay rates

We define

$$|in\rangle : f_1(\vec{p}), \dots, f_n(\vec{p}) \text{ in momentum space}$$

where  $f_i(\vec{p})$  is a narrow peak centered around  $\vec{k}_i$ . And

$$|out\rangle : f'_1(\vec{p}), \dots, f'_{n'}(\vec{p}) \text{ in momentum space}$$

where  $f'_i(\vec{p})$  is a narrow peak centered around  $\vec{k}'_i$ .

According to the LSZ formula:

$$\begin{aligned} \langle out|in\rangle &= i^{n+n'} \int \widetilde{dp_1} f_1(\vec{p}_1) \dots \int \widetilde{dp_n} f_n(\vec{p}_n) \int \widetilde{dp'_1} f'^*_1(\vec{p}'_1) \dots \int \widetilde{dp'_{n'}} f'^*_{n'}(\vec{p}'_{n'}) \int d^4 x_1 e^{ip_1 x_1} (m^2 - \partial_1^2) \dots \int d^4 x_n e^{ip_n x_n} (m^2 - \partial_n^2) \\ &\quad \int d^4 x'_1 e^{ip'_1 x'_1} (m^2 - \partial_1'^2) \dots \int d^4 x'_{n'} e^{ip'_{n'} x'_{n'}} (m^2 - \partial_{n'}'^2) \langle 0 | T \varphi(x_1) \dots \varphi(x_n) \varphi(x'_1) \dots \varphi(x'_{n'}) | 0 \rangle \end{aligned}$$

While in the on-shell scheme for the overall state of  $\varphi(x)$ :

$$i^{n+n'} \int d^4 x_1 \dots d^4 x'_1 \dots e^{ip_1 x_1 + \dots + ip'_1 x'_1 - \dots} (m^2 - \partial_1^2) \dots (m^2 - \partial_{n'}'^2) \dots \langle 0 | T \varphi(x_1) \dots \varphi(x'_1) \dots | 0 \rangle = i(2\pi)^4 \delta(p_{in} - p'_{out}) \mathcal{J}(\vec{p}'_1, \dots; \vec{p}_1, \dots)$$

with

$$p_{in} = p_1 + \dots + p_n, \quad p'_{out} = p'_1 + \dots + p'_{n'}$$

and

$$\widetilde{dp} = \widetilde{dp_1} \dots \widetilde{dp_n}, \quad \widetilde{dp'} = \widetilde{dp'_1} \dots \widetilde{dp'_{n'}}$$

hence

$$\langle out|in\rangle = \int \widetilde{dp} \widetilde{dp'} f_1(\vec{p}_1) \dots f_n(\vec{p}_n) f'^*_1(\vec{p}'_1) \dots f'^*_{n'}(\vec{p}'_{n'}) i(2\pi)^4 \delta(p_{in} - p'_{out}) \mathcal{J}(\vec{p}'_1, \dots; \vec{p}_1, \dots)$$

and

$$\langle in|in\rangle = \int \widetilde{dp} |f_1(\vec{p}_1)|^2 \dots |f_n(\vec{p}_n)|^2, \quad \langle out|out\rangle = \int \widetilde{dp'} |f'_1(\vec{p}'_1)|^2 \dots |f'_{n'}(\vec{p}'_{n'})|^2$$

Define an incoming  $n$ -body wave function

$$\psi_{in}(\vec{p}_1, \dots, \vec{p}_n) = \sum_Q f_1(\vec{p}_{Q1}) \cdots f_n(\vec{p}_{Qn})$$

and an outgoing  $n'$ -body wave function

$$\psi_{out}(\vec{p}'_1, \dots, \vec{p}'_{n'}) = \sum_{Q'} f'_1(\vec{p}'_{Q'1}) \cdots f'_{n'}(\vec{p}'_{Q'n'})$$

then the above results could be re-expressed by the wave functions

$$\begin{aligned} \langle out|in \rangle &= \frac{1}{n!n'!} \int \widetilde{dp} \widetilde{dp'} \psi_{out}^*(\vec{p}') \psi_{in}(\vec{p}) i(2\pi)^4 \delta(p_{in} - p'_{out}) \mathcal{J}(\vec{p}'; \vec{p}) \\ \langle in|in \rangle &= \frac{1}{n!} \int \widetilde{dp} |\psi_{in}(\vec{p})|^2, \quad \langle out|out \rangle = \frac{1}{n'!} \int \widetilde{dp'} |\psi_{out}(\vec{p}')|^2, \quad \vec{p} = (\vec{p}_1, \dots, \vec{p}_n) \end{aligned}$$

If we try to express  $|in\rangle$  as superpositions of outgoing states,

$$|in\rangle = \sum_{n''=1}^{\infty} |n''\text{-body outgoing states}\rangle = \sum_{n''=1}^{\infty} |n''\text{-body out}\rangle$$

and

$$\langle out1|out2 \rangle = \frac{1}{n'!} \int \widetilde{dp'} \psi_{out1}^*(p') \psi_{out2}(p')$$

Since for different  $n'$ , the outgoing states are orthogonal,

$$\langle out | \sum_{n''=1}^{\infty} |n''\text{-body out}\rangle = \sum_{n''} \langle out | n''\text{-body out}\rangle = \langle out | n'\text{-body outgoing component}\rangle$$

Combining the  $\langle out|in \rangle$  and  $\langle out1|out2 \rangle$ , we conclude that the wavefunction of the  $n'$ -body outgoing component is

$$\psi(\vec{p}') = \frac{1}{n!} \int \widetilde{dp} i(2\pi)^4 \delta(p_{in} - p'_{out}) \mathcal{J}(p'; p) \psi_{in}(p)$$

And

$$\langle in|in \rangle = \sum_{n''} \langle n''\text{-body out} | n''\text{-body out} \rangle$$

If we set

$$\int \widetilde{dp} |f_i(\vec{p})|^2 = 1$$

then

$$\langle in|in \rangle = 1, \quad 1 = \sum_{n''} \langle n''\text{-body out} | n''\text{-body out} \rangle$$

and using the wavefunction,

$$P_{n' \leftarrow n} = \langle n'\text{-body out} | n'\text{-body out} \rangle = \frac{1}{n'!} \int \widetilde{dp'} |\psi(p')|^2 = \frac{1}{n'!n!n!} \int \widetilde{dp'} \widetilde{dp} \widetilde{dq} (2\pi)^8 \delta(p_{in} - p_{out}) \delta(q_{in} - p_{in}) \mathcal{J}^*(p'; p) \mathcal{J}(p'; q) \psi_{in}^*(p) \psi_{in}(q)$$

Or by substituting the  $\delta(q_{in} - p_{in})$  part with the FT form, we have

$$P_{n' \leftarrow n} = \frac{1}{n'!} \int \widetilde{dp'} |\mathcal{J}(p'; k)|^2 (2\pi)^4 \delta(p_{in} - p'_{out}) \int d^4a \int \widetilde{dp}_1 \cdots \widetilde{dp}_n \widetilde{dq}_1 \cdots \widetilde{dq}_n f_1^*(\vec{p}_1) \cdots f_n^*(\vec{p}_n) f_1(\vec{q}_1) \cdots f_n(\vec{q}_n) e^{i(q_{in} - p_{in}) \cdot a}$$

Define

$$g_i(a) = \int \widetilde{dq} f_i(\vec{q}) e^{iq \cdot a}$$

then

$$P_{n' \leftarrow n} = \frac{1}{n'!} \int \widetilde{dp'} |\mathcal{J}(p'; k)|^2 (2\pi)^4 \delta(p_{in} - p'_{out}) \int d^4a |g_1(a)| |g_2(a)|^2 \cdots |g_n(a)|^2$$

we have

$$|g_i(a)|^2 \propto \text{probability density of the } i\text{th incoming particle at position } \vec{a} \text{ at time } a^0$$

For fixed  $a^0$ ,

$$\int d^3a |g_i(a)|^2 = \dots = \int d\vec{q} \frac{1}{2\omega_{\vec{q}}} |f_i(\vec{q})|^2 \approx \frac{1}{2k_i^0}$$

therefore the probability density

$$\rho_i(a) \approx 2k_i^0 |g_i(a)|^2$$

while the  $i$ th particle's probability density in its rest frame

$$\rho_i^{(0)}(a) = \sqrt{1 - v_i^2} \rho_i(a) = 2m |g_i(a)|^2$$

which has used

$$k_i^0 = \frac{m}{\sqrt{1 - v_i^2}}$$

thus

$$|g_i(a)|^2 = \frac{\rho_i^{(0)}(a)}{2m}$$

therefore we have the generalized amplitude

$$P = \frac{1}{S} \int \widetilde{dp'_1} \dots \widetilde{dp'_{n'}} |\mathcal{J}|^2 (2\pi)^4 \delta(k_1 + \dots + k_n - p'_1 - \dots - p'_{n'}) \int d^4a \frac{\rho_1^{(0)}(a) \dots \rho_n^{(0)}(a)}{2M_1 \dots 2M_n}$$

with

$$S = \prod_i n'_i!, \quad n' = \sum_i n'_i$$

Now specialize to  $n = 2$ :

For fixed target space  $\vec{v}_2 = 0$ ,

$$dP = \widetilde{dp'_1} \dots \widetilde{dp'_{n'}} |\mathcal{J}|^2 (2\pi)^4 \delta(k_{in} - p'_{out}) \int d^4a \frac{\rho_1^{(0)}(a) \rho_2^{(0)}(a)}{4M_1 M_2}$$

Let

$$j_1 \equiv \frac{\rho_1^{(0)}(a)}{\sqrt{1 - v_1^2}} v_1$$

then

$$\int d^4a \frac{\rho_1^{(0)}(a) \rho_2^{(0)}(a)}{4M_1 M_2} = \int d^4a \frac{\sqrt{1 - v_1^2} j_1}{4M_1 M_2 v_1} \rho_2^{(0)}(a) = \frac{\sqrt{1 - v_1^2} j_1}{4M_1 M_2 v_1} T = \frac{j_1}{4|\vec{k}_1| M_2} T$$

where  $T$  is the time duration of the bombardment, and  $\vec{k}_1$  is the linear momentum of the first particle.

The differential cross section

$$d\sigma \equiv \frac{dP}{j_1 T} = \frac{|\mathcal{J}|^2}{4|\vec{k}_1|_{FT} M_2} \widetilde{dp'_1} \dots \widetilde{dp'_{n'}} (2\pi)^4 \delta(k_1 + k_2 - p'_{out})$$

From the dimension we can see that

$$[d\sigma] \sim x^{d-2} \text{ in } d \text{ dimensional spacetime}$$

Define the square of energy in C.O.M frame

$$s \equiv -(k_1 + k_2)^2$$

then

$$4|\vec{k}_1|_{FT} M_2 = 2\sqrt{s^2 - 2(M_1^2 + M_2^2)s + (M_1^2 - M_2^2)^2} = 4|\vec{k}_1|_{C.O.M} \sqrt{s} = 4|\vec{k}_2|_{C.O.M} \sqrt{s}$$

and the total cross section

$$\sigma \equiv \frac{1}{S} \int d\sigma$$

Consider a  $1 \rightarrow n'$  procedure:

$$P_{n' \leftarrow 1} = \frac{1}{S} \int \widetilde{dp'_1} \cdots \widetilde{dp'_{n'}} |\mathcal{J}|^2 (2\pi)^4 \delta(k_1 - p'_1 - \cdots - p'_{n'}) \int d^4 a \frac{\rho_1^{(0)}(a)}{2M_1}$$

where

$$S = n'_1! \cdots n'_M!, \quad n' = n'_1 + \cdots + n'_M$$

And we have

$$\int d^4 a \frac{\rho_1^{(0)}(a)}{2M_1} = \int d^4 a \frac{\rho_1(a) \sqrt{1-v_1^2}}{2M_1} = \frac{\sqrt{1-v_1^2}}{2M_1} \int da^0 \underbrace{\int d^3 a \rho_1(a)}_{=1}$$

thus

$$P = \frac{1}{S} \int \widetilde{dp'_1} \cdots \widetilde{dp'_{n'}} |\mathcal{J}|^2 (2\pi)^4 \delta(k_1 - p'_1 - \cdots - p'_{n'}) \frac{\sqrt{1-v_1^2} \int dt}{2M_1}$$

The decay rate

$$\Gamma \equiv \frac{dP}{dt} = \frac{1}{S} \frac{\sqrt{1-v_1^2}}{2M_1} \int \widetilde{dp'_1} \cdots \widetilde{dp'_{n'}} |\mathcal{J}|^2 (2\pi)^4 \delta(k_1 - p'_1 - \cdots - p'_{n'})$$

and the decay rate in the rest frame of particle 1

$$\Gamma_0 \equiv \frac{dP}{d\tau} = \frac{dP}{\sqrt{1-v_1^2} dt} = \frac{1}{S} \frac{1}{2M_1} \int \widetilde{dp'_1} \cdots \widetilde{dp'_{n'}} |\mathcal{J}|^2 (2\pi)^4 \delta(k_1 - p'_1 - \cdots - p'_{n'})$$

The mean lifetime

$$\bar{t} \equiv \frac{1}{\Gamma}$$

and the mean lifetime in rest frame (proper lifetime)

$$\bar{t}_0 = \frac{1}{\Gamma_0}$$

we have

$$\bar{t} = \frac{\bar{t}_0}{\sqrt{1-v_1^2}}$$

Consider a field having 4-momentum  $P^\mu$  with

$$[P^\mu, P^\nu] = 0$$

Define

$$(2\pi)^4 \delta(P - k) \equiv (2\pi)^4 \delta(p^0 - k^0) \delta(p^1 - k^1) \delta(p^2 - k^2) \delta(p^3 - k^3)$$

and in another inertial frame,

$$\delta(\bar{P} - \bar{k}) = \prod_{\mu=0}^3 \delta(\Lambda^\mu_\nu (p^\nu - k^\nu)) = \frac{1}{|\det \Lambda|} \delta(P - k) = \delta(P - k)$$

Consider a state  $|s\rangle$ . Define the **spectral function**

$$\tilde{g}_s(k) \equiv \langle s | (2\pi)^4 \delta(P - k) | s \rangle$$

and define

$$g_s(x) \equiv \langle s | e^{ip \cdot x} | s \rangle$$

then

$$\int \frac{d^4 k}{(2\pi)^4} \tilde{g}_i(k) e^{ik \cdot x} = \int \frac{d^4 k}{(2\pi)^4} \langle s | (2\pi)^4 \delta(P - k) | s \rangle e^{ik \cdot x} = \langle s | \int d^4 k \delta(P - k) e^{ik \cdot x} | s \rangle = \langle s | e^{ip \cdot x} | s \rangle = g_s(x)$$

Similarly,

$$\int d^4 x g_s(x) e^{-ik \cdot x} = \tilde{g}_s(k)$$

Consider an eigenstate as

$$F(x)|\psi\rangle = \lambda|\psi\rangle$$

If we define

$$|\psi'\rangle = \text{translation of } |\psi\rangle \text{ by } -x$$

then

$$F(0)|\psi\rangle = \lambda|\psi'\rangle$$

Using the Heisenberg equation we have

$$e^{-iP \cdot x} F(0) e^{iP \cdot x} |\psi\rangle = \lambda |\psi\rangle \implies F(0) e^{iP \cdot x} |\psi\rangle = \lambda e^{iP \cdot x} |\psi\rangle$$

thus

$$|\psi'\rangle = e^{iP \cdot x} |\psi\rangle : \text{translation of } |\psi\rangle \text{ by } -x$$

Similarly

$$e^{-iP \cdot x} |\psi\rangle : \text{translation of } |\psi\rangle \text{ by } +x$$

Since

$$e^{-iP \cdot x} |\psi\rangle = e^{iP^0 x^0 - i\vec{p} \cdot \vec{x}} |\psi\rangle, \quad P^0 \equiv H, x^0 \equiv \Delta t$$

thus

$$e^{iH\Delta t} |\psi\rangle : \text{time translation of } |\psi\rangle \text{ by } \Delta t$$

Define

$$|q\rangle \equiv |Q \text{ is measured to be } q \text{ at time } 0\rangle$$

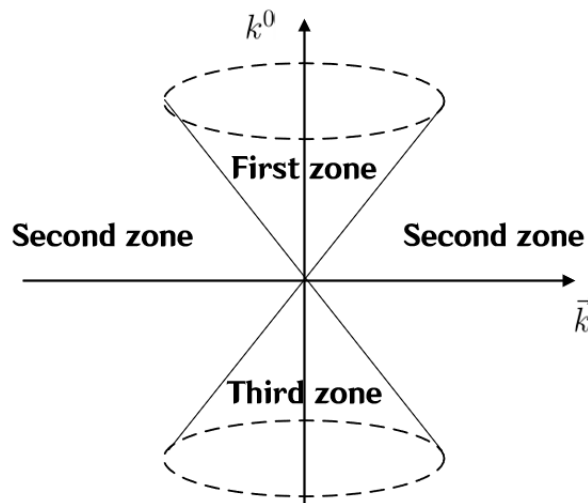
The expectation of  $F$  at time  $t$  is

$$\langle q | e^{iHt} F(0) e^{-iHt} | q \rangle$$

If  $t > 0$ , then in Heisenberg picture,

$$e^{-iHt} |q\rangle = |Q \text{ is measured to be } q \text{ at time } t\rangle$$

For 4-momentum  $k$ , we define 3 zones:



- First zone:  $k^0 > 0$  and  $k^{0^2} - |\vec{k}|^2 > 0$

- Second zone:  $k^{0^2} - |\vec{k}|^2 < 0$
- Third zone:  $k^{0^2} < 0$  and  $k^{0^2} - |\vec{k}|^2 > 0$

Therefore

$$\tilde{g}_s(k) = 0, \quad \text{if } k \in \text{second zone or third zone or their border}$$

Some properties of  $\tilde{g}_s(k)$ :

$$\tilde{g}_s(k) \geq 0$$

$$\int \frac{d^4 k}{(2\pi)^4} \tilde{g}_s(k) = g_s(0) = \langle s|s \rangle$$

And  $\tilde{g}_s(k)$  is real,

$$\tilde{g}_s(k)^* = \tilde{g}_s(k) \implies g_s(x)^* = g_s(-x)$$

and

$$g_{e^{-iP \cdot x}|s\rangle}(x) = \langle s|e^{iP \cdot x} e^{iP \cdot x} e^{-iP \cdot x}|s\rangle = \langle s|e^{iP \cdot x}|s\rangle = g_s(x)$$

If  $|s\rangle \neq 0$  and  $g_s(-x) = g_s(x)$  for all spacelike  $x$ , we say that  $|s\rangle$  is a **b-state**.

If  $|s\rangle \neq 0$  and  $g_s(-x) = -g_s(x)$  for all spacelike  $x$ , we say that  $|s\rangle$  is a **f-state**.

For a hermitian scalar field,

$$\varphi(x) = e^{-iP \cdot x} \varphi(0) e^{iP \cdot x}$$

while  $\varphi(x)$  may NOT satisfy KG equation.

Consider the spectral function, according to the requirements and Lorentz invariance,

$$\tilde{g}_{\varphi(0)|0\rangle}(k) = A(2\pi)^4 \delta(k) + 2\pi \rho(-k^2) \theta(k^0)$$

where

$$\rho(s) = 0 \text{ if } s < 0$$

Since

$$\tilde{g}_{\varphi(0)|0\rangle}(k) = \langle 0|\varphi(0)(2\pi)^4 \delta(P - k) \varphi(0)|0\rangle$$

Substituting with

$$1 = |0\rangle\langle 0| + \sum_{\alpha} |\alpha\rangle\langle \alpha|$$

we have

$$\tilde{g}_{\varphi(0)|0\rangle}(k) = \langle 0|\varphi(0)(2\pi)^4 \delta(P - k) 1 \varphi(0)|0\rangle = \langle 0|\varphi(0)|0\rangle^2 (2\pi)^4 \delta(k) + \sum_{\alpha} \langle 0|\varphi(0)(2\pi)^4 \delta(P - k) |\alpha\rangle \langle \alpha|\varphi(0)|0\rangle$$

Thus

$$A = \langle 0|\varphi(0)|0\rangle^2 = \langle 0|\varphi(x)|0\rangle^2$$

$g_{\varphi(x)|0\rangle}$ : invariant under proper ortho L.T.'s

This can be proven as follows. Since

$$\rho(-k^2) = \int_0^\infty ds \rho(s) \delta(k^2 + s)$$

thus

$$g_{\varphi(0)|0\rangle}(x) = A + \int_0^\infty ds \rho(s) \int \frac{d^3 k}{(2\pi)^3 2\sqrt{|\vec{k}|^2 + s}} e^{-i\sqrt{|\vec{k}|^2 + s}x^0 + i\vec{k} \cdot \vec{x}}$$

Therefore

$$g_{\varphi(0)|0\rangle}(\Lambda x) = g_{\varphi(0)|0\rangle}(x), \quad \text{if } \Lambda \text{ is proper orthochronous L.T.}$$

and

$$g_{\varphi(0)|0}(x) = g_{\varphi(0)|0}(-x), \quad \text{if } x \text{ is spacelike}$$

hence  $\varphi(0)|0\rangle$  is a b-state. And

$$\varphi(x)|0\rangle = e^{-iP \cdot x} \varphi(0) e^{iP \cdot x} |0\rangle = e^{-iP \cdot x} \varphi(0) |0\rangle$$

Therefore  $\varphi(x)|0\rangle$  is a b-state.

Now we consider

$$\langle 0 | \varphi(x) \varphi(y) | 0 \rangle = \langle 0 | e^{-iP \cdot x} \varphi(0) e^{iP \cdot x} e^{-iP \cdot y} \varphi(0) e^{iP \cdot y} | 0 \rangle = \langle 0 | \varphi(0) e^{iP \cdot (x-y)} \varphi(0) | 0 \rangle = g_{\varphi(0)|0}(x-y)$$

thus

$$\langle 0 | \varphi(x) \varphi(y) | 0 \rangle = \langle 0 | \varphi(y) \varphi(x) | 0 \rangle \text{ if } x, y \text{ are spacelike separated}$$

The exact propagator

$$\Delta(x-y) = \langle 0 | T \varphi(x) \varphi(y) | 0 \rangle$$

with the FT

$$\hat{\Delta}(k) = \int d^4x \Delta(x) e^{-ik \cdot x}$$

since

$$\Delta(x) = \langle 0 | \varphi(x) \varphi(0) | 0 \rangle \theta(x^0) + \langle 0 | \varphi(0) \varphi(x) | 0 \rangle \theta(-x^0) = A + \int_0^\infty ds \rho(s) \int \frac{-i d^4k}{(2\pi)^4} \frac{e^{ik \cdot x}}{k^2 + s - i\varepsilon}$$

thus

$$\hat{\Delta}(k) = iA(2\pi)^4 \delta(k) + \int_0^\infty ds \rho(s) \frac{1}{k^2 + s - i\varepsilon}$$

we have

$$\text{Im } \hat{\Delta}(k) = A(2\pi)^4 \delta(k) + \int_0^\infty ds \rho(s) \pi \delta(k^2 + s) = A(2\pi)^4 \delta(k) + \pi \rho(-k^2)$$

Consider the Lorentz form

$$\rho(s) = \frac{\Gamma_0 S}{4\pi m \left[ (\sqrt{s} - m)^2 + \left( \frac{\Gamma_0}{2} \right)^2 \right]} + (\text{background})$$

which is the broadened function of

$$\delta(s - m^2) = \frac{1}{2m} \delta(\sqrt{s} - m)$$

and  $\Gamma_0$  is the FWHM.

And

$$\hat{\Delta}(k) = \frac{S}{k^2 + \left( m - \frac{i\Gamma_0}{2} \right)^2} + (\text{background})$$

Then

$$\langle 0 | T \varphi(x) \varphi(y) | 0 \rangle \approx \frac{Sm}{4\pi^2 i \tau} K_1 \left[ \left( \frac{\Gamma_0}{2} + im \right) \tau \right]$$

If  $\tau \gg m^{-1}$ , then

$$\langle 0 | T \varphi(x) \varphi(y) | 0 \rangle \sim e^{-im\tau - \frac{\Gamma_0}{2}\tau}, \quad |\langle 0 | T \varphi(x) \varphi(y) | 0 \rangle|^2 \sim e^{-\Gamma_0 \tau}$$

If we use the natural unit.

$$\hbar = c = 1$$

then the length dimension

$$\{m\} = -1, \quad \{x^\mu\} = +1, \quad \{\partial^\mu\} = -1, \quad \{d^d x\} = +d$$

and the action

$$S = \int d^d x \left[ -\frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} m^2 \varphi^2 + \sum_{n=3}^M \frac{g_n}{n!} \varphi^n \right]$$

Since  $\hbar$  is dimensionless, thus

$$\{S\} = 0 \implies \{\mathcal{L}\} = -d$$

and

$$\{m^2 \varphi^2\} = -d \implies \{\varphi\} = \frac{2-d}{2}$$

and that

$$\{g_n \varphi^n\} = -d \implies \{g_n\} = n \frac{d-2}{2} - d$$

If all  $\{g\}'s \leq 0$ : renormalizable

If at least one  $\{g\} > 0$ : nonrenormalizable

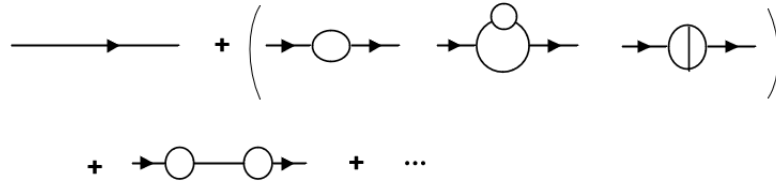
Thus  $\varphi^3$  in  $d = 6$  is interesting. In this spacetime,

$$\hat{\Delta}(x_1 - x_2) = \langle 0 | T \varphi(x_1) \varphi(x_2) | 0 \rangle$$

and

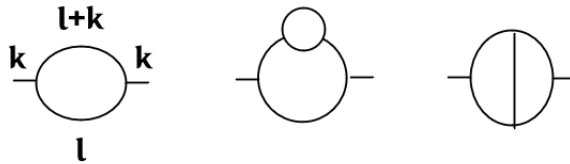
$$\hat{\Delta}(k) = \int d^6 x e^{-ik \cdot x} \Delta(x), \quad k \cdot x = -k^0 x^0 + k^1 x^1 + \dots + k^5 x^5$$

which could be portrayed as



The three diagrams in the bracket are **1PI** diagrams (One-Particle Irreducible diagram, which does not fall into two pieces if we cut one internal line).

We define  $\Pi(k^2)$  as sum of 1PI diagrams having two external free propagators with these two external free propagators removed, which is



Then

$$\begin{aligned} \hat{\Delta}(k) &= \hat{\Delta}(k) + \hat{\Delta}(k) \Pi(k^2) \hat{\Delta}(k) + \hat{\Delta}(k) \Pi(k^2) \hat{\Delta}(k) \Pi(k^2) \hat{\Delta}(k) + \dots \\ &= \hat{\Delta}(k) \frac{1}{1 - \Pi(k^2) \hat{\Delta}(k)} = \frac{1}{\hat{\Delta}(k)^{-1} - \Pi(k^2)} = \frac{1}{k^2 + m^2 - i\varepsilon - \Pi(k^2)} \end{aligned}$$

and that

$$\Pi(k^2) = \frac{g^2}{2} \int \frac{-id^6 l}{(2\pi)^6} \frac{1}{(l^2 + m^2 - i\varepsilon)((l+k)^2 + m^2 - i\varepsilon)} + O(g^4)$$

which is quadratically divergent.



Define  $m$  to be the physical mass, then we consider the **Lehmann-Kallen form** of  $\hat{\Delta}$ :

$$\hat{\Delta}(k) = iA(2\pi)^6 \delta(k) + \int_0^\infty ds \rho(s) \frac{1}{k^2 + s^2 - i\varepsilon}$$

where

$$A = \langle 0 | \varphi(x) | 0 \rangle^2$$

$$\rho(s) = \delta(s - m^2) + \underbrace{\rho_{Srednicki}(s)}_{\text{vanishes if } s < 4m^2 \text{ and there're no bound states}}$$

We can choose  $\varphi$  such that

$$A = 0$$

which is the on-shell scheme for  $\varphi$ . Thus

$$\hat{\Delta}(k) = \frac{1}{k^2 + m^2 - i\varepsilon} + \underbrace{\int_{4m^2}^{+\infty} ds \frac{\rho_{Srednicki}(s)}{k^2 + s - i\varepsilon}}_{\text{analytical when } k^2 \text{ is around } -m^2}$$

hence  $-m^2$  is a pole of  $\hat{\Delta}$  as a function of  $k^2$ , with residue 1. Therefore

$$\Pi(-m^2) = 0, \quad \Pi(k^2) = \Pi'(-m^2)(k^2 + m^2) + O[(k^2 + m^2)^2], \quad \text{when } k^2 \text{ is close to } -m^2$$

thus

$$\hat{\Delta}(k) = \frac{1}{k^2 + m^2 - i\varepsilon - \Pi'(-m^2)(k^2 + m^2) + \dots}$$

with residue

$$\frac{1}{1 - \Pi'(-m^2)}$$

Since we require the residue be 1,

$$\Pi'(-m^2) = 0$$

We define

$$\partial_k^2 = g^{\mu\nu} \frac{\partial}{\partial k^\mu} \frac{\partial}{\partial k^\nu}$$

then

$$\partial_k^2 \partial_k^2 \frac{1}{(l+k)^2 + m^2 - i\varepsilon} = \frac{384m^4}{[(l+k)^2 + m^2 - i\varepsilon]^5}$$

So  $\partial_k^2 \partial_k^2 \Pi_{1\text{ loop}}(k^2)$  is UV (ultraviolet) convergent, and is a well defined function of  $k^2$ . Therefore

$$\Pi_{1\text{ loop}}(k^2) = \Pi_{1\text{ loop}}^{\text{particular}}(k^2) + b_0 + b_1 k^2$$

which is called the freedom of integration.

We now do Wick rotation with respect to  $k^0$ , which is counterclockwise,

$$\Pi_{1\text{ loop}}(-k^{02} + |\vec{k}|^2) = \frac{g^2}{2} \int \frac{d^5 l}{(2\pi)^5} \int \frac{-i dl^0}{2\pi} \frac{1}{(-l^{02} + |\vec{l}|^2 + m^2 - i\varepsilon)[-(l^0 + k^0)^2 + |\vec{l} + \vec{k}|^2 + m^2 - i\varepsilon]}$$

If  $l^0, l^0 + k^0$  belong to the 1st or 3rd quadrant, then

$$\text{Im } l^{02} \geq 0, \quad \text{Im } (l^0 + k^0)^2 \geq 0$$

Therefore if do rotation CCW, then it won't hit the poles in the integral.

Let

$$l^6 \equiv l_6 \equiv i l^0, \quad k^6 \equiv k_6 \equiv i k^0$$

Since  $l^0 : -i\infty \rightarrow +i\infty$  after Wick rotation, we have

$$l^6 : +\infty \rightarrow -\infty$$

Therefore

$$\Pi_{1\text{ loop}}(k^{0^2} + k^{1^2} + \dots + k^{6^2}) = \frac{g^2}{2} \underbrace{\int \frac{d^5 l}{(2\pi)^5} \int_{-\infty}^{+\infty} \frac{dl^6}{2\pi}}_{\int \frac{d^6 l_E}{(2\pi)^6}} \frac{1}{(l^2 + m^2)[(l+k)^2 + m^2]}$$

Using Feynman's trick:

$$\frac{1}{A_1 \dots A_n} = \int dF_n (x_1 A_1 + \dots + x_n A_n)^{-n}$$

with

$$\int dF_n = (n-1)! \int_0^1 dx_1 \dots dx_n \delta(x_1 + \dots + x_n - 1)$$

Hence,

$$\begin{aligned} \frac{1}{(l^2 + m^2)[(l+k)^2 + m^2]} &= \int_0^1 dx \{x[(l+k)^2 + m^2] + (l-x)(l^2 + m^2)\}^{-2} \\ &= \int_0^1 dx (l^2 + 2xl \cdot k + xk^2 + m^2)^{-2} \\ &= \int_0^1 dx \underbrace{[(l+xk)^2]_q + \underbrace{x(1-x)k^2 + m^2}_D}^{-2} \end{aligned}$$

then

$$\begin{aligned} \Pi_{1\text{ loop}}(k^{0^2} + k^{1^2} + \dots + k^{6^2}) &= \frac{g^2}{2} \int_0^1 dx \int \frac{d^6 l_E}{(2\pi)^6} \frac{1}{(q^2 + D)^2} \\ &= \frac{g^2}{2} \int_0^1 dx \left[ \int_{q < Q} \frac{d^6 l_E}{(2\pi)^6} \frac{1}{(q^2 + D)^2} + \int_{q > Q} \frac{d^6 l_E}{(2\pi)^6} \frac{1}{(q^2 + D)^2} \right] \end{aligned}$$

For the first integral, using d-ball we can obtain

$$\int_{q < Q} \frac{d^6 l_E}{(2\pi)^6} \frac{1}{(q^2 + D)^2} = \int_0^Q \frac{\pi^3 q^5 dq}{(2\pi)^6} \frac{1}{(q^2 + D)^2}$$

By integrating we have

$$\begin{aligned} \Pi_{1\text{ loop}}(k^2) &= \frac{g^2}{128\pi^3} \left\{ \int_0^1 dx \left[ D \ln D + \frac{Q^2}{2} + D \left( \frac{1}{2} - 2 \ln Q \right) + O\left(\frac{1}{Q^2}\right) + \int_Q^\infty \dots \right] \right\} \\ &= \frac{g^2}{128\pi^3} \int_0^1 dx D \ln D + b_1 + b_2 k^2 \end{aligned}$$

Let

$$\alpha \equiv \frac{g^2}{(4\pi)^3}$$

and imposing

$$\Pi_{1\text{ loop}}(-m^2) = \Pi'_{1\text{ loop}}(-m^2) = 0$$

we have

$$\Pi(k^2) = \frac{\alpha}{12} \left[ (k^2 + 4m^2)^{\frac{3}{2}} \frac{2 \operatorname{arcsinh} \frac{\sqrt{k^2}}{2m}}{\sqrt{k^2}} + (3 - 2\pi\sqrt{3})m^2 + (3 - \pi\sqrt{3})k^2 \right] + O(\alpha^2)$$

Rotating back to almost real frequency, or we can also replace  $m^2$  by  $m^2 - i\varepsilon$ .

Since

$$\text{If } |k^2| \gg m^2, \quad \frac{\Pi(k^2)}{k^2 + m^2} \approx \begin{cases} \frac{\alpha}{12} \left( \ln \frac{k^2}{m^2} + c_1 \right) + O(\alpha^2), & k^2 \gg m^2 \\ \frac{\alpha}{12} \left( -i\pi + \ln \frac{|k^2|}{m^2} + c_1 \right) + O(\alpha^2), & k^2 \gg m^2 \end{cases}, \quad c_1 = 3 - \pi\sqrt{3}$$

Then

$$\hat{\Delta}(k) \approx \frac{1}{k^2 + m^2 - i\varepsilon} \left( 1 + \frac{\alpha}{12} \ln \frac{k^2}{m^2} \right) \approx \frac{1}{k^2 + m^2 - i\varepsilon} \left( \frac{k^2}{m^2} \right)^{\gamma_\varphi}, \quad \gamma_\varphi = \frac{\alpha}{12} + O(\alpha^2)$$

where  $\gamma_\varphi$  is the **anomalous dimension** of  $\varphi$ .

## Loop corrections to the vertex

Define the **vertex function**

$$V_3(k_1, k_2, k_3) = \text{sum of 1PI diagrams with 3 external free propagators}$$

with

$$k_1 + k_2 + k_3 = 0$$

thus  $V_3(k_1, k_2, k_3)$  could be portrayed as

$$+ \quad \text{loop diagram} \quad + \quad O(g^5)$$

then

$$\begin{aligned} V_3(k_1, k_2, k_3) &= g + g^2 \underbrace{\int \frac{-i d^6 l}{(2\pi)^6} \hat{\Delta}(l) \hat{\Delta}(l + k_2) \hat{\Delta}(l - k_1)}_{\text{logarithmically divergent}} + O(g^5) \\ &= g + g^2 \int \frac{d^6 l_E}{(2\pi)^6} \int dF_3 [x_2(l - k_1)^2 + x_1(l + k_2)^2 + x_3 l^2 + m^2]^{-2} + O(g^5) \end{aligned}$$

with

$$\int dF_3 \equiv 2 \int_0^1 dx_1 dx_2 dx_3 \delta(x_1 + x_2 + x_3 - 1)$$

Also

$$\frac{\partial}{\partial k_1^\mu} \int \frac{-i d^6 l}{(2\pi)^6} \hat{\Delta}(l) \hat{\Delta}(l + k_2) \hat{\Delta}(l - k_1)$$

is convergent. Therefore

$$\int \frac{-i d^6 l}{(2\pi)^6} \hat{\Delta}(l) \hat{\Delta}(l + k_2) \hat{\Delta}(l - k_3) \in \{f(k_1, k_2, k_3) + b'_0\}, \quad b'_0 \text{ is free}$$

Let

$$q \equiv l - x_2 k_1 + x_1 k_2, \quad D \equiv x_1 x_2 k_3^2 + x_2 x_3 k_1^2 + x_1 x_3 k_2^2 + m^2$$

then

$$\begin{aligned} \int \frac{-i d^6 l}{(2\pi)^6} \hat{\Delta}(l) \hat{\Delta}(l + k_2) \hat{\Delta}(l - k_3) &= \int dF_3 \int_{q < Q} \frac{d^6 q_E}{(2\pi)^6} (q^2 + D)^{-3} + \int dF_3 \int_{q > Q} \frac{d^6 q_E}{(2\pi)^6} (q^2 + D)^{-3} \\ &= \left[ -\frac{1}{128\pi^3} \int dF_3 \ln D \right] \underbrace{-\frac{3}{256\pi^3} + \frac{\ln Q}{64\pi^3} + o(Q)}_{+b'_0} + \int dF_3 \int_Q^\infty \dots \end{aligned}$$

A simple definition is that

$$g \equiv V_3(0, 0, 0)$$

hence

$$V_{3-1\text{-loop}}(0, 0, 0) = 0 \implies b'_0 = +\frac{1}{128\pi^3} \int dF_3 \ln m^2$$

therefore,

$$V_3(k_1, k_2, k_3) = g \left[ 1 - \frac{\alpha}{2} \int dF_3 \ln \frac{D}{m^2} + O(\alpha^2) \right]$$

When  $|k_1^2| \sim |k_2^2| \sim |k_3^2| \gg m^2$ ,

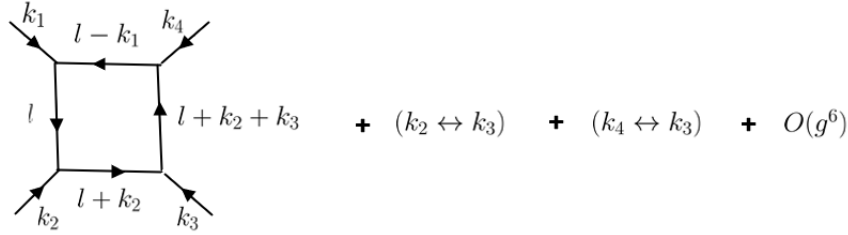
$$\frac{V_3}{g} \approx 1 - \frac{\alpha}{2} \left( \ln \frac{k_i^2}{m^2} + O(1) \right) + O(\alpha^2)$$

Similarly, define the  $n$ -vertex function

$V_n(k_1, \dots, k_m) \equiv$  sum of 1PI diagrams with  $n$  external lines with the external propagators removed

with  $n \geq 3$ .

For example, for  $V_4$ ,



then

$$V_4 = g^4 \int \frac{-i d^6 l}{(2\pi)^6} \hat{\Delta}(l) \hat{\Delta}(l+k_2) \hat{\Delta}(l+k_2+k_3) \hat{\Delta}(l-k_1) + (k_2 \leftrightarrow k_3) + (k_4 \leftrightarrow k_3) + O(g^6)$$

with

$$\hat{\Delta}(l) = \frac{1}{l^2 + m^2 - i\varepsilon}$$

Define

$$D_{1234} = x_1 x_4 k_1^2 + x_2 x_4 k_2^2 + x_2 x_3 k_3^2 + x_1 x_3 k_4^2 + x_1 x_2 (k_1 + k_2)^2 + x_3 x_4 (k_2 + k_3)^2 + m^2$$

then by using Feynman's trick,

$$V_4 = \frac{g^4}{6(4\pi)^3} \int dF_4 \left( \frac{1}{D_{1234}} + \frac{1}{D_{1324}} + \frac{1}{D_{1243}} \right) + O(g^6)$$

For  $n \geq 4$ , 1 loop  $V_n$  is convergent for  $\varphi^3$  in 6 spacetime dimensions.

Consider

$$\mathcal{L} = -\frac{1}{2} (\partial^\mu \varphi) (\partial_\mu \varphi) - \frac{1}{2} \omega_0^2 \varphi^2 + \sum_{n=3}^M \frac{g_n}{n!} \varphi^n$$

in  $d$  spacetime dimensions.

We study a  $\text{CONNECTED}(N_3 = 1)$  diagram in this theory, which has

- $E$  external lines
- $V_3$  3-point vertices
- $V_4$  4-point vertices
- $\dots$
- $V_M$   $M$ -point vertices

Then

$$N_0 = E + \sum_{n=3}^M V_n, \quad N_1 = \frac{1}{2} \left( E + \sum_{n=3}^M n V_n \right)$$

Using

$$N_0 - N_1 + N_2 - N_3 = 0$$

we have

$$N_2 = 1 + \frac{1}{2} \left( -E + \sum_n (n-2)V_n \right) = 1 - \frac{E}{2} \sum_n \frac{n-2}{2} V_n$$

and the number of internal lines

$$I = N_1 - E = \frac{1}{2} \left( -E + \sum_n n V_n \right)$$

The diagram has  $N_2$  independent momentums to integrate over, each of which has  $d$  spacetime variables, totally  $N_2 d$ .

Each internal line brings a free propagator, which scales like  $k^{-2}$ .

Hence the integral scales with power

$$D = N_2 d - 2I$$

which is the **superficial degree of divergence**. If  $D \geq 0$ , then the diagram is divergent; If  $D < 0$ , then convergent.

Substituting  $N_2, I$  we have

$$D = (d + E - \frac{dE}{2}) - \sum_{n=3}^M V_n \left( d + n - \frac{dn}{2} \right) = [g_E] - \sum_{n=3}^M V_n [g_n]$$

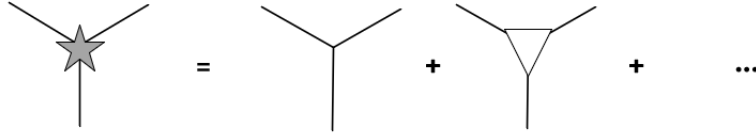
with

$$[g_n] = d + n - \frac{dn}{2} = d - \frac{n}{2}(d-2)$$

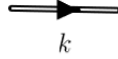
which is the mass-dimension of  $g_n$ . And

$$[g_E] \geq 0 \quad \text{if} \quad E \leq \frac{2d}{d-2}$$

We define **exact vertex** as:



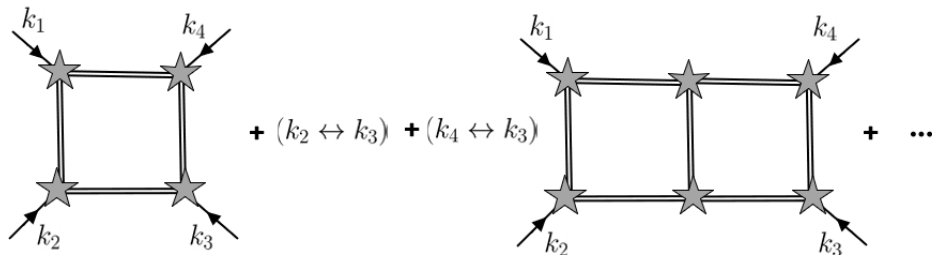
which equals  $V_3(k_1, k_2, k_3)$ . And define the **exact propagator**



which equals

$$\frac{1}{k^2 + m^2 - i\varepsilon - \Pi(k^2)}$$

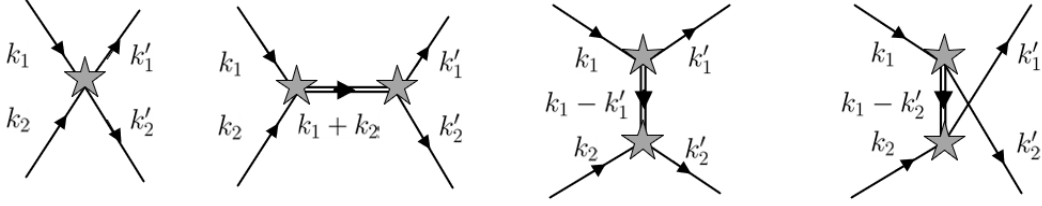
Skeleton expansion for  $V_n (n \geq 4)$ : For  $n = 4, V_4$  equals



Consider  $n_i$  incoming particles and  $n_f$  outgoing particles. Define

$\mathcal{J}$  = sum of tree level diagrams made of  $\hat{\Delta}$ ,  $V_3$ ,  $V_4$ ,  $V_5, \dots$

For example,  $\mathcal{J}_{2 \leftarrow 2}$  equals



which is

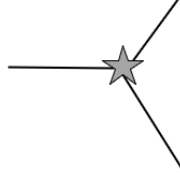
$$R^{\frac{4}{2}} [V_4(k_1, k_2, -k'_1, -k'_2) + V_3(k_1, k_2, -k_1 - k_2) \hat{\Delta}(k_1 + k_2) V_3(k_1 + k_2, -k'_1, -k'_2) + (k_2 \leftrightarrow -k'_1) + (k_2 \leftrightarrow -k'_2)]$$

where  $R$  is the residue of the exact propagator,

$$\hat{\Delta}(k) = \frac{R}{k^2 + m^2 - i\varepsilon} + O(1)$$

Each external line:  $R^{\frac{1}{2}}$

Consider  $\langle 0 | T \varphi \varphi \varphi | 0 \rangle$ , which is



which equals

$$\hat{\Delta} V_3 \hat{\Delta} \hat{\Delta} \approx R^{\frac{3}{2}}$$

thus

$$V_3 \propto R^{-\frac{3}{2}}$$

In general

$$V_n \propto R^{-\frac{n}{2}}$$

Thus  $\mathcal{J}$  is independent of  $R$ .

And for  $\mathcal{J}_{n_f \leftarrow n_i}$ , the coefficient is in general

$$R^{\frac{n_i + n_f}{2}}$$

## Schwinger-Dyson equations & Ward-Takahashi identities

Consider the action

$$S = S[\varphi_1(x), \dots, \varphi_N(x)] = \int d^d x \mathcal{L}$$

and the variance

$$\varphi_a(x) \rightarrow \varphi_a(x) + \eta_a(x)$$

while  $\eta_a(x)$  is independent of  $\varphi$ . And define

$$D\varphi = \prod_{a,x} d\varphi_a(x)$$

For any variables  $(a_1, x_1), (a_2, x_2), \dots, (a_n, x_n)$ , we consider

$$\int D\varphi e^{iS[\varphi]} \varphi_{a_1}(x_1) \cdots \varphi_{a_n}(x_n) = \int \underbrace{D(\varphi + \eta)}_{D\varphi} e^{iS[\varphi + \eta]} [\varphi_{a_1}(x_1) + \eta_{a_1}(x_1)] \cdots [\varphi_{a_n}(x_n) + \eta_{a_n}(x_n)]$$

And

$$S[\varphi + \eta] - S[\varphi] = \int W_a(x) \eta_a(x) d^d x + O(\eta^2)$$

which contains infinite sum over  $a$ . Thus

$$W_a(x) = \frac{\delta S}{\delta \varphi_a(x)}$$

If

$$S = \int d^d x \mathcal{L}[\varphi_a(x), \partial_\mu \varphi_a(x)]$$

where  $\mathcal{L}$  contains  $N + Nd$  variables. Then

$$\frac{\delta S}{\delta \varphi_a(x)} = \frac{\partial \mathcal{L}}{\partial \varphi_a(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial [\partial_\mu \varphi_a(x)]}$$

Hence

$$\begin{aligned} 0 &= \int D\varphi e^{i(S[\varphi] + \int W_a(x) \eta_a(x) d^d x)} \prod_{i=1}^n [\varphi_{a_i}(x_i) + \eta_{a_i}(x_i)] - \int D\varphi e^{iS[\varphi]} \prod_{i=1}^n \varphi_{a_i}(x_i) + O(\eta^2) \\ &= \int D\varphi e^{iS[\varphi]} \left[ \int iW_a(x) \eta_a(x) d^d x \prod_{i=1}^n \varphi_{a_i}(x_i) + \sum_{j=1}^n \varphi_{a_1}(x_1) \cdots \eta_{a_j}(x_j) \cdots \varphi_{a_n}(x_n) \right] \\ &= \int d^d x \eta_a(x) \underbrace{\int D\varphi e^{iS[\varphi]} \left[ iW_a(x) \varphi_{a_1}(x_1) \cdots \varphi_{a_n}(x_n) + \sum_{j=1}^n \varphi_{a_1}(x_1) \cdots \delta_{a,a_j} \delta(x - x_j) \cdots \varphi_{a_n}(x_n) \right]}_{X_a(x)} + O(\eta^2) \end{aligned}$$

Therefore, for any  $\eta_a(x)$  the above expression equals 0, thus

$$X_a(x) = 0$$

Dividing by  $\int D\varphi e^{iS[\varphi]}$  we have

$$i \langle 0 | T \frac{\delta S}{\delta \varphi_a(x)} \varphi_{a_1}(x_1) \cdots \varphi_{a_n}(x_n) | 0 \rangle + \sum_{j=1}^n \langle 0 | T \varphi_{a_1}(x_1) \cdots \delta_{a,a_j} \delta(x - x_j) \cdots \varphi_{a_n}(x_n) | 0 \rangle = 0$$

which is the **Schwinger-Dyson equations**.

For example, consider the  $\varphi^3$  theory

$$S = \int d^4 x \left( -\frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{\omega_0^2}{2} \varphi^2 + \frac{g}{6} \varphi^3 \right)$$

with

$$\frac{\delta S}{\delta \varphi} = \frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} = (\partial^2 - \omega_0^2) \varphi + \frac{g}{2} \varphi^2$$

For  $n = 0$ ,

$$\langle 0 | T \left[ (\partial^2 - \omega_0^2) \varphi + \frac{g}{2} \varphi^2 \right] | 0 \rangle = 0$$

For  $n = 1$ ,

$$i \langle 0 | T \left[ (\partial^2 - \omega_0^2) \varphi(x) + \frac{g}{2} \varphi(x)^2 \right] \varphi(y) | 0 \rangle + \delta(x - y) = 0$$

or to re-express as

$$i(\omega_0^2 - \partial_x^2) \langle 0 | T \varphi(x) \varphi(y) | 0 \rangle - \frac{ig}{2} \langle 0 | T \varphi(x)^2 \varphi(y) | 0 \rangle = \delta(x - y)$$

In KG,  $g = 0$ , then

$$i(\omega_0^2 - \partial_x^2) \langle 0 | T \varphi(x) \varphi(y) | 0 \rangle = \delta(x - y)$$

and that

$$\langle 0 | T \varphi(x) \varphi(y) | 0 \rangle = \int \frac{-i d^6 k}{(2\pi)^6} \frac{e^{ik \cdot (x-y)}}{k^2 + \omega_0^2 - i\epsilon}$$

which by taking derivatives we get the same expression  $\delta(x - y)$ .

This time we consider the variance of  $\mathcal{L}$ . Consider

$$\varphi_a(x) \rightarrow \varphi_a(x) + \varepsilon_a(x)$$

where  $\varepsilon_a(x)$  is infinitesimal. And if  $\mathcal{L}$  changes as

$$\mathcal{L}(x) \rightarrow \mathcal{L}(x) + \partial_\mu K^\mu(x)$$

for some  $K^\mu$ , may or may not be 0.

Then we say that we have a continuous symmetry.

If

$$\mathcal{L} = \mathcal{L}(\varphi_a(x), \partial_\mu \varphi_a(x))$$

then

$$\begin{aligned} \partial_\mu K^\mu(x) &= \mathcal{L}(\varphi_a(x) + \varepsilon_a(x), \partial_\mu(\varphi_a(x) + \varepsilon_a(x))) - \mathcal{L}(\varphi_a(x), \partial_\mu \varphi_a(x)) \\ &= \frac{\partial \mathcal{L}}{\partial \varphi_a(x)} \varepsilon_a(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a(x))} \partial_\mu \varepsilon_a(x) \\ &= \left( \underbrace{\frac{\partial \mathcal{L}}{\partial \varphi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)}}_{\frac{\delta S}{\delta \varphi_a(x)}} \right) \varepsilon_a(x) + \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \varepsilon_a(x) \right) \end{aligned}$$

hence

$$\frac{\delta S}{\delta \varphi_a(x)} \varepsilon_a(x) = \partial_\mu (K^\mu - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \varepsilon_a(x)) = -\partial_\mu j^\mu$$

where

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \varepsilon_a(x) - K^\mu$$

is **Noether current**.

The SD (Schwinger-Dyson) equation:

$$i \langle 0 | T \frac{\delta S}{\delta \varphi_a(x)} \varphi_{a_1}(x_1) \cdots \varphi_{a_n}(x_n) | 0 \rangle + \sum_{j=1}^n \langle 0 | T \varphi_{a_1}(x_1) \cdots \delta_{a,a_j} \delta(x - x_j) \cdots \varphi_{a_n}(x_n) | 0 \rangle = 0$$

which could be expressed as  $SD_a$ . We consider

$$\sum_a SD_a \times \varepsilon_a(x) = 0$$

which is

$$-i \partial_\mu \langle 0 | j^\mu(x) \varphi_{a_1}(x_1) \cdots \varphi_{a_n}(x_n) | 0 \rangle + \sum_{j=1}^n \langle 0 | T \varphi_{a_1}(x_1) \cdots \varepsilon_{a_j}(x) \delta(x - x_j) \cdots \varphi_{a_n}(x_n) | 0 \rangle = 0$$

We now consider other integration schemes of  $\varphi^3$  theory (in 6 dimensional spacetime).

Previous scheme:

$$\begin{cases} \Pi(-m^2) = \Pi'(-m^2) = 0 \\ V_3(0, 0, 0) = g \end{cases}$$

We consider another scheme which is  **$\mu$ -scheme** :

$$\begin{cases} \Pi(\mu^2) = \Pi'(\mu^2) = 0 \\ V_3(k_1, k_2, k_3) = g(\mu) \quad \text{if} \quad k_1^2 = k_2^2 = k_3^2 = \mu^2 \end{cases}$$

In the  $\mu$ -scheme, the Lagrangian density



$$\mathcal{L} = -\frac{1}{2}\partial_\nu\varphi\partial^\nu\varphi - \frac{\omega_0(\mu)^2}{2}\varphi^2 + \frac{g(\mu)}{6}\varphi^3$$

where  $\omega_0 \neq m$  usually.

The exact propagator

$$\Delta(x_1, x_2) = \langle 0 | T \varphi(x_1) \varphi(x_2) | 0 \rangle \implies \Delta(k) \propto \varphi^2$$

and

$$\langle 0 | T \varphi(x_1) \varphi(x_2) \varphi(x_3) | 0 \rangle = \int (i d^6 y_1) (i d^6 y_2) (i d^6 y_3) \Delta(x_1 - y_1) \Delta(x_2 - y_2) \Delta(x_3 - y_3) \tilde{V}_3(y_1, y_2, y_3)$$

thu

$$\tilde{V}_3 \propto \varphi^{-3} \implies V_3 \propto \varphi^{-3}$$

Generally,

$$V_n \propto \varphi^{-n}$$

Consider a small change in  $\mu$ , then we denote the change of  $\varphi(x)$  as

$$\varphi(x)|_{\mu(1+\varepsilon)} = (1 - \gamma_p \varepsilon) \varphi(x)|_\mu, \quad \varepsilon \rightarrow 0$$

thus

$$\Delta(k)|_{\mu(1+\varepsilon)} = (1 - 2\gamma_p \varepsilon) \Delta(k)|_\mu$$

$$V_3(k_1, k_2, k_3)|_{\mu(1+\varepsilon)} = (1 + 3\gamma_p \varepsilon) V_3(k_1, k_2, k_3)|_\mu$$

Or to rewrite as

$$\mu \varepsilon \frac{d}{d\mu} \Delta(k) = -2\gamma_p \varepsilon \Delta(k)_\mu \implies \frac{d}{d \ln \mu} \Delta(k) = -2\gamma_p \Delta(k) \implies \frac{d}{d \ln \mu} \Delta^{-1}(k) = +2\gamma_p \Delta^{-1}(k)$$

and

$$\frac{d}{d \ln \mu} V_3(k_1, k_2, k_3) = +3\gamma_p V_3(k_1, k_2, k_3)$$

The derivative with respect to  $\ln \mu$  could be written as

$$\frac{d}{d \ln \mu} = \frac{\partial}{\partial \ln \mu} + \frac{d\omega_0}{d \ln \mu} \frac{\partial}{\partial \omega_0} + \frac{d\alpha}{d \ln \mu} \frac{\partial}{\partial \alpha}, \quad \alpha \equiv \frac{g^2}{(4\pi)^3}$$

then we obtain the **Callan-Symanzik equations**

$$\begin{cases} \left( \frac{\partial}{\partial \ln \mu} + \frac{d\alpha}{d \ln \mu} \frac{\partial}{\partial \alpha} + \frac{d\omega_0}{d \ln \mu} \frac{\partial}{\partial \omega_0} - 2\gamma_p \right) \hat{\Delta}^{-1}(k) = 0 \\ \left( \frac{\partial}{\partial \ln \mu} + \frac{d\alpha}{d \ln \mu} \frac{\partial}{\partial \alpha} + \frac{d\omega_0}{d \ln \mu} \frac{\partial}{\partial \omega_0} - 3\gamma_p \right) V_3(k_1, k_2, k_3) = 0 \\ \dots \\ \left( \frac{\partial}{\partial \ln \mu} + \frac{d\alpha}{d \ln \mu} \frac{\partial}{\partial \alpha} + \frac{d\omega_0}{d \ln \mu} \frac{\partial}{\partial \omega_0} - n\gamma_p \right) V_n(k_1, \dots, k_n) = 0 \end{cases}$$

For this Lagrangian density,

$$\hat{\Delta}^{-1}(k) = k^2 + \omega_0^2 - \Pi(k^2)$$

where

$$\Pi(k^2) = \frac{\alpha}{2} \left( \int_0^1 D \ln D dx + b_1 + b_2 k^2 \right) + O(\alpha^2), \quad D = x(1-x)k^2 + \omega_0^2$$

Imposing

$$\Pi(\mu^2) = \Pi'(\mu^2) = 0$$

we get

$$\Pi(k^2) = \frac{\alpha}{2} \left[ -\frac{1}{6}(k^2 - \mu^2) + \int_0^1 D \ln \frac{D}{x(1-x)\mu^2 + \omega_0^2} dx \right] + O(\alpha^2)$$

thus

$$\hat{\Delta}^{-1}(k) = k^2 + \omega_0^2 - \frac{\alpha}{2} \left[ -\frac{1}{6}(k^2 - \mu^2) + \int_0^1 D \ln \frac{D}{x(1-x)\mu^2 + \omega_0^2} dx \right] + O(\alpha^2)$$

Since

$$\hat{\Delta}^{-1}(k) = 0 \quad \text{at} \quad k^2 = -m^2$$

which means  $\hat{\Delta}(k)$  has a pole at  $k^2 = -m^2$  with residue  $\neq 1$ . (No longer on-shell scheme)

And

$$V_3(k_1, k_2, k_3) = g \left[ 1 - \frac{\alpha}{2} \left( \int dF_3 \ln D_3 + b_3 \right) + O(\alpha^2) \right], \quad D_3 = x_1 x_2 k_3^2 + x_2 x_3 k_1^2 + x_3 x_1 k_2^2 + \omega_0^2$$

Imposing

$$V_3(k_1, k_2, k_3) = g \quad \text{at} \quad k_1^2 = k_2^2 = k_3^2 = \mu^2$$

we see that

$$V_3(k_1, k_2, k_3) = g \left[ 1 - \frac{\alpha}{2} \int dF_3 \ln \frac{D}{(x_1 x_2 + x_2 x_3 + x_3 x_1)\mu^2 + \omega_0^2} + O(\alpha^2) \right]$$

Substituting  $\hat{\Delta}^{-1}(k)$  and  $V_3(k_1, k_2, k_3)$  into Callan-Symanzik equations we have

$$\begin{cases} \gamma_\varphi = \frac{1}{4} f\left(\frac{\omega_0}{\mu}\right) \alpha + O(\alpha^2) \\ \gamma_{\omega_0} = \frac{d \ln \omega_0}{d \ln \mu} = \frac{1}{4} \left( 1 + \frac{\mu^2}{\omega_0^2} \right) f\left(\frac{\omega_0}{\mu}\right) \alpha + O(\alpha^2) \\ \frac{d\alpha}{d \ln \mu} = \left[ \frac{3}{2} f\left(\frac{\omega_0}{\mu}\right) - 2h\left(\frac{\omega_0}{\mu}\right) \right] \alpha^2 + O(\alpha^3) \end{cases}$$

where

$$f(\xi) = \frac{1}{3} - 2\xi^2 + \frac{8\xi^4}{\sqrt{1+4\xi^2}} \operatorname{arcsinh} \frac{1}{2\xi}, \quad h(\xi) = \int dF_3 \frac{x_1 x_2 + x_2 x_3 + x_3 x_1}{x_1 x_2 + x_2 x_3 + x_3 x_1 + \xi^2}$$

When  $\mu \rightarrow \infty$ ,  $\frac{\omega_0}{\mu} \rightarrow 0$ ,

$$\begin{cases} \gamma_\varphi = \frac{\alpha}{12} + O(\alpha^2) \\ \gamma_{\omega_0} = \frac{1}{4} \left( 1 + \frac{\mu^2}{\omega_0^2} \right) \left( \frac{1}{3} - 2\frac{\omega_0^2}{\mu^2} + \dots \right) \alpha + O(\alpha^2) \\ \frac{d\alpha}{d \ln \mu} = -\frac{3}{2} \alpha^2 + O(\alpha^3) \equiv \beta(\alpha) \end{cases}$$

where we defined the  $\beta$  function above. The  $\beta$  function determines the running of the coupling constant.

If we ignore the  $\alpha^3$  term we have

$$\frac{d\alpha}{\alpha^2} = -\frac{3}{2} d \ln \mu \implies -\frac{1}{\alpha} = C - \frac{3}{2} \ln \mu$$

If  $\alpha = \alpha_0$  when  $\mu = \mu_0$ , then

$$\alpha(\mu) = \frac{\alpha_0}{1 + \frac{3}{2} \alpha_0 \ln \frac{\mu}{\mu_0}}$$

We see that

$$\alpha(\mu) \rightarrow 0 \quad \text{at} \quad \ln \frac{\mu}{\mu_0} \rightarrow \infty$$

which is called **asymptotic freedom**.

Also we can see from

$$\hat{\Delta}_{\mu(1+\varepsilon)}^{-1}(k^2) = (1 + 2\gamma_\varphi\varepsilon)\hat{\Delta}_\mu^{-1}(k^2)$$

$$\hat{\Delta}_\mu^{-1}(\mu^2) = \mu^2 + \omega_0^2 - \Pi(\mu^2) = \mu^2 + \omega_0^2 \approx \mu^2, \quad \mu \gg \omega_0$$

hence

$$\frac{\hat{\Delta}_{\mu(1+\varepsilon)}^{-1}(\mu^2)}{\hat{\Delta}_\mu^{-1}(\mu^2)} \approx 1 + 2(1 - \gamma_p)\varepsilon$$

which means that

$$\hat{\Delta}(k) \propto k^{-2+2\gamma_\varphi} \quad \text{at} \quad k \sim \mu \gg \omega_0$$

thus

$$\langle \varphi(\vec{r}_1) \varphi(\vec{r}_2) \rangle \sim r^{-4-2\gamma_\varphi}$$

## Majorana theory

Consider KG equation:

$$-\partial_\mu \partial^\mu \varphi + m^2 \varphi = 0$$

If we define

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \end{pmatrix} \equiv \begin{pmatrix} \varphi \\ \frac{1}{m} \partial^0 \varphi \\ \frac{1}{m} \partial^1 \varphi \\ \frac{1}{m} \partial^2 \varphi \\ \frac{1}{m} \partial^3 \varphi \end{pmatrix}$$

then we can rewrite KG eq as:

$$-\partial_0 \varphi_2 - \partial_1 \varphi_3 - \partial_2 \varphi_4 - \partial_3 \varphi_5 + m \varphi_1 = 0$$

with restrictions

$$\begin{cases} -\partial^0 \varphi + m \varphi_2 = 0 \\ -\partial^1 \varphi + m \varphi_3 = 0 \\ -\partial^2 \varphi + m \varphi_4 = 0 \\ -\partial^3 \varphi + m \varphi_5 = 0 \end{cases}$$

We try to find a term like

$$[(\cdots) \partial_0 + (\cdots) \partial_1 + (\cdots) \partial_2 + (\cdots) \partial_3 + m] \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_5 \end{pmatrix} = 0$$

thus we consider the term

$$(C^\mu \partial_\mu + m) \varphi(x) = 0, \quad \varphi(x) = \begin{pmatrix} \varphi_1(x) \\ \vdots \\ \varphi_N(x) \end{pmatrix}$$

and  $C^0, C^1, C^2, C^3$  are  $N \times N$  **real matrices**.

And we require

$$\varphi_a(x)^* = \varphi_a(x), \quad \varphi_a(x)^\dagger = \varphi_a(x), \quad 1 \leq a \leq N$$

If  $k_\mu k^\mu = -m^2$ , then there exists solution

$$\varphi(x) = u e^{ik \cdot (x-a)} + u^* e^{-ik \cdot (x-a)}$$

then putting into the equation we have

$$(ik_\mu C^\mu + m)u = 0 \implies \det(ik_\mu C^\mu + m1) = 0$$

and if  $k_\mu k^\mu \neq -m^2$ , then

$$\det(ik_\mu C^\mu + m1) \neq 0$$

We are free to choose a transformation:

$$\varphi = S \varphi'$$

where  $S$  is a  $N \times N$  real matrix satisfying  $\det S \neq 0$ . Then

$$(S^{-1} C^\mu S \partial_\mu + m) \varphi' = 0$$

Search for theory:

- $N = 1$ : impossible
- $N = 2$ : impossible
- $N = 3$ : impossible
- $N = 4$ : there is one theory called **Majorana theory**

In this theory,

$$C^0 = \begin{pmatrix} & & -1 \\ & +1 & \\ +1 & & \end{pmatrix}, \quad C^1 = \begin{pmatrix} & +1 & \\ +1 & & \\ & -1 & -1 \end{pmatrix}$$

$$C^2 = \begin{pmatrix} +1 & & \\ & -1 & \\ & & +1 \\ & & & -1 \end{pmatrix}, \quad C^3 = \begin{pmatrix} & & -1 \\ & & \\ -1 & & \\ & -1 & \end{pmatrix}$$

with the **Majorana equation**

$$(C^\mu \partial_\mu + m)\varphi = 0$$

The C-matrices has some properties,

$$\frac{1}{2}\{C^\mu, C^\nu\} = g^{\mu\nu}1 \implies \frac{1}{2}\{S^{-1}C^\mu S, S^{-1}C^\nu S\} = g^{\mu\nu}1$$

and

$$\det(ik_\mu C^\mu + m1) = (k_\mu k^\mu + m^2)^2$$

The Majorana equation can be turned in to KG eq:

$$(c^\nu \partial_\nu - m)(c^\mu \partial_\mu - m) = 0 \implies (g^{\mu\nu} \partial_\nu \partial_\mu - m^2)\varphi = 0 \implies (\partial^2 - m^2)\varphi_a = 0, \quad 1 \leq a \leq 4$$

Consider the transformation. For scalar field,

$$\bar{\varphi}(x) = \varphi(\Lambda^{-1}x)$$

and for Majorana field,

$$\bar{\varphi}(x) = L(\Lambda)\varphi(\Lambda^{-1}x)$$

where  $L(\Lambda)$  is  $4 \times 4$  real matrix.

If  $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu$ , then  $L = 1$ .

The  $L(\Lambda)$  must satisfy

$$L(\Lambda_2)L(\Lambda_1) = L(\Lambda_2\Lambda_1)$$

which means  $L(\Lambda)$  form a 4-dimensional representation of proper orthochronous Lorentz group.

Consider infinitesimal transformation

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \theta^\mu{}_\nu, \quad \theta^\mu{}_\nu \text{ infinitesimal}$$

and we express  $L$  as

$$L(\delta^\mu{}_\nu + \theta^\mu{}_\nu) = 1 + \frac{i}{2}\theta_{\mu\nu}S^{\mu\nu}$$

where

$$\theta_{\mu\nu} \equiv g_{\mu\rho}\theta^\rho{}_\nu$$

and  $S^{\mu\nu}$  are  $4 \times 4$  matrices, where  $S_{ab}^{\mu\nu}$  ( $1 \leq a, b \leq 4$ ) are purely imaginary or zero.

And since for Lorentz infinitesimal transformations

$$\theta_{\mu\nu} = -\theta_{\nu\mu}$$

we have

$$S^{\mu\nu} = -S^{\nu\mu}$$

hence only 6 independent components:  $S^{0i}, S^{12}, S^{23}, S^{31}$ .

Consider the Majorana eq for  $\bar{\varphi}$ . We see that

$$(C^\mu \partial_\mu + m)\bar{\varphi}(x) = L(\Lambda) \left[ (\Lambda^{-1})^\mu{}_\nu L^{-1}(\Lambda) C^\nu L(\Lambda) \frac{\partial \varphi(y)}{\partial y^\mu} + m\varphi(y) \right]$$

If

$$(\Lambda^{-1})^\mu{}_\nu L^{-1}(\Lambda) C^\nu L(\Lambda) = C^\mu$$

then  $\bar{\varphi}(x)$  satisfies Majorana eq. Or to re-express as

$$L^{-1}(\Lambda) C^\mu L(\Lambda) = \Lambda^\mu{}_\nu C^\nu$$

hence we get

$$[C^\mu, iS^{\rho\sigma}] = g^{\mu\rho} C^\sigma - g^{\mu\sigma} C^\rho$$

However, these equations cannot uniquely decide  $S^{\rho\sigma}$ , since

$$[C^\mu, S^{\rho\sigma} - S'^{\rho\sigma}] = 0 \implies S^{\rho\sigma} - S'^{\rho\sigma} = b1$$

If we impose  $\text{Tr } S^{\rho\sigma} = 0$  then we obtain

$$S^{\mu\nu} = -\frac{i}{4} [C^\mu, C^\nu]$$

with commutators

$$[S^{\mu\nu}, S^{\rho\sigma}] = i[g^{\mu\rho} S^{\nu\sigma} - (\mu \leftrightarrow \nu)] - (\rho \leftrightarrow \sigma)$$

And we find that

$$S^{ij} (i \neq j) : \text{eigenvalues } \pm \frac{1}{2}, \pm \frac{1}{2} \quad S^{0i} (i = 1, 2, 3) : \text{eigenvalues } \pm \frac{i}{2}, \pm \frac{i}{2}$$

Now we consider  $\varphi_a(x)$ :

$$\varphi_a^\dagger(x) = \varphi_a(x), \quad 1 \leq a \leq 4$$

with transformation

$$\bar{\varphi}_a(x) = \mathcal{U}(\Lambda)^{-1} \varphi_a(x) \mathcal{U}(\Lambda) = \sum_{b=1}^4 L_{ab}(\Lambda) \varphi_b(\Lambda^{-1}x)$$

where  $\mathcal{U}(\Lambda)$  satisfies

$$\mathcal{U}(\Lambda)\mathcal{U}(\Lambda') = \mathcal{U}(\Lambda\Lambda')$$

And for infinitesimal transformations

$$\mathcal{U}(1 + \theta) = 1 + \frac{i}{2} \theta_{\mu\nu} M^{\mu\nu}, \quad M^{\mu\nu} = -M^{\nu\mu}$$

then we define

$$M^{12} = J_3, \quad M^{23} = J_2, \quad M^{31} = J_1, \quad K_i = M^{i0}$$

with commutators

$$\begin{cases} [J_i, J_j] = +i\varepsilon_{ijk} J_k \\ [J_i, K_j] = +i\varepsilon_{ijk} K_k \\ [K_i, K_j] = -i\varepsilon_{ijk} J_k \end{cases}$$

thus

$$[\varphi_a(x), M^{\mu\nu}] = \frac{1}{i}(x^\mu \partial^\nu - x^\nu \partial^\mu) \varphi_a(x) + \sum_b S_{ab}^{\mu\nu} \varphi_b(x)$$

where the last term is for Majorana field, while for scalar field it is zero.

Hence for  $J_3$ ,

$$[\varphi_a, J_3] = \frac{1}{i}(x^1 \partial^2 - x^2 \partial^1) \varphi_a(x) + \sum_b S_{ab}^{12} \varphi_b(x)$$

Define

$$\begin{cases} O_1(t) \equiv \int d^3x (\varphi_1(x) + i\varphi_2(x)) \\ O_1^\dagger(t) \equiv \int d^3x (\varphi_1(x) - i\varphi_2(x)) \\ O_2(t) \equiv \int d^3x (\varphi_3(x) + i\varphi_4(x)) \\ O_2^\dagger(t) \equiv \int d^3x (\varphi_3(x) - i\varphi_4(x)) \end{cases}$$

we can see that

$$[J_3, O_i(t)] = -\frac{1}{2} O_i(t)$$

and define new fields

$$\psi_L = \begin{pmatrix} \varphi_1 + i\varphi_2 \\ \varphi_3 + i\varphi_4 \end{pmatrix}, \quad \psi_R = \begin{pmatrix} \varphi_1 - i\varphi_2 \\ \varphi_3 - i\varphi_4 \end{pmatrix}$$

then

$$\begin{aligned} \mathcal{U}(\Lambda)^{-1} \psi_L(x) \mathcal{U}(\Lambda) &= L_L(\Lambda) \psi_L(\Lambda^{-1}x), \quad L_L(1+\theta) = 1 + \frac{i}{2} \theta_{\mu\nu} S_L^{\mu\nu} \\ \mathcal{U}(\Lambda)^{-1} \psi_R(x) \mathcal{U}(\Lambda) &= L_R(\Lambda) \psi_R(\Lambda^{-1}x), \quad L_R(1+\theta) = 1 + \frac{i}{2} \theta_{\mu\nu} S_R^{\mu\nu} \end{aligned}$$

with

$$L_R(\Lambda) = L_L(\Lambda)^*, \quad S_L^{\mu\nu} = S_R^{\mu\nu*}$$

Moreover we define

$$N_i = \frac{1}{2}(J_i - iK_i), \quad N_i^\dagger = \frac{1}{2}(J_i + iK_i)$$

then

$$\begin{cases} [N_i, N_j] = i\varepsilon_{ijk} N_k \\ [N_i^\dagger, N_j^\dagger] = i\varepsilon_{ijk} N_k^\dagger \\ [N_i, N_j^\dagger] = 0 \end{cases}$$

Now we consider another approach. Going back to the Majorana equation,

$$(C^\mu \partial_\mu + m)\varphi = 0, \quad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix}$$

with

$$\varphi_a^\dagger = \varphi_a$$

Assume

$$[\varphi_a(x), \varphi_b(y)]_\sigma = f_{ab}(x-y), \quad 1 \leq a, b \leq 4$$

with the commutator

$$[A, B]_\sigma = AB - \sigma BA, \quad \sigma = \pm 1$$

And assume

$$f_{ab}(x - y) = 0 \quad \text{if} \quad (x - y)^2 > 0$$

Consider the FT and IFT

$$\tilde{f}_{ab}(k) = \int d^4x f_{ab}(x) e^{-ik \cdot x}, \quad f_{ab}(x) = \int \frac{d^4k}{(2\pi)^4} \tilde{f}_{ab}(k) e^{ik \cdot x}$$

It could be proved that

$$\tilde{f}_{ab}(k)^* = \tilde{f}_{ba}(k)$$

Consider the commutator between  $\varphi$  and Majorana equation,

$$[C_{ac}^\mu \partial_\mu \varphi_c(x) + m \varphi_a(x), \varphi_b(0)]_\sigma = 0$$

then

$$C_{ac}^\mu \partial_\mu f_{cb}(x) + m f_{ab}(x) = 0$$

or in matrix notation

$$C^\mu \partial_\mu f(x) + m f(x) = 0 \implies (ik_\mu C^\mu + m1) \tilde{f}(k) = 0$$

But

$$\det(ik_\mu C^\mu + m1) = (k^2 + m^2)^2$$

with the transformation rule

$$\mathcal{U}(\Lambda)^{-1} \varphi_a(x) \mathcal{U}(\Lambda) = L(\Lambda)_{ab} \varphi_b(\Lambda^{-1}x)$$

Particularly for scalar field  $\phi(x)$ , we have

$$\phi(a)|s\rangle = \lambda|s\rangle \implies \phi(\Lambda a)[\mathcal{U}(\Lambda)|s\rangle] = \lambda[\mathcal{U}(\Lambda)|s\rangle] \implies \mathcal{U}(\Lambda)^{-1} \phi(\Lambda a) \mathcal{U}(\Lambda)|s\rangle = \lambda|s\rangle = \phi(a)|s\rangle$$

thus

$$\mathcal{U}(\Lambda)^{-1} \phi(\Lambda a) \mathcal{U}(\Lambda) = \phi(a), \quad \mathcal{U}(\Lambda)^{-1} \phi(x) \mathcal{U}(\Lambda) = \phi(\Lambda^{-1}x)$$

Since  $f_{ab}(x)$  is a c-number we have

$$f_{ab}(x) = \mathcal{U}(\Lambda)^{-1} f_{ab}(x) \mathcal{U}(\Lambda) = \mathcal{U}(\Lambda)^{-1} [\varphi_a(x), \varphi_b(0)]_\sigma \mathcal{U}(\Lambda) = [\mathcal{U}(\Lambda)^{-1} \varphi_a(x) \mathcal{U}(\Lambda), \mathcal{U}(\Lambda)^{-1} \varphi_b(0) \mathcal{U}(\Lambda)]_\sigma = 0$$

thus

$$f_{ab}(x) = [L_{ac}(\Lambda) \varphi_c(\Lambda^{-1}x), L_{bd}(\Lambda) \varphi_d(0)]_\sigma = L_{ac}(\Lambda) f_{cd}(\Lambda^{-1}x) L_{db}^T(\Lambda)$$

or in matrix notation

$$f(x) = L(\Lambda) f(\Lambda^{-1}x) L^T(\Lambda) \implies f(\Lambda y) = L(\Lambda) f(y) L^T(\Lambda) \implies \tilde{f}(\Lambda k) = L(\Lambda) \tilde{f}(k) L^T(\Lambda)$$

Consider

$$k^\mu = (k^0, 0, 0, 0)$$

then

$$\tilde{f}(k^0, 0, 0, 0) = 2\pi \delta(k^{0^2} - m^2) \cdot A$$

where  $A$  depends on  $\text{sgn}(k^0)$ . The infinitesimal rotation

$$L(\Lambda) = L(\theta_1, \theta_2, \theta_3) = 1 + i\theta_1 S^{23} + i\theta_2 S^{13} + i\theta_3 S^{12}$$

and the requirement is that  $\tilde{f}$  doesn't change,

$$\tilde{f}(k^0, 0, 0, 0) = (1 + i\theta_1 S^{23} + i\theta_2 S^{13} + i\theta_3 S^{12}) \tilde{f}(k^0, 0, 0, 0) (1 + i\theta_1 S^{23} + i\theta_2 S^{13} + i\theta_3 S^{12})^T$$



i.e.

$$\theta_1(S^{23}\tilde{f} + \tilde{f}S^{23T}) + \theta_2(S^{31}\tilde{f} + \tilde{f}S^{31T}) + \theta_3(S^{12}\tilde{f} + \tilde{f}S^{12T}) = 0$$

for all  $\theta_1, \theta_2, \theta_3$ . Thus

$$S^{23}\tilde{f} + \tilde{f}S^{23T} = 0, \quad S^{31}\tilde{f} + \tilde{f}S^{31T} = 0, \quad S^{12}\tilde{f} + \tilde{f}S^{12T} = 0$$

which requires

$$\tilde{f}(k^0, 0, 0, 0) = 2\pi\delta(k^{0^2} - m^2)(\xi_0 1 + \xi_1 C^0 + \xi_2 C^1 C^2 C^3 + \xi_3 C^0 C^1 C^2 C^3)$$

By using

$$(ik_\mu C^\mu + m1)\tilde{f}(k) = 0$$

we obtain that

$$\xi_1 = i\xi_0 \operatorname{sgn}(k^0), \quad \xi_3 = i\xi_2 \operatorname{sgn}(k^0)$$

and by using

$$\tilde{f}(k)^\dagger = \tilde{f}(k)$$

we have

$$\xi_0^* = \xi_0, \quad \xi_i^* = -\xi_i, \quad 1 \leq i \leq 3$$

But using  $\xi_3 = i\xi_2 \operatorname{sgn}(k^0)$  we obtain

$$\xi_2 = \xi_3 = 0$$

Hence

$$\tilde{f}(k^0, 0, 0, 0) = 2\pi\delta(k^{0^2} - m^2)[\theta(k^0)\alpha(1 + iC^0) + \theta(-k^0)\beta(1 - iC^0)]$$

By using

$$\tilde{f}(\Lambda k) = L(\Lambda)\tilde{f}(k)L(\Lambda)^T$$

we could get the result for arbitrary  $k$ :

$$\tilde{f}(k) = 2\pi\delta(k^2 + m^2)\frac{\alpha\theta(k^0) - \beta\theta(-k^0)}{m}(\not{k} + im)C^0, \quad \not{k} = k_\mu C^\mu$$

Using IFT we get

$$f(x) = \left(-\frac{i\alpha}{m}\frac{\partial I(x)}{\partial x^\mu} + i\beta\frac{\partial I(-x)}{\partial x^\mu}\right)C^\mu C^0 + [i\alpha I(x) - i\beta I(-x)]C^0$$

where

$$I(x) = \int \widetilde{\mathrm{d}p} e^{ipx} = \frac{m}{4\pi^2} \frac{\mathrm{K}_1[m\sqrt{|\vec{x}|^2 - (x^0 - i\varepsilon)^2}]}{\sqrt{|\vec{x}|^2 - (x^0 - i\varepsilon)^2}}$$

and that  $I(-x) = I(x)$  if  $x^2 > 0$ . Since

$$f(x) = 0 \quad \text{if} \quad x^2 > 0$$

we have

$$\alpha = \beta$$

Hence

$$\begin{aligned} \tilde{f}(k) &= \frac{\alpha}{m} 2\pi\delta(k^2 + m^2)(\not{k} + im)C^0 \\ f(x) &= -\frac{2\alpha}{m} \frac{\partial I_2(x)}{\partial x^\mu} C^\mu C^0 + 2\alpha I_2(x) C^0 \end{aligned}$$

with

$$I_2(x) = -\text{Im } I(x)$$

which is nonzero if  $x^2 \leq 0$ , and  $I_2(-x) = -I_2(x)$ .

Since  $C^\mu C^0$  is symmetric and  $C^0$  is antisymmetric, we have

$$f(-x) = f(x)^T$$

or

$$f_{ba}(y-x) = f_{ab}(x-y)$$

or returning to  $\varphi$ ,

$$[\varphi_b(y), \varphi_a(x)]_\sigma = [\varphi_a(x), \varphi_b(y)]_\sigma \implies (1+\sigma)[\varphi_a(x)\varphi_b(y) - \varphi_b(y)\varphi_a(x)] = 0$$

In order to avoid classical field, we require

$$\sigma = -1$$

namely,

$$\varphi_a(x)\varphi_b(y) + \varphi_b(y)\varphi_a(x) = f_{ab}(x-y)$$

or written as

$$\{\varphi_a(x), \varphi_b(y)\} = f_{ab}(x-y)$$

i.e.

$$\{\varphi_a(x), \varphi_b(0)\} = -\frac{2\alpha}{m} \frac{\partial I_2(x)}{\partial x^\mu} (C^\mu C^0)_{ab} + 2\alpha I_2(x) C_{ab}^0$$

Since

$$I_2(t, \vec{x}) \rightarrow 0\delta(\vec{x}), \frac{\partial I_2(t, \vec{x})}{\partial t} \rightarrow \frac{1}{2}\delta(\vec{x}), \frac{\partial I_2(t, \vec{x})}{\partial x^i} \rightarrow 0\delta(x), \quad t \rightarrow \pm 0$$

thus

$$\{\varphi_a(0, \vec{x}), \varphi_b(0, \vec{0})\} = -\frac{\alpha}{m} \delta(\vec{x}) (C^0 C^0)_{ab}$$

Using  $C^0 C^0 = -1$  and considering  $\vec{x}, \vec{y}$  we obtain

$$\{\varphi_a(t, \vec{x}), \varphi_b(t, \vec{x})\} = \frac{\alpha}{m} \delta(\vec{x} - \vec{y}) \delta_{ab}$$

But  $\varphi_a(x)$  is hermitian, leading to  $\varphi_a(x)^2$  positive definite, thus we require

$$\alpha > 0$$

Choosing amplitude of  $\varphi(x)$  such that  $\alpha = m$ , we obtain that

$$\{\varphi_a(t, \vec{x}), \varphi_b(t, \vec{y})\} = \delta_{ab} \delta(\vec{x} - \vec{y})$$

Going back to Majorana eq,

$$(C^\mu \partial_\mu + m)\varphi = 0$$

Multiplying by  $C^0$ ,

$$C^0(C^0 \partial_0 + C^i \partial_i + m)\varphi = 0$$

simple calculating we have

$$\frac{\partial \varphi}{\partial t} = (C^0 C^i \partial_i + m C^0)\varphi, \quad \frac{\partial \varphi_a}{\partial t} = (C^0 C^i)_{ab} \partial_i \varphi_b + m C_{ab}^0 \varphi_b$$

We guess the corresponding Hamiltonian has the form

$$H(t) = \int d^3x \eta \varphi_a(x) [(C^0 C^i)_{ab} \partial_i \varphi_b + m C_{ab}^0 \varphi_b]$$

from which we get

$$i[H(t), \varphi_a(x)] = -2i\eta [(C^0 C^i)_{ab} \partial_i \varphi_b + m C_{ab}^0 \varphi_b]$$

but we require

$$\frac{\partial \varphi_a(x)}{\partial t} = i[H(t), \varphi_a(x)]$$

therefore

$$-2i\eta = 1 \implies \eta = \frac{i}{2}$$

and the Hamiltonian

$$H = \frac{i}{2} \int d^3x \varphi^T C^0 (C^i \partial_i + m1) \varphi, \quad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix}$$

from which we see

$$H^\dagger = H$$

For momentum, we guess

$$p^i = \eta^i \int d^3x \varphi_a \partial_i \varphi_a$$

then from

$$-i[p^i, \varphi_a] = \partial_i \varphi_a$$

we get

$$\eta^i = -\frac{i}{2}$$

therefore

$$p^i(t) = -\frac{i}{2} \int d^3x \varphi_a(t, \vec{x}) \partial_i \varphi_a(t, \vec{x})$$

and we see that

$$p^{i\dagger} = p^i$$

We now consider the FT of  $\varphi$ , with

$$\varphi(x) = \int \frac{d^4k}{(2\pi)^4} \tilde{\varphi}(k) e^{ik \cdot x}$$

from Majorana eq we have

$$(ik_\mu C^\mu + m1) \tilde{\varphi}(k) = 0$$

with

$$\varphi_a(x)^\dagger = \varphi_a(x) \implies \tilde{\varphi}_a(k)^\dagger = \tilde{\varphi}_a(-k)$$

Considering on-shell scheme, we rewrite  $k$  as  $\vec{p}$ ,

$$(ip_\mu C^\mu + m1) \tilde{\varphi}(\vec{p}) = 0$$

and we denote

$$u_s(\vec{p}), \quad s = \pm 1$$

with  $s = \pm 1$  representing upper and lower half of on-shell scheme respectively. Hence we can express  $\tilde{\varphi}$  as

$$\tilde{\varphi}(p) = 2\pi\delta(p^2 + m^2)[b_+(\vec{p})u_+(\vec{p}) + b_-(\vec{p})u_-(\vec{p})], \quad \text{if } p^0 > 0$$

or extending

$$\tilde{\varphi}(p) = 2\pi\delta(p^2 + m^2)[\theta(p^0) \sum_{s=\pm} b_s(\vec{p})u_s(\vec{p}) + \theta(-p^0) \sum_{s=\pm} b_s^\dagger(-\vec{p})u_s^*(-\vec{p})]$$

hence

$$\varphi(x) = \sum_{s=\pm} \int \widetilde{d\vec{p}} [b_s(\vec{p})u_s(\vec{p})e^{ip \cdot x} + b_s^\dagger(\vec{p})u_s^*(\vec{p})e^{-ip \cdot x}]$$

with

$$(i\not{p} + m)u_s(\vec{p}) = 0$$

Particularly for  $\vec{p} = 0$ ,

$$S_3 u_s(\vec{0}) = \frac{s}{2} u_s(\vec{0})$$

with

$$S_3 = S_{12} = -\frac{i}{4}[C^1, C^2]$$

thus we choose

$$u_+(\vec{0}) = \sqrt{\frac{m}{2}} \begin{pmatrix} +1 \\ -i \\ +1 \\ +i \end{pmatrix}, \quad u_-(\vec{0}) = \sqrt{\frac{m}{2}} \begin{pmatrix} -1 \\ -i \\ +1 \\ -i \end{pmatrix}$$

For general  $\vec{p}$ , consider transformation

$$\Lambda : p'^\mu = \Lambda^\mu_\nu p^\nu \implies p'_\mu = (\Lambda^{-1})^\nu_\mu p_\nu$$

thus

$$\begin{aligned} L^{-1}(\Lambda)(ip'_\nu C^\nu + m)L(\Lambda)u &= [iL^{-1}(\Lambda)p'_\nu C^\nu L(\Lambda) + m]u \\ &= [ip_\mu \underbrace{L^{-1}(\Lambda)(\Lambda^{-1})^\mu_\nu C^\nu L(\Lambda)}_{C^\mu} + m]u \\ &= (ip_\mu C^\mu + m)u \end{aligned}$$

meaning that if

$$(ip_\mu C^\mu + m)u = 0$$

then

$$(ip'_\mu C^\mu + m)L(\Lambda)u = 0$$

thus if

$$(i\not{p} + m)u_s(\vec{0}) = 0$$

for  $p = (m, \vec{0})$ , then

$$(i\not{p}' + m)L(\Lambda)u_s(\vec{0}) = 0$$

for  $p' = \Lambda p$ , with

$$u_s(\vec{p}) = L(\Lambda)u_s(\vec{0})$$

Consider general  $p$  with

$$p = \begin{pmatrix} m \cosh \eta \\ m \sinh \eta \sin \theta \cos \phi \\ m \sinh \eta \sin \theta \sin \phi \\ m \sinh \eta \cos \theta \end{pmatrix}, \quad \eta > 0$$

then for the boost part,

$$L(\Lambda) = e^{i\eta \sin \theta \cos \phi S^{10} + i\eta \sin \theta \sin \phi S^{20} + i\eta \cos \theta S^{30}}$$

hence

$$u_+(\vec{p}) = \frac{1}{2\sqrt{m+p^0}} \begin{pmatrix} m+p^0-p^1-ip^2-p^3 \\ -i(m+p^0+p^1+ip^2-p^3) \\ m+p^0-p^1-ip^2+p^3 \\ i(m+p^0+p^1+ip^2+p^3) \end{pmatrix}, \quad u_-(\vec{p}) = \frac{1}{2\sqrt{m+p^0}} \begin{pmatrix} -(m+p^0-p^1-ip^2-p^3) \\ -i(m+p^0-p^1+ip^2-p^3) \\ m+p^0+p^1-ip^2+p^3 \\ -i(m+p^0-p^1+ip^2+p^3) \end{pmatrix}$$

and we can prove that

$$\not{p}u_s(\vec{p}) = imu_s(\vec{p}), \quad \not{p}u_s^*(\vec{p}) = -imu_s^*(\vec{p})$$

and by introducing

$$\bar{u} \equiv u^\dagger iC^0, \quad C_5 \equiv C^0 C^1 C^2 C^3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \end{pmatrix}$$

we have

$$\bar{u}_s(\vec{p})\not{p} = im\bar{u}_s(\vec{p})$$

$$u_{s'}^\dagger(\vec{p})u_s(\vec{p}) = 2p^0\delta_{s's}, \quad u_{s'}^\dagger(\vec{p})iC^0u_s(\vec{p}) = u_{s'}^*(\vec{p})u_s(\vec{p}) = 0, \quad \bar{u}_{s'}(\vec{p})u_s(\vec{p}) = 2m\delta_{s's}$$

and

$$C_5^2 = -1, \quad \{C_5, C^\mu\} = 0$$

and that

$$\{X, C^\mu\} = 0 \implies X \propto C_5$$

also

$$C_5 u_s(\vec{p}) = -isu_{-s}^*(\vec{p})$$

and

$$C_5, S^{\mu\nu} = 0, \quad [C_5, C^\mu C^\nu] = 0, \quad C_5 C^\mu C^\nu = -C^\mu C_5 C^\nu = +C^\mu C^\nu C_5 \\ 2m\bar{u}_{s'}(\vec{p}')C^\mu u_s(\vec{p}) = -i\bar{u}_{s'}(\vec{p}')[(p' + p)^\mu - 2iS^{\mu\nu}(p' - p)_\nu]u_s(\vec{p})$$

It can be shown that

$$\int d^3x e^{-ip \cdot x} u_s^\dagger(\vec{p}) \varphi(x) = b_s(\vec{p}), \quad \int d^3x e^{+ip \cdot x} u_s^T(\vec{p}) \varphi(x) = b_s^\dagger(\vec{p})$$

and

$$\{b_s(\vec{p}), b_{s'}(\vec{p}')\} = 0, \quad \{b_s(\vec{p}), b_{s'}^\dagger(\vec{p}')\} = 2\omega_{\vec{p}}(2\pi)^3 \delta(\vec{p} - \vec{p}') \delta_{ss'}$$

The results we obtained before

$$H = p^0 = \frac{i}{2} \int d^3x \varphi^T \underbrace{C^0 (C^i \partial_i + m)}_{\not{\varphi}} \varphi = -\frac{i}{2} \int d^3x \varphi^T \partial^0 \varphi \\ p^i = -\frac{i}{2} \int d^3x \varphi^T \partial^i \varphi$$

To generalize,

$$p^\mu = -\frac{i}{2} \int d^3x \varphi^T \partial^\mu \varphi = \sum_s \int \widetilde{dp} \frac{p^T}{2} [b_s^\dagger(\vec{p}) b_s(\vec{p}) - b_s(\vec{p}) b_s^\dagger(\vec{p})] = Const + \sum_s \int \widetilde{dp} p^\mu b_s^\dagger(\vec{p}) b_s(\vec{p})$$

Also,

$$b_s(\vec{p})|0\rangle = 0 \text{ for all } (s, \vec{p})$$

with the convention

$$\langle 0|0\rangle = 1$$

we can denote

$$b_s^\dagger(\vec{p})|0\rangle : \text{single-particle state}$$

$$b_{s_1}^\dagger(\vec{p}_1) b_{s_2}^\dagger(\vec{p}_2)|0\rangle : \text{2-particle states}$$

$$b_{s_1}^\dagger(\vec{p}_1) b_{s_2}^\dagger(\vec{p}_2) b_{s_3}^\dagger(\vec{p}_3)|0\rangle : \text{3-particle states}$$

Consider  $N$  Majorana fields

$$\varphi_a^{(1)}(x), \quad \dots, \varphi_a^{(N)}(x) \quad 1 \leq a \leq 4$$

which have mass  $m_1, \dots, m_N$  respectively. The Majorana eq

$$(C^\mu \partial_\mu + m_i) \varphi^{(i)} = 0, \quad 1 \leq i \leq N$$

If  $N = 2$  and  $m_1 = m_2 = m$ , it is called **Dirac field**. In this condition we define

$$\varphi_a^{(1)}(x) \equiv \varphi_a(x), \quad \varphi_a^{(2)}(x) \equiv \zeta_a(x)$$

then the Majorana eq

$$\begin{cases} (C^\mu \partial_\mu + m_i) \varphi(x) = 0 \\ (C^\mu \partial_\mu + m_i) \zeta(x) = 0 \end{cases}$$

which belongs to **Dirac theory**.

The commutator

$$\{\varphi_a(t, \vec{x}), \varphi_b(t, \vec{y})\} = \delta_{ab} \delta(\vec{x} - \vec{y}), \quad \{\zeta_a(t, \vec{x}), \zeta_b(t, \vec{y})\} = \delta_{ab} \delta(\vec{x} - \vec{y}), \quad \{\varphi_a(t, \vec{x}), \zeta_b(t, \vec{y})\} = 0$$

Combining the Majorana eqs:

$$(C^\mu \partial_\mu + m) \frac{\varphi(x) + i\zeta(x)}{\sqrt{2}} = 0$$

Define

$$\Psi(x) \equiv \frac{\varphi(x) + i\zeta(x)}{\sqrt{2}}$$

then we have the **Dirac equation**

$$(C^\mu \partial_\mu + m) \Psi(x) = 0$$

and we define

$$\not{\partial} \equiv C^\mu \partial_\mu, \quad \not{p} \equiv C^\mu p_\mu$$

then the eq becomes

$$(\not{\partial} + m) \Psi = 0$$

with the commutators

$$\{\Psi_a(t, \vec{x}), \Psi_b(t, \vec{y})\} = 0, \quad \{\Psi_a^\dagger(t, \vec{x}), \Psi_b^\dagger(t, \vec{y})\} = 0, \quad \{\Psi_a(t, \vec{x}), \Psi_b^\dagger(t, \vec{y})\} = \delta_{ab} \delta(\vec{x} - \vec{y})$$

and we have the expansion

$$\varphi(x) = \sum_{s=\pm} \int \widetilde{d\vec{p}} \left[ b_s^{(\varphi)}(\vec{p}) u_s(\vec{p}) e^{ipx} + b_s^{(\varphi)\dagger}(\vec{p}) u_s^*(\vec{p}) e^{-ipx} \right]$$

$$\zeta(x) = \sum_{s=\pm} \int \widetilde{d\vec{p}} \left[ b_s^{(\zeta)}(\vec{p}) u_s(\vec{p}) e^{ipx} + b_s^{(\zeta)\dagger}(\vec{p}) u_s^*(\vec{p}) e^{-ipx} \right]$$

and

$$\Psi(x) = \frac{1}{\sqrt{2}}(\varphi + i\zeta) = \sum_{s=\pm} \int \widetilde{d\vec{p}} \left[ b_s(\vec{p}) u_s(\vec{p}) e^{ipx} + d_s^\dagger(\vec{p}) u_s^*(\vec{p}) e^{-ipx} \right]$$

where

$$b_s(\vec{p}) = \frac{1}{\sqrt{2}}[b_s^{(\varphi)}(\vec{p}) + i b_s^{(\zeta)}(\vec{p})], \quad d_s(\vec{p}) = \frac{1}{\sqrt{2}}[b_s^{(\varphi)}(\vec{p}) - i b_s^{(\zeta)}(\vec{p})]$$

Reversely we have expansions for  $b_s^{(\varphi)}, b_s^{(\zeta)}$

$$b_s^{(\varphi)}(\vec{p}) = \int d^3x e^{-ipx} u_s^\dagger(\vec{p}) \varphi(x), \quad b_s^{(\zeta)}(\vec{p}) = \int d^3x e^{-ipx} u_s^\dagger(\vec{p}) \zeta(x)$$

thus

$$b_s(\vec{p}) = \int d^3x e^{-ipx} u_s^\dagger(\vec{p}) \frac{\varphi + i\zeta}{\sqrt{2}} = \int d^3x e^{-ipx} u_s^\dagger(\vec{p}) \Psi(x)$$

$$d_s(\vec{p}) = \int d^3x e^{-ipx} u_s^\dagger(\vec{p}) \frac{\varphi - i\zeta}{\sqrt{2}} = \int d^3x e^{-ipx} \frac{\varphi^T - i\zeta^T}{\sqrt{2}} u_s^*(\vec{p}) = \int d^3x e^{-ipx} \Psi^\dagger(x) u_s^*(\vec{p})$$

with commutators

$$\{b_s(\vec{p}), b_{s'}(\vec{p}')\} = 0, \quad \{b_s(\vec{p}), d_{s'}^\dagger(\vec{p}')\} = 0$$

$$\{b_s(\vec{p}), b_{s'}^\dagger(\vec{p}')\} = 2\omega_p(2\pi)^3 \delta(\vec{p} - \vec{p}') \delta_{ss'}, \quad \{d_s(\vec{p}), d_{s'}^\dagger(\vec{p}')\} = 2\omega_p(2\pi)^3 \delta(\vec{p} - \vec{p}') \delta_{ss'}$$

Consider the momentum and Hamiltonian:

$$P^\mu = P^{(\varphi)\mu} + P^{(\zeta)\mu}$$

we have

$$H \equiv P^0 = \int d^3x \Psi^\dagger i C^0 (C^i \partial_i + m) \Psi = -i \int d^3x \Psi^\dagger \partial^0 \Psi$$

$$P^i = -i \int d^3x \Psi^\dagger \partial^i \Psi$$

Or to summarize,

$$P^\mu = -i \int d^3x \Psi^\dagger \partial^\mu \Psi$$

Or in terms of  $b_s, d_s$ :

$$P^\mu = \sum_{s=\pm} \int \widetilde{d\vec{p}} p^\mu [b_s^\dagger(\vec{p}) b_s(\vec{p}) - d_s(\vec{p}) d_s^\dagger(\vec{p})] = \text{Const} + \sum_{s=\pm} \int \widetilde{d\vec{p}} p^\mu [b_s^\dagger(\vec{p}) b_s(\vec{p}) + d_s^\dagger(\vec{p}) d_s(\vec{p})]$$

Define vacuum states as

$$b_s(\vec{p})|0\rangle = d_s(\vec{p})|0\rangle = 0, \quad \langle 0|0\rangle = 1$$

Consider

$$Q \equiv \int d^3x \Psi^\dagger \Psi$$

We can see

$$[Q, \Psi] = -\Psi, \quad [Q, \Psi^\dagger] = +\Psi^\dagger$$

and

$$e^{i\alpha Q} \Psi e^{-i\alpha Q} = e^{-i\alpha} \Psi, \quad e^{i\alpha Q} \Psi^\dagger e^{-i\alpha Q} = e^{+i\alpha} \Psi^\dagger$$

also

$$e^{i\alpha Q} H e^{-i\alpha Q} = e^{-i\alpha} H \implies [Q, H] = 0$$

which means  $Q$  is a conserved charge. Using  $b_s, d_s$  we have

$$Q = \sum_{s=\pm} \int \widetilde{d}p [b_s^\dagger(\vec{p}) b_s(\vec{p}) + d_s(\vec{p}) d_s^\dagger(\vec{p})] = \sum_{s=\pm} \int \widetilde{d}p [b_s^\dagger(\vec{p}) b_s(\vec{p}) - d_s^\dagger(\vec{p}) d_s(\vec{p})] + \text{Const}$$

The classical electromagnetic field satisfies

$$\begin{cases} \nabla \cdot \vec{E} = \rho \\ \nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{J} \\ \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \end{cases}$$

Introduce the field tensor

$$F_{\mu\nu} : \quad F_{00} = 0, F_{i0} = -F_{0i} = E_i; \quad F_{ij} = \varepsilon_{ijk} B_k$$

which is anti-symmetric:

$$F_{\mu\nu} = -F_{\nu\mu}$$

then the first two Maxwell eqs become

$$\partial_\nu F^{\mu\nu} = J^\mu$$

where

$$J^\mu \equiv (\rho, \vec{J})$$

and the last two become

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0$$

Now we quantize the fields, and we require  $F_{\mu\nu}$  to be Hermitian:

$$F_{\mu\nu}(x)^\dagger = F_{\mu\nu}(x)$$

Free EM-field:

$$\begin{cases} \partial_\nu F^{\mu\nu} = 0 \\ \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0 \end{cases}$$

Now we define

$$[F^{\mu\nu}(x), F^{\rho\sigma}(y)]_s = f^{\mu\nu\rho\sigma}(x-y)$$

where

$$[A, B]_s = AB - sBA$$

and we assume

$$f^{\mu\nu\rho\sigma}(x) = 0 \quad \text{if} \quad x^2 > 0$$

with the antisymmetry of  $F$  we have

$$f^{\mu\nu\rho\sigma}(x) = -f^{\nu\mu\rho\sigma}(x) = -f^{\mu\nu\sigma\rho}(x)$$



and

$$f^{\mu\nu\rho\sigma*}(x) = f^{\rho\sigma\mu\nu}(-x)$$

Also using free EM-field eqs we have

$$\begin{aligned}\partial_\nu f^{\mu\nu\rho\sigma}(x) &= 0, & \partial_\sigma f^{\mu\nu\rho\sigma}(x) &= 0 \\ \partial^\alpha f^{\mu\nu\rho\sigma}(x) + \overleftarrow{\alpha\mu\nu} + \overleftarrow{\alpha\mu\nu} &= 0, & \partial^\alpha f^{\mu\nu\rho\sigma}(x) + \overleftarrow{\alpha\rho\sigma} + \overleftarrow{\alpha\rho\sigma} &= 0\end{aligned}$$

where the arrows mean  $\alpha\mu\nu \rightarrow \mu\nu\alpha$ .

Consider transformation  $U(\Lambda)$  for field.

Scalar field:

$$\mathcal{U}(\Lambda)^{-1}\varphi(x)\mathcal{U}(\Lambda) = \varphi(\Lambda^{-1}x)$$

Majorana field:

$$\mathcal{U}(\Lambda)^{-1}\varphi(x)\mathcal{U}(\Lambda) = \mathcal{U}(\Lambda)\varphi(\Lambda^{-1}x)$$

EM field:

$$\mathcal{U}(\Lambda)^{-1}F^{\mu\nu}(x)\mathcal{U}(\Lambda) = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma F^{\rho\sigma}(\Lambda^{-1}x)$$

hence

$$f^{\mu\nu\rho\sigma}(x) = \Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'}\Lambda^\rho{}_{\rho'}\Lambda^\sigma{}_{\sigma'} f^{\mu'\nu'\rho'\sigma'}(\Lambda^{-1}x)$$

Consider the parity transformation  $\mathcal{U}(\mathcal{P})$ ,

$$\mathcal{P}^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}_{\mu\nu}$$

According to symmetries we have

$$\begin{aligned}\mathcal{U}(\mathcal{P})^{-1}\vec{E}(x)\mathcal{U}(\mathcal{P}) &= -\vec{E}(\mathcal{P}^{-1}x) \\ \mathcal{U}(\mathcal{P})^{-1}\vec{B}(x)\mathcal{U}(\mathcal{P}) &= \vec{B}(\mathcal{P}^{-1}x) \\ \mathcal{U}(\mathcal{P})^{-1}\rho(x)\mathcal{U}(\mathcal{P}) &= \rho(\mathcal{P}^{-1}x) \\ \mathcal{U}(\mathcal{P})^{-1}\vec{J}(x)\mathcal{U}(\mathcal{P}) &= -\vec{J}(\mathcal{P}^{-1}x)\end{aligned}$$

From the first two we have

$$\mathcal{U}(\mathcal{P})^{-1}F^{\mu\nu}(x)\mathcal{U}(\mathcal{P}) = \mathcal{P}^\mu{}_\rho\mathcal{P}^\nu{}_\sigma F^{\rho\sigma}(\mathcal{P}^{-1}x)$$

and henceforth

$$f^{\mu\nu\rho\sigma}(x) = \mathcal{P}^\mu{}_{\mu'}\mathcal{P}^\nu{}_{\nu'}\mathcal{P}^\rho{}_{\rho'}\mathcal{P}^\sigma{}_{\sigma'} f^{\mu'\nu'\rho'\sigma'}(\mathcal{P}^{-1}x)$$

Now we consider FT

$$f^{\mu\nu\rho\sigma}(x) = \int \frac{d^4k}{(2\pi)^4} e^{ikx} \tilde{f}^{\mu\nu\rho\sigma}(k)$$

For  $\tilde{f}$  we have the similar properties:

$$\begin{aligned}\tilde{f}^{\mu\nu\rho\sigma}(k) &= -\tilde{f}^{\nu\mu\rho\sigma}(k) = -\tilde{f}^{\mu\nu\sigma\rho}(k) \\ \tilde{f}^{\mu\nu\rho\sigma*}(k) &= \tilde{f}^{\rho\sigma\mu\nu}(k) \\ k_\nu \tilde{f}^{\mu\nu\rho\sigma} &= 0, & k_\sigma \tilde{f}^{\mu\nu\rho\sigma} &= 0 \\ k^\alpha \tilde{f}^{\mu\nu\rho\sigma}(k) + \overleftarrow{\alpha\mu\nu} + \overleftarrow{\alpha\mu\nu} &= 0, & k^\alpha \tilde{f}^{\mu\nu\rho\sigma}(k) + \overleftarrow{\alpha\rho\sigma} + \overleftarrow{\alpha\rho\sigma} &= 0\end{aligned}$$

and its transformation rule:

$$\tilde{f}^{\mu\nu\rho\sigma}(\Lambda k) = \Lambda^\mu_{\mu'} \Lambda^\nu_{\nu'} \Lambda^\rho_{\rho'} \Lambda^\sigma_{\sigma'} \tilde{f}^{\mu'\nu'\rho'\sigma'}(k)$$

Also from the last property above,

$$\underbrace{k_\alpha k^\alpha}_{k^2} \tilde{f}^{\mu\nu\rho\sigma}(k) + \underbrace{k_\alpha k^\mu}_{0} \tilde{f}^{\nu\alpha\rho\sigma}(k) + \underbrace{k_\alpha k^\nu}_{0} \tilde{f}^{\alpha\mu\rho\sigma}(k) = 0$$

thus

$$\tilde{f}^{\mu\nu\rho\sigma}(k) = [h_+^{\mu\nu\rho\sigma}(\vec{k})\theta(k^0) + h_-^{\mu\nu\rho\sigma}(\vec{k})\theta(-k^0)]2\pi\delta(-k^{0^2} + |\vec{k}|^2) \quad \text{if } \vec{k} \neq 0$$

and

$$f^{\mu\nu\rho\sigma}(x) = FT \left\{ [h_+^{\mu\nu\rho\sigma}(\vec{k})\theta(k^0) + h_-^{\mu\nu\rho\sigma}(\vec{k})\theta(-k^0)]2\pi\delta(-k^{0^2} + |\vec{k}|^2) \right\} + Q^{\mu\nu\rho\sigma}(x)$$

where

$$\partial_\alpha \partial^\alpha Q^{\mu\nu\rho\sigma}(x) = 0$$

We can write

$$h_\pm^{\mu\nu\rho\sigma}(\vec{k}) = \begin{cases} \text{sgn}(\nu - \mu) \text{sgn}(\sigma - \rho) X_{\mu\nu, \rho\sigma}^{(\pm)}(\vec{k}), & \mu \neq \nu \text{ and } \rho \neq \sigma \\ 0, & \mu = \nu \text{ or } \rho = \sigma \end{cases}$$

where

$$\overline{01} = \overline{10} = 1, \quad \overline{02} = \overline{20} = 2, \quad \overline{03} = \overline{30} = 3, \quad \overline{23} = \overline{32} = 4, \quad \overline{13} = \overline{31} = 5, \quad \overline{12} = \overline{21} = 6$$

Now we consider a special condition:

$$\vec{k} = (0, 0, b), \quad k^\mu = (\pm b, 0, 0, b)$$

Since  $\vec{k}$  now is invariant rotating about  $z$ -axis, with the rotation

$$R^\mu_{\nu} = \begin{pmatrix} 1 & & & & \\ & \cos \theta & \sin \theta & & \\ & -\sin \theta & \cos \theta & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}_{\mu\nu}$$

we have

$$\tilde{f}^{\mu\nu\rho\sigma}(k^0, 0, 0, b) = R^\mu_{\mu'} R^\nu_{\nu'} R^\rho_{\rho'} R^\sigma_{\sigma'} \tilde{f}^{\mu'\nu'\rho'\sigma'}(k^0, 0, 0, b)$$

and we obtain that

$$X^{(+)}(0, 0, b) = b^2 \begin{pmatrix} u & iw & 0 & -iw & -u & 0 \\ -iw & u & 0 & -u & iw & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ iw & -u & 0 & u & -iw & 0 \\ -u & -iw & 0 & iw & u & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad X^{(-)}(0, 0, b) = b^2 \begin{pmatrix} u' & -iw' & 0 & -iw' & u' & 0 \\ iw' & u' & 0 & u' & iw' & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ iw' & u' & 0 & u' & iw' & 0 \\ u' & -iw' & 0 & -iw' & u' & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where

$$u, u', w, w' \in \mathbb{R}$$

For general  $k$ , using the transformation we obtain

$$\tilde{f}^{\mu\nu\rho\sigma}(k) = \{[uT^{\mu\nu\rho\sigma}(k) + iwS^{\mu\nu\rho\sigma}(k)]\theta(k^0) + [u'T^{\mu\nu\rho\sigma}(k) + iw'S^{\mu\nu\rho\sigma}(k)]\theta(-k^0)\}2\pi\delta(k^2), \quad \text{if } \vec{k} \neq 0$$

where

$$T^{\mu\nu\rho\sigma}(k) \equiv [g^{\mu\rho}k^\nu k^\sigma - (\mu \leftrightarrow \nu)] - (\rho \leftrightarrow \sigma)$$

$$S^{\mu\nu\rho\sigma}(k) = \frac{1}{2}(\varepsilon^{\tau\nu\rho\sigma}k_\tau k^\mu + \varepsilon^{\mu\tau\rho\sigma}k_\tau k^\nu - \varepsilon^{\mu\nu\tau\sigma}k_\tau k^\rho - \varepsilon^{\mu\nu\rho\tau}k_\tau k^\sigma)$$

and using FT we have

$$f^{\mu\nu\rho\sigma}(x) = Q^{\mu\nu\rho\sigma}(x) + \{[g^{\mu\rho}(x^2 g^{\nu\sigma} - 4x^\nu x^\sigma) - (\rho \leftrightarrow \sigma)] - (\mu \leftrightarrow \nu)\} \\ \left\{ \frac{u}{2\pi^2[|\vec{x}|^2 - (x^0 - i\varepsilon)^2]^3} + \frac{u'}{2\pi^2[|\vec{x}|^2 - (x^0 + i\varepsilon)^2]^3} \right\}$$

Using the condition

$$f^{\mu\nu\rho\sigma}(x) = 0, \quad \text{if } x^2 > 0$$

we have

$$u' + u = 0, \quad Q^{\mu\nu\rho\sigma}(x) = 0$$

hence

$$\tilde{f}^{\mu\nu\rho\sigma}(k) = u T^{\mu\nu\rho\sigma}(k) \text{sgn}(k^0) 2\pi \delta(k^2) \\ f^{\mu\nu\rho\sigma}(x) = \frac{i u}{2\pi} \text{sgn}(x^0) (g^{\mu\rho} \partial^\nu \partial^\sigma - g^{\mu\sigma} \partial^\nu \partial^\rho - g^{\nu\rho} \partial^\mu \partial^\sigma + g^{\nu\sigma} \partial^\mu \partial^\rho) \delta(x^2)$$

From this expression we have

$$f^{\rho\sigma\mu\nu}(-x) = -f^{\mu\nu\rho\sigma}(x)$$

hence

$$[F^{\rho\sigma}(y), F^{\mu\nu}(x)]_s = -[F^{\mu\nu}(x), F^{\rho\sigma}(y)]_s \Leftrightarrow (1-s)[F^{\mu\nu}(x)F^{\rho\sigma}(y) + F^{\rho\sigma}(y)F^{\mu\nu}(x)] = 0$$

To make

$$F^{\mu\nu}(x)F^{\rho\sigma}(y) + F^{\rho\sigma}(y)F^{\mu\nu}(x)$$

not identically zero (otherwise classical EM-field), we require

$$s = 1$$

Consider the  $x^0 \rightarrow 0$  limit, since

$$\begin{cases} \text{sgn}(x^2) \partial^0 \partial^0 \delta(x^2) \rightarrow 0 \delta(\vec{x}) \\ \text{sgn}(x^0) \partial^0 \partial^i \delta(x^2) \rightarrow -2\pi \partial^i \delta(\vec{x}) \\ \text{sgn}(x^0) \partial^i \partial^j \delta(x^2) \rightarrow 0 \delta(\vec{x}) \end{cases}$$

thus we have the commutators

$$\begin{cases} [E_i(t, \vec{x}), E_j(t, \vec{y})] = 0 \\ [B_i(t, \vec{x}), B_j(t, \vec{y})] = 0 \\ [E_i(t, \vec{x}), B_j(t, \vec{y})] = -i u \varepsilon_{ijk} \frac{\partial \delta(\vec{x} - \vec{y})}{\partial x^k} \\ [B_i(t, \vec{x}), E_j(t, \vec{y})] = +i u \varepsilon_{ijk} \frac{\partial \delta(\vec{x} - \vec{y})}{\partial x^k} \end{cases}$$

Now we consider the Hamiltonian. We guess

$$H(t) = \eta \int d^3x (E^2 + B^2)$$

Using the Heisenberg eqs,

$$\dot{E}_i(t, \vec{x}) = i[H, E_i(t, \vec{x})] = 2\eta u \varepsilon_{ijk} \frac{\partial}{\partial x_j} B_k(t, \vec{x})$$

$$\dot{B}_i(t, \vec{x}) = i[H, B_i(t, \vec{x})] = -2\eta u \varepsilon_{ijk} \frac{\partial}{\partial x_j} E_k(t, \vec{x})$$

Or

$$\dot{\vec{E}} = 2\eta u \nabla \times \vec{B}, \quad \dot{\vec{B}} = -2\eta u \nabla \times \vec{E}$$

According to the Maxwell eqs we need

$$2\eta u = 1$$

Let's take

$$\eta = \frac{1}{2}$$

then

$$u = 1$$

Now consider the linear momentum, according to classical Maxwell eq:

$$\vec{P} = \int d^3x \vec{E} \times \vec{B}$$

In quantum circumstances,

$$P^i = \eta' \varepsilon_{ijk} \int d^3x E_j B_k$$

Using Heisenberg eqs,

$$\begin{aligned} \frac{\partial}{\partial x^i} E_j &= -i[P^i, E_j] = \eta' \left( \frac{\partial E_j}{\partial x^i} - \delta_{ij} \nabla \cdot \vec{E} \right) \\ \frac{\partial}{\partial x^i} B_j &= -i[P^i, B_j] = \eta' \left( \frac{\partial B_j}{\partial x^i} - \delta_{ij} \nabla \cdot \vec{B} \right) \end{aligned}$$

According to Maxwell eqs

$$\nabla \cdot \vec{E} = 0, \quad \nabla \cdot \vec{B} = 0$$

we require

$$\eta' = 1$$

thus

$$\vec{P} = \int d^3x \vec{E} \times \vec{B}$$

is also valid in quantum theory.

Returning back to eqs about  $F^{\mu\nu}$ ,

$$\begin{cases} \partial_\nu F^{\mu\nu} = 0 \\ \partial^\mu F^{\nu\rho} + \partial^\nu F^{\rho\mu} + \partial^\rho F^{\mu\nu} = 0 \end{cases}$$

Taking  $\partial_\rho$  to the second eq, we have

$$\partial^2 F^{\mu\nu} = 0 \implies k^2 \tilde{F}^{\mu\nu}(k) = 0$$

therefore we can expand as

$$F^{\mu\nu}(x) = \sum_\lambda \int d\vec{k} \left[ a_\lambda(\vec{k}) \varepsilon_\lambda^{\mu\nu*}(\vec{k}) e^{ikx} + a_\lambda^\dagger(\vec{k}) \varepsilon_\lambda^{\mu\nu}(\vec{k}) e^{-ikx} \right]$$

with

$$\varepsilon_\lambda^{\mu\nu}(\vec{k}) = -\varepsilon_\lambda^{\nu\mu}(\vec{k})$$

and satisfies

$$\begin{cases} k_\nu \varepsilon_\lambda^{\mu\nu}(\vec{k}) = 0 \\ k^\mu \varepsilon_\lambda^{\nu\rho}(\vec{k}) + k^\nu \varepsilon_\lambda^{\rho\mu}(\vec{k}) + k^\rho \varepsilon_\lambda^{\mu\nu}(\vec{k}) = 0 \end{cases}$$

which has two linearly independent solutions (labeled with  $\lambda$ ) for each  $\vec{k} \neq 0$ .

To construct  $\varepsilon_{\lambda}^{\mu\nu}(\vec{k})$ , we consider

$$\vec{u}_1 \perp \vec{u}_2, \quad \vec{u}_1 \perp \hat{k}, \vec{u}_2 \perp \hat{k}, \quad \vec{u}_1 \times \vec{u}_2 = \hat{k}$$

where  $\hat{k} \equiv \frac{\vec{k}}{|\vec{k}|}$ . Then define

$$\vec{e}_{\lambda}(\hat{k}) \equiv \frac{1}{\sqrt{2}}(\vec{u}_1 - \lambda i \vec{u}_2), \quad \lambda = \pm 1$$

which is

$$\vec{e}_+ = \frac{1}{\sqrt{2}}(\vec{u}_1 - i \vec{u}_2), \quad \vec{e}_- = \frac{1}{\sqrt{2}}(\vec{u}_1 + i \vec{u}_2)$$

thus

$$\hat{k} \times \vec{e}_{\lambda}(\hat{k}) = \lambda i \vec{e}_{\lambda}(\hat{k}), \quad \hat{k} \cdot \vec{e}_{\lambda}(\hat{k}) = 0$$

hence we can construct as

$$\begin{cases} \varepsilon_{\lambda}^{00}(\vec{k}) = 0 \\ \varepsilon_{\lambda}^{0i}(\vec{k}) = -\varepsilon_{\lambda}^{i0}(\vec{k}) = |\vec{k}| e_{\lambda}^i(\hat{k}) \\ \varepsilon_{\lambda}^{ij}(\vec{k}) = \lambda i \varepsilon_{ijk} |\vec{k}| e_{\lambda}^k(\hat{k}) \end{cases}$$

therefore

$$F^{\mu\nu}(x) = \sum_{\lambda} \int \widetilde{d\vec{k}} [a_{\lambda}(\vec{k}) \varepsilon_{\lambda}^{\mu\nu*}(\vec{k}) e^{ikx} + a_{\lambda}^{\dagger}(\vec{k}) \varepsilon_{\lambda}^{\mu\nu}(\vec{k}) e^{-ikx}]$$

and

$$\begin{cases} \vec{E}(x) = \sum_{\lambda=\pm} \int \widetilde{d\vec{k}} |\vec{k}| \left[ a_{\lambda}(\vec{k}) \vec{e}_{\lambda}^*(\vec{k}) e^{ikx} + a_{\lambda}^{\dagger}(\vec{k}) \vec{e}_{\lambda}(\vec{k}) e^{-ikx} \right] \\ \vec{B}(x) = \sum_{\lambda=\pm} \int \widetilde{d\vec{k}} \vec{k} \times \left[ a_{\lambda}(\vec{k}) \vec{e}_{\lambda}^*(\vec{k}) e^{ikx} + a_{\lambda}^{\dagger}(\vec{k}) \vec{e}_{\lambda}(\vec{k}) e^{-ikx} \right] \end{cases}$$

Also we can calculate  $a_{\lambda}(\vec{k})$  from

$$a_{\lambda}(\vec{k}) = \vec{e}_{\lambda}(\hat{k}) \cdot \int d^3x e^{-ikx} [\vec{E}(x) + i\lambda \vec{B}(x)], \quad a_{\lambda}^{\dagger}(\vec{k}) = \vec{e}_{\lambda}^*(\hat{k}) \cdot \int d^3x e^{ikx} [\vec{E}(x) - i\lambda \vec{B}(x)]$$

where  $k^0 = |\vec{k}|$ .

And we have the commutators

$$[a_{\lambda}(\vec{k}), a_{\lambda'}(\vec{k}')] = 0, \quad [a_{\lambda}^{\dagger}(\vec{k}), a_{\lambda'}^{\dagger}(\vec{k}')] = 0, \quad [a_{\lambda}(\vec{k}), a_{\lambda'}^{\dagger}(\vec{k}')] = (2\pi)^3 2|\vec{k}| \delta(\vec{k} - \vec{k}') \delta_{\lambda\lambda'}$$

with  $a_{\lambda}(\vec{k})$  the photon annihilation operator, and  $a_{\lambda}^{\dagger}(\vec{k})$  the photon creation operator.

Substituting the mode expansions of  $\vec{E}$ ,  $\vec{B}$ , and using properties of  $\vec{e}_{\lambda}(\vec{k})$  we get

$$H = \sum_{\lambda=\pm} \int \widetilde{d\vec{k}} \frac{|\vec{k}|}{2} [a_{\lambda}^{\dagger}(\vec{k}) a_{\lambda}(\vec{k}) + a_{\lambda}(\vec{k}) a_{\lambda}^{\dagger}(\vec{k})] = \sum_{\lambda=\pm} \int \widetilde{d\vec{k}} |\vec{k}| a_{\lambda}^{\dagger}(\vec{k}) a_{\lambda}(\vec{k}) + Const$$

and

$$\vec{p} = \sum_{\lambda=\pm} \int \widetilde{d\vec{k}} \frac{\vec{k}}{2} [a_{\lambda}^{\dagger}(\vec{k}) a_{\lambda}(\vec{k}) + a_{\lambda}(\vec{k}) a_{\lambda}^{\dagger}(\vec{k})] = \sum_{\lambda=\pm} \int \widetilde{d\vec{k}} \vec{k} a_{\lambda}^{\dagger}(\vec{k}) a_{\lambda}(\vec{k}) + Const$$

So  $a_{\lambda}(\vec{k})$  annihilates a photon with energy  $|\vec{k}|$  and momentum  $\vec{k}$ ,  $a_{\lambda}^{\dagger}(\vec{k})$  creates such a photon.

The vacuum state (ground state of the EM field) is denoted  $|0\rangle$  with

$$a_\lambda(\vec{k})|0\rangle = 0, \quad \langle 0|0\rangle = 1$$

We have single-photon states

$$a_\lambda^\dagger(\vec{k})|0\rangle$$

and two-photon states

$$a_{\lambda_1}^\dagger(\vec{k}_1)a_{\lambda_2}^\dagger(\vec{k}_2)|0\rangle$$

three-photon states

$$a_{\lambda_1}^\dagger(\vec{k}_1)a_{\lambda_2}^\dagger(\vec{k}_2)a_{\lambda_3}^\dagger(\vec{k}_3)|0\rangle$$

and so on.

Consider a proper orthochronous Lorentz transformation  $\Lambda$ :

$$\mathcal{U}(\Lambda)^{-1}F^{\mu\nu}(x)\mathcal{U}(\Lambda) = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma F^{\rho\sigma}(\Lambda^{-1}x)$$

and it could be expressed as

$$\mathcal{U}(\Lambda) = e^{\frac{i}{2}\theta_{\mu\nu}M^{\mu\nu}}$$

where  $M^{\mu\nu}$  are generators of the transformation in the Hilbert space. And

$$M^{12} = J_3, \quad M^{23} = J_1, \quad M^{31} = J_2$$

which are components of the angular momentum operator.

For infinitesimal Lorentz transformations we get

$$[F^{\rho\sigma}(x), M^{\mu\nu}] = \frac{1}{i}(x^\mu\partial^\nu - x^\nu\partial^\mu)F^{\rho\sigma}(x) + \frac{1}{i}(g^{\rho\mu}F^{\nu\sigma} - g^{\rho\nu}F^{\mu\sigma} + g^{\sigma\mu}F^{\rho\nu} - g^{\sigma\nu}F^{\rho\mu})$$

thus for any 3-vector  $\vec{u}$  we have

$$[\vec{E}(x), \vec{u} \cdot \vec{J}] = \vec{u} \cdot [\vec{x} \times (-i\nabla)]\vec{E}(x) + i\vec{u} \times \vec{E}(x), \quad [\vec{B}(x), \vec{u} \cdot \vec{J}] = \vec{u} \cdot [\vec{x} \times (-i\nabla)]\vec{B}(x) + i\vec{u} \times \vec{B}(x)$$

hence

$$[\hat{k} \cdot \vec{J}, a_\lambda^\dagger(\vec{k})] = +\lambda a_\lambda^\dagger(\vec{k})$$

If we set the angular momentum of the vacuum state to be zero, then we have

$$\hat{k} \cdot \vec{J}[a_\lambda^\dagger(\vec{k})|0\rangle] = \lambda a_\lambda^\dagger(\vec{k})|0\rangle$$

So  $a_\lambda^\dagger(\vec{k})|0\rangle$  is an eigenstate of  $\hat{k} \cdot \vec{J}$ , with eigenvalue  $\lambda$ .

$\lambda$ : projection of the angular momentum of the photon along its direction of motion.

But the projection of orbital angular momentum along the direction of motion is

$$\hat{k} \cdot (\vec{r} \times \vec{k}) = 0$$

So  $\lambda$  is the projection of the **spin** of the photon along its direction.

$\lambda$  is called **helicity**.

helicity  $> 0$ : right-handed helicity  $< 0$ : left-handed

Using the mode expansion we have

$$\langle 0|F^{\mu\nu}(x)F^{\rho\sigma}(y)|0\rangle = \int \widetilde{d\vec{k}} T^{\mu\nu\rho\sigma}(k) e^{ik(x-y)} = \frac{\{g^{\mu\rho}[(x-y)^2 g^{\nu\sigma} - 4(x-y)^\nu(x-y)^\sigma] - (\rho \leftrightarrow \sigma)\} - [\mu \leftrightarrow \nu]}{2\pi^2[|\vec{x} - \vec{y}|^2 - (x^0 - y^0 - i\varepsilon)^2]^3}$$

In particular,

$$\langle 0|E_i(t, \vec{x})E_j(t, \vec{0})|0\rangle = \langle 0|B_i(t, \vec{x})B_j(t, \vec{0})|0\rangle = \frac{2\hat{x}^i\hat{x}^j - \delta_{ij}}{\pi^2 r^4}$$

where  $r = |\vec{x}|$ . And

$$\langle 0|E_i(t, \vec{x})B_j(t, \vec{0})|0\rangle = 0$$

For the time-ordered product

$$TF^{\mu\nu}(x)F^{\rho\sigma}(y) = \theta(x^0 - y^0)F^{\mu\nu}(x)F^{\rho\sigma}(y) + \theta(y^0 - x^0)F^{\rho\sigma}(y)F^{\mu\nu}(x)$$

we have the "Gauge-invariant propagator" of the photon

$$\begin{aligned}\langle 0|TF^{\mu\nu}(x)F^{\rho\sigma}(y)|0\rangle &= \int \widetilde{d^4k} T^{\mu\nu\rho\sigma}(k)[\theta(x^0 - y^0)e^{ik(x-y)} + \theta(y^0 - x^0)e^{ik(y-x)}] \\ &= \int \frac{-id^4k}{(2\pi)^4} \frac{T^{\mu\nu\rho\sigma}(k)}{k^2 - i\varepsilon} e^{ik(x-y)} \\ &\equiv \Delta^{\mu\nu\rho\sigma}(x - y)\end{aligned}$$

with

$$\hat{\Delta}^{\mu\nu\rho\sigma}(k) = \int id^4x \Delta^{\mu\nu\rho\sigma}(x) e^{-ikx} = \frac{T^{\mu\nu\rho\sigma}(k)}{k^2 - i\varepsilon}$$

where

$$T^{\mu\nu\rho\sigma} = \sum_{\lambda=\pm} \varepsilon_{\lambda}^{\mu\nu*}(\vec{k}) \varepsilon_{\lambda}^{\rho\sigma}(\vec{k}) = g^{\mu\rho} k^{\nu} k^{\sigma} + g^{\nu\sigma} k^{\mu} k^{\rho} - g^{\mu\rho} k^{\mu} k^{\sigma} - g^{\nu\sigma} k^{\nu} k^{\rho}$$

is the same polynomial as we encountered in the FT of the commutator in EM-field.

And

$$\tilde{f}^{\mu\nu\rho\sigma}(k) = \int d^4x e^{-ikx} [F^{\mu\nu}(x), F^{\rho\sigma}(0)] = T^{\mu\nu\rho\sigma}(k) \text{sgn}(k^0) 2\pi\delta(k^2)$$

Later we'll also introduce the "Gauge-dependent propagator"

$$\Delta^{\mu\nu}(x - y) \equiv \langle 0|TA^{\mu}(x)A^{\nu}(y)|0\rangle$$

where

$$F^{\mu\nu}(x) = \partial^{\mu}A^{\nu}(x) - \partial^{\nu}A^{\mu}(x)$$

So far, we have studied the Majorana, Dirac, and Maxwell theories and their quantizations (in the absence of charges and currents for the EM-field).

Next, we study the path integrals for these theories. First study the path integral for Dirac, which is a theory of FERMIONS.

Path integral of fermions necessitates the introduction of "a-numbers" (anticommuting numbers), called **Grassmann variables**.

In relativistic QFT: Path integral is more elegant than Hamiltonian approach because Lagrangian density and action are Lorentz invariant. So we'll do path integral for spin- $\frac{1}{2}$  and spin-1 particles. But spin- $\frac{1}{2}$  particles are fermions.

Recall path integral for spin-0 particles:

quantum mechanical system having  $N$  degrees of freedom: conjugate variables  $Q_1, Q_2, \dots, Q_N; P_1, P_2, \dots, P_N$ , with

$$[Q_j, Q_k] = 0, \quad [P_j, P_k] = 0, \quad [Q_j, P_k] = i\delta_{jk}$$

The  $Q_1, \dots, Q_N$  can be diagonalized simultaneously:

$$|q\rangle = |q_1, \dots, q_N\rangle$$

such that

$$Q_j|q\rangle = q_j|q\rangle, \quad 1 \leq j \leq N$$

Resolution of identity

$$1 = \int d^N q |q\rangle \langle q|, \quad d^N q = dq_1 \cdots dq_N$$

Path integral is about the smart way of using this identity. In particular,

$$\langle q | e^{-iH(t-t')} | q' \rangle = \int d^N q^{(1)} \cdots d^N q^{(M)} \langle q | e^{-iH(t-t_M)} | q^{(M)} \rangle \langle q^{(M)} | \cdots | q^{(1)} \rangle \langle q^{(1)} | e^{-iH(t_1-t')} | q' \rangle$$

When  $\Delta t$  is small, we can use Suzuki-Trotter decomposition to calculate the motion elements of  $e^{-iH\Delta t}$ .

Consider a quantum mechanical system with operators

$$c_1, \dots, c_N; \quad c_1^\dagger, \dots, c_N^\dagger$$

satisfying

$$\{c_j, c_k\} = 0, \quad \{c_j^\dagger, c_k^\dagger\} = 0, \quad \{c_j, c_k^\dagger\} = \delta_{jk}$$

thus  $c_1, \dots, c_N$  have a common eigenstate

$$|\alpha\rangle \equiv |\alpha_1, \dots, \alpha_N\rangle$$

such that

$$c_j |\alpha\rangle = \alpha_j |\alpha\rangle, \quad 1 \leq j \leq N$$

$\alpha_j$ 's are "numbers", so  $\alpha_j |\alpha\rangle$  is an eigenstate of  $c_k$ :

$$c_k (c_j |\alpha\rangle) = c_k (\alpha_j |\alpha\rangle) = \alpha_k \alpha_j |\alpha\rangle$$

Similarly

$$c_j c_k |\alpha\rangle = \alpha_j \alpha_k |\alpha\rangle$$

Add the two eqs we get

$$(c_k c_j + c_j c_k) |\alpha\rangle = (\alpha_k \alpha_j + \alpha_j \alpha_k) |\alpha\rangle$$

But

$$c_k c_j + c_j c_k = \{c_j, c_k\} = 0$$

thus

$$(\alpha_k \alpha_j + \alpha_j \alpha_k) |\alpha\rangle = 0$$

We can satisfy this by requiring

$$\alpha_k \alpha_j + \alpha_j \alpha_k = 0$$

The  $\alpha_1, \dots, \alpha_N$  are thus called "anticommuting numbers" or "a-numbers", more commonly known as **Grassmann variables**.

Grassmann variables are neither real numbers nor complex numbers, they're new mathematical objects.

*tsn's philosophy:*

*Grassmann variables and ordinary numbers both exist in the Platonic universe of abstract concepts, but they do NOT exist in physical nature.*

*You've seen 1 apple, 2 apples, 1 tree, 2 trees, 1 dog, 2 dogs, and so on.*

*But you've never seen the abstract 1 or 2, devoid of physical objects.*

*Likewise, Grassmann variables do NOT exist in nature. But we have THEORETICAL FREEDOM to invent any bizarre quantities as long as we don't run into contradictions.*

*If the quantity is USEFUL, we just use it.*

Properties of a single Grassmann variable  $\xi$ :



$$\xi = 0, \quad \text{but } \xi^2 = 0$$

No ordinary number satisfy the two conditions simultaneously.

Also we define

$$\xi x = x\xi$$

for any ordinary number  $x$ .

Now consider a smooth function of  $\xi$  such as

$$e^{\xi x} = 1 + \xi x + \frac{(\xi x)^2}{2!} + \dots$$

Since

$$(\xi x)^2 = \xi x \xi x = \xi \xi x x = 0 x^2 = 0$$

we have

$$e^{\xi x} = 1 + \xi x$$

In general, any smooth function of  $\xi$  is of the form

$$f(\xi) = a + \xi b$$

Now introduce the integral over Grassmann variable

$$\int d\xi f(\xi)$$

Axioms: Linearity, Invariance when the integrand is translated

Linearity

$$\int d\xi [f(\xi)u + g(\xi)v] = \left[ \int d\xi f(\xi) \right] u + \left[ \int d\xi g(\xi) \right] v$$

Invariance when the integrand is translated:

$$\int d\xi f(\xi + u) = \int d\xi f(\xi) \implies \int d\xi (\xi + u) = \int d\xi \xi \implies \int d\xi u = 0 \implies \int d\xi 1 = 0$$

thus

$$\int d\xi (a + \xi b) = \left( \int d\xi \xi \right) b$$

Define

$$\int d\xi \xi \equiv 1$$

then

$$\int d\xi (a + \xi b) = b$$

From  $\int d\xi \xi \equiv 1$  we see that the dimension of  $d\xi$  is the INVERSE of the dimension of  $\xi$ .

In general,

$$\int d\xi f(\xi) = b = \frac{\partial}{\partial \xi} (a + \xi b) = \frac{\partial}{\partial \xi} f(\xi)$$

The integral over a Grassmann variable is the same as the derivative with respect to it.

Change of variable:  $\xi = -3\eta$ , we have

$$b = \int d\xi(a + \xi b) = \int d(-3\eta)(a - 3\eta b)$$

But

$$\int d\eta(a - 3\eta b) = -3b$$

thus

$$d(-3\eta) = -\frac{1}{3}d\eta$$

In general

$$d(u\xi) = \frac{1}{u}d\xi$$

for any nonzero complex number  $u$ . Also we define

$$(d\xi)\xi = -\xi d\xi$$

Now consider  $N$  independent Grassmann variables  $\xi_1, \dots, \xi_N$  with

$$\xi_j \xi_k + \xi_k \xi_j = 0, \quad \xi u = u\xi$$

where  $u$  is any ordinary number. So  $\xi_j$  commutes with any product of an EVEN number of Grassmann variables.

And  $\xi_j$  anticommutes with any product of an ODD number of Grassmann variables.

The square of any linear combination of Grassmann variables is also zero:

$$(u_1 \xi_1 + \dots + u_N \xi_N)^2 = 0$$

where  $u_1, \dots, u_N$  are ordinary numbers.

If

$$1, \quad \xi_j, \quad \xi_j \xi_k (j < k), \quad \xi_j \xi_k \xi_l (j < k < l), \quad \dots, \quad \xi_1 \dots \xi_N$$

$2^N$  quantities are all linearly independent, we say that  $\xi_1, \dots, \xi_N$  are independent Grassmann variables. They form a  $2^N$ -dimensional algebra under addition, multiplication among themselves, and multiplication by complex numbers.

An arbitrary element in the algebra can be reduced to the form

$$f(\xi) = a + \xi_i b_i + \frac{1}{2!} \xi_i \xi_j c_{ij} + \dots + \frac{1}{N!} \xi_{i_1} \dots \xi_{i_N} d_{i_1 \dots i_N}$$

where repeated indices are to be summed over automatically, and  $c_{ij}, \dots, d_{i_1 \dots i_N}$  are completely antisymmetric.

We can easily show

$$d_{i_1 \dots i_N} = d \cdot \varepsilon_{i_1 \dots i_N}$$

where  $\varepsilon_{i_1 \dots i_N}$  is Levi-civita symbol, which is completely anti symmetric and

$$\varepsilon_{1,2,\dots,N} \equiv 1$$

So

$$f(\xi) = a + \xi_i b_i + \frac{1}{2!} \xi_i \xi_j c_{ij} + \dots + \xi_1 \dots \xi_N d$$

Define

$$d^N \xi \equiv d\xi_N d\xi_{N-1} \dots d\xi_1$$

then

$$\int d^N \xi f(\xi) = d$$

Also define

$$d\xi_i d\xi_j = -d\xi_j d\xi_i, \quad d\xi_i \xi_j = -\xi_j d\xi_i$$

Formulas: If  $J_{jk}$  are complex numbers, then

$$(J_{11}\xi_1 + J_{12}\xi_2)(J_{21}\xi_1 + J_{22}\xi_2) = \det \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \xi_1 \xi_2$$

In general, for  $N$  Grassmann variables  $\xi_1, \dots, \xi_N$  with

$$\xi'_i = \sum_{j=1}^N J_{ij} \xi_j$$

with

$$\xi'_1 \cdots \xi'_N = (\det J) \xi_1 \cdots \xi_N$$

and with the function

$$f(\xi') = a + \xi'_i b_i + \frac{1}{2!} \xi'_i \xi'_j c_{ij} + \cdots + \xi'_1 \cdots \xi'_N d$$

we define

$$\int d^N(J\xi) f(\xi') \equiv \int d^N \xi' f(\xi') = d$$

On the other hand

$$f(\xi') = a + \xi_i \tilde{b}_i + \frac{1}{2!} \xi_i \xi_j \tilde{c}_{ij} + \cdots + \xi_1 \cdots \xi_N (\det J) d$$

So

$$\int d^N \xi f(\xi') = (\det J) d \implies \int d^N(J\xi) f(\xi') = d = \frac{1}{\det J} \int d^N \xi f(\xi')$$

Hence

$$d^N(J\xi) = (\det J)^{-1} d^N \xi$$

This is the generalization of  $d(u\xi_1) = \frac{1}{u} d\xi_1$  for a single Grassmann variable.

Let  $M$  be any  $N \times N$  antisymmetric matrix of complex number:

$$M_{ij} = -M_{ji}$$

with

$$\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_N \end{pmatrix}, \quad \xi^T = (\xi_1 \quad \cdots \quad \xi_N), \quad \frac{1}{2} \xi^T M \xi = \sum_{ij} \frac{1}{2} \xi_i M_{ij} \xi_j$$

we can show that

$$\int d^N \xi e^{\frac{1}{2} \xi^T M \xi} = \text{pf}(M)$$

where  $\text{pf}(M)$  is the **Pfaffian** of the antisymmetric matrix  $M$ :

$$\text{pf}(M) = \begin{cases} 0, & N \text{ is odd} \\ \frac{1}{N!!} \sum_{\sigma \in S_N} \text{sgn}(\sigma) M_{\sigma(1), \sigma(2)} M_{\sigma(3), \sigma(4)} \cdots M_{\sigma(N-1), \sigma(N)}, & N \text{ is even} \end{cases}$$

In general, for any antisymmetric matrix  $M$ ,

$$[\text{pf}(M)]^2 = \det M$$

Suppose that  $\alpha_1, \dots, \alpha_N$  and  $\beta_1, \dots, \beta_N$  are  $2N$  Grassmann variables, and  $M$  is any  $N \times N$  matrix of complex numbers, with

$$\alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_N \end{pmatrix}$$

we have

$$\int d\beta_1 d\alpha_1 d\beta_2 d\alpha_2 \cdots d\beta_N d\alpha_N e^{\alpha^T M \beta} = \det M$$

To prove this, we introduce

$$\gamma = M\beta \implies \beta = M^{-1}\gamma$$

and that

$$\begin{aligned} d\beta_1 d\alpha_1 d\beta_2 d\alpha_2 d\beta_3 d\alpha_3 &= (d\beta_3 d\alpha_3)(d\beta_2 d\alpha_2)(d\beta_1 d\alpha_1) \\ &= d\beta_3 (d\beta_2 d\alpha_2)(d\beta_1 d\alpha_1) d\alpha_3 \\ &= d\beta_3 d\beta_2 (d\beta_1 d\alpha_1) d\alpha_2 d\alpha_3 \end{aligned}$$

or

$$d\beta_1 d\alpha_1 d\beta_2 d\alpha_2 d\beta_3 d\alpha_3 = d\beta_3 d\beta_2 d\beta_1 d\alpha_1 d\alpha_2 d\alpha_3$$

In general

$$\prod_{i=1}^N (d\beta_i d\alpha_i) = (d\beta_N \cdots d\beta_1)(d\alpha_1 \cdots d\alpha_N)$$

hence

$$\begin{aligned} \int \prod_{i=1}^N (d\beta_i d\alpha_i) e^{\alpha^T M \beta} &= \int (d\beta_N \cdots d\beta_1)(d\alpha_1 \cdots d\alpha_N) e^{\sum_i \alpha_i \gamma_i} \\ &= \int d^N(M^{-1}\gamma) d\alpha_1 \cdots d\alpha_N e^{\alpha_1 \gamma_1} \cdots e^{\alpha_N \gamma_N} \\ &= \int d^N(M^{-1}\gamma) d\alpha_1 \cdots d\alpha_N (1 + \alpha_1 \gamma_1) \cdots (1 + \alpha_N \gamma_N) \\ &= \int d^N(M^{-1}\gamma) \int d\alpha_1 (1 + \alpha_1 \gamma_1) \cdots \int d\alpha_N (1 + \alpha_N \gamma_N) \\ &= \det M \int d^N \gamma \gamma_1 \cdots \gamma_N \\ &= \det M \end{aligned}$$

Q.E.D

Now we consider shifted Gaussian integral:

$$\alpha^T \rightarrow \alpha^T + \xi^T M^{-1}, \quad \beta \rightarrow \beta + M^{-1}\eta$$

thus

$$\det M = \int \prod_{i=1}^N (d\beta_i d\alpha_i) e^{(\alpha^T + \xi^T M^{-1})M(\beta + M^{-1}\eta)} \implies \int \prod_{i=1}^N (d\beta_i d\alpha_i) e^{\alpha^T M \beta + \xi^T \beta + \alpha^T \eta} = (\det M) e^{-\xi^T M^{-1} \eta}$$

And introducing

$$D\beta D\alpha = \left( \prod_{i=1}^N d\beta_i d\alpha_i \right)$$

we have

$$\begin{aligned} &\int D\beta D\alpha (1 + \xi_1 \beta_1) \cdots (1 + \xi_N \beta_N) (1 + \alpha_1 \eta_1) \cdots (1 + \alpha_N \eta_N) e^{\alpha^T M \beta} \\ &= (\det M) \left[ 1 - \xi_i (M^{-1})_{ij} \eta_j + \frac{1}{2!} \xi_i (M^{-1})_{ij} \eta_j \xi_k (M^{-1})_{kl} \eta_l + \cdots \right] \end{aligned}$$

If

$$\det M \neq 0$$

then we can expand both sides in powers of  $\xi_i, \eta_j$  and match the coefficients to find the **Wick's theorem** for Grassmann variables:

$$\frac{\int D\beta D\alpha \beta_i \alpha_j e^{\alpha^T M \beta}}{\int D\beta D\alpha e^{\alpha^T M \beta}} = -(M^{-1})_{ij} \equiv \Delta_{ij}$$

$$\frac{\int D\beta D\alpha \beta_i \alpha_j \beta_k \alpha_l e^{\alpha^T M \beta}}{\int D\beta D\alpha e^{\alpha^T M \beta}} = \Delta_{ij} \Delta_{kl} - \Delta_{il} \Delta_{kj}$$

In general,

$$\frac{\int D\beta D\alpha \beta_{i_1} \alpha_{j_1} \cdots \beta_{i_a} \alpha_{j_a} e^{\alpha^T M \beta}}{\int D\beta D\alpha e^{\alpha^T M \beta}} = \sum_{\sigma \in S_a} \text{sgn}(\sigma) \Delta_{i_1, j_{\sigma(1)}} \cdots \Delta_{i_a, j_{\sigma(a)}}$$

and

$$\frac{\int D\beta D\alpha \alpha_{i_1} \cdots \alpha_{i_a} \beta_{j_1} \cdots \beta_{j_b} e^{\alpha^T M \beta}}{\int D\beta D\alpha e^{\alpha^T M \beta}} = 0, \quad \text{if } a \neq b$$

For any "real" Grassmann variables  $x_1, \dots, x_N, y_1, \dots, y_N$ , define the "complex conjugate" operation  $*$  such that

$$x_i^* = x_i, \quad y_i^* = y_i$$

$$(u + v)^* = u^* + v^*, \quad u, v \in G$$

where  $G$  is the  $2^{2N}$  dimensional algebra formed by  $x_1, \dots, x_N, y_1, \dots, y_N$ .

$$(uv)^* = v^* u^*, \quad u, v \in G$$

If  $a$  is a complex number,

$$a^* \equiv \text{complex conjugate of } a$$

Define

$$\psi_i \equiv \frac{1}{\sqrt{2}}(x_i + iy_i), \quad \psi_i^* \equiv \frac{1}{\sqrt{2}}(x_i - iy_i)$$

Then

$$\begin{pmatrix} \psi_i \\ \psi_i^* \end{pmatrix} = J \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \quad J = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix}$$

and

$$d\psi_i d\psi_i^* = (\det J)^{-1} dx_i dy_i = i dx_i dy_i$$

But

$$\psi_i^* \psi_i = (\det J) y_i x_i = -i y_i x_i$$

thus

$$\int d\psi_i d\psi_i^* \psi_i^* \psi_i = 1$$

So the complex Grassmann variables obey the usual integration rules as real Grassmann variables.

Now define

$$\psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix}, \quad \psi^\dagger = (\psi_1^* \quad \psi_2^* \quad \cdots \quad \psi_N^*)$$

Then since  $\psi_1, \dots, \psi_N, \psi_1^*, \dots, \psi_N^*$  are independent Grassmann variables, we have

$$d^N \psi d^N \psi^* e^{\psi^\dagger M \psi} = \det M$$

where

$$d^N \psi d^N \psi^* \equiv \prod_{i=1}^N d\psi_i d\psi_i^* = d\psi_N \cdots d\psi_1 d\psi_1^* \cdots d\psi_N^* = d\psi_1 \cdots d\psi_N d\psi_N^* \cdots d\psi_1^*$$

So the **Wick's theorem** for complex Grassmann variables

$$\frac{\int d^N \psi d^N \psi^* \psi_i \psi_j^* e^{\psi^\dagger M \psi}}{\int d^N \psi d^N \psi^* e^{\psi^\dagger M \psi}} = -(M^{-1})_{ij} \equiv \Delta_{ij}$$

In general,

$$\frac{\int d^N \psi d^N \psi^* \psi_{i_1} \psi_{j_1}^* \cdots \psi_{i_a} \psi_{j_a}^* e^{\psi^\dagger M \psi}}{\int d^N \psi d^N \psi^* e^{\psi^\dagger M \psi}} = \sum_{\sigma \in S_a} \text{sgn}(\sigma) \Delta_{i_1, j_{\sigma(1)}} \cdots \Delta_{i_a, j_{\sigma(a)}}$$

$$\frac{\int d^N \psi d^N \psi^* \psi_{i_1}^* \cdots \psi_{i_a}^* \psi_{j_1} \cdots \psi_{j_b} e^{\psi^\dagger M \psi}}{\int d^N \psi d^N \psi^* e^{\psi^\dagger M \psi}} = 0, \quad \text{if } a \neq b \text{ and } \det M \neq 0$$

## Path integral for fermions

Consider a fermion system having operators

$$\hat{\psi}_1, \dots, \hat{\psi}_N, \hat{\psi}_1^\dagger, \dots, \hat{\psi}_N^\dagger$$

satisfying

$$\{\hat{\psi}_n, \hat{\psi}_m\} = 0, \quad \{\hat{\psi}_n^\dagger, \hat{\psi}_m^\dagger\} = 0, \quad \{\hat{\psi}_n, \hat{\psi}_m^\dagger\} = \delta_{nm}, \quad 1 \leq n, m \leq N$$

General form of Hamiltonian:

$$\hat{H}(t) = \sum_{k=0}^{2N} \sum_{j=0}^k A_{a_1, \dots, a_j; a_{j+1}, \dots, a_k}^{(k,j)}(t) \hat{\psi}_{a_1}^\dagger \cdots \hat{\psi}_{a_j}^\dagger \hat{\psi}_{a_{j+1}} \cdots \hat{\psi}_{a_k} = \hat{H}(t)^\dagger$$

which is the **normal ordered form**, in which the  $\hat{\psi}^\dagger$ 's precede  $\hat{\psi}$ 's.

This form is convenient for path integration. The Heisenberg eq:

$$\frac{d\hat{\psi}_n}{dt} = i[\hat{H}(t), \hat{\psi}_n], \quad \frac{d\hat{\psi}_n^\dagger}{dt} = i[\hat{H}(t), \hat{\psi}_n^\dagger]$$

Define "empty" state  $|e\rangle$ :

$$\hat{\psi}_n |e\rangle = 0, \quad 1 \leq n \leq N \quad \langle e|e\rangle = 1$$

$|e\rangle$  may NOT be vacuum state, since  $|0\rangle$  is ground state, but  $|e\rangle$  is in general not eigenstate of  $\hat{H}(t)$ .

Define "complex" Grassmann variables  $\alpha_1, \dots, \alpha_N, \alpha_1^*, \dots, \alpha_N^*$  (independent), and

$$\{\alpha_n, \hat{\psi}_m\} = \{\alpha_n^*, \hat{\psi}_m\} = \{\alpha_n, \hat{\psi}_m^\dagger\} = \{\alpha_n^*, \hat{\psi}_m^\dagger\} = 0$$

and the projection operator onto  $|e\rangle$ :

$$(\hat{\psi}_1 \hat{\psi}_1^\dagger) \cdots (\hat{\psi}_N \hat{\psi}_N^\dagger) = |e\rangle \langle e|$$

so

$$\alpha_n |e\rangle \langle e| = |e\rangle \langle e| \alpha_n, \quad \alpha_n^* |e\rangle \langle e| = |e\rangle \langle e| \alpha_n^*$$

To facilitate later calculations we make stronger assumptions:

$$\alpha_n |e\rangle = |e\rangle \alpha_n, \quad \alpha_n^* |e\rangle = |e\rangle \alpha_n^*, \quad \alpha_n \langle e| = \langle e| \alpha_n, \quad \alpha_n^* \langle e| = \langle e| \alpha_n^*$$

Define fermionic coherent state:

$$|\alpha\rangle \equiv e^{-\frac{1}{2} \sum_{n=1}^N \alpha_n^* \alpha_n - \sum_{n=1}^N \alpha_n \hat{\psi}_n^\dagger} |e\rangle$$

thus

$$|\alpha\rangle = \left(1 - \frac{1}{2} \alpha_1^* \alpha_1\right) \cdots \left(1 - \frac{1}{2} \alpha_N^* \alpha_N\right) (1 - \alpha_1 \hat{\psi}_1^\dagger) \cdots (1 - \alpha_N \hat{\psi}_N^\dagger) |e\rangle$$

We can show that

$$\hat{\psi}_n |\alpha\rangle = \alpha_n |\alpha\rangle$$

To prove this we consider 1-degree freedom:

$$\hat{\psi}(1 - \alpha \hat{\psi}^\dagger) |e\rangle = (\hat{\psi} - \hat{\psi} \alpha \hat{\psi}^\dagger) |e\rangle = \alpha \hat{\psi} \hat{\psi}^\dagger |e\rangle = \alpha |e\rangle$$

and

$$\alpha(1 - \alpha \hat{\psi}^\dagger) |e\rangle = \alpha |e\rangle$$

so

$$\hat{\psi}(1 - \alpha \hat{\psi}^\dagger) |e\rangle = \alpha(1 - \alpha \hat{\psi}^\dagger) |e\rangle$$

For  $N$ -degree freedom: Use  $\hat{\psi}_1$  as example

$$\begin{aligned} \hat{\psi}_1 |\alpha\rangle &= \left(1 - \frac{1}{2} \alpha_1^* \alpha_1\right) \cdots \left(1 - \frac{1}{2} \alpha_N^* \alpha_N\right) (\hat{\psi}_1 - \hat{\psi}_1 \alpha_1 \hat{\psi}_1^\dagger) \cdots (1 - \alpha_N \hat{\psi}_N^\dagger) |e\rangle \\ &= \left(1 - \frac{1}{2} \alpha_1^* \alpha_1\right) \cdots \left(1 - \frac{1}{2} \alpha_N^* \alpha_N\right) \alpha_1 \hat{\psi}_1 \hat{\psi}_1^\dagger (1 - \alpha_2 \hat{\psi}_2^\dagger) \cdots (1 - \alpha_N \hat{\psi}_N^\dagger) |e\rangle \\ &= \alpha_1 \left(1 - \frac{1}{2} \alpha_1^* \alpha_1\right) \cdots \left(1 - \frac{1}{2} \alpha_N^* \alpha_N\right) (1 - \alpha_2 \hat{\psi}_2^\dagger) \cdots (1 - \alpha_N \hat{\psi}_N^\dagger) |e\rangle \\ &= \alpha_1 \left(1 - \frac{1}{2} \alpha_2^* \alpha_2\right) \cdots \left(1 - \frac{1}{2} \alpha_N^* \alpha_N\right) (1 - \alpha_2 \hat{\psi}_2^\dagger) \cdots (1 - \alpha_N \hat{\psi}_N^\dagger) |e\rangle \end{aligned}$$

and

$$\begin{aligned} \alpha_1 |\alpha\rangle &= \alpha_1 \left(1 - \frac{1}{2} \alpha_1^* \alpha_1\right) \cdots \left(1 - \frac{1}{2} \alpha_N^* \alpha_N\right) (1 - \alpha_1 \hat{\psi}_1^\dagger) \cdots (1 - \alpha_N \hat{\psi}_N^\dagger) |e\rangle \\ &= \alpha_1 \left(1 - \frac{1}{2} \alpha_2^* \alpha_2\right) \cdots \left(1 - \frac{1}{2} \alpha_N^* \alpha_N\right) (1 - \alpha_1 \hat{\psi}_1^\dagger) \cdots (1 - \alpha_N \hat{\psi}_N^\dagger) |e\rangle \\ &= \alpha_1 \left(1 - \frac{1}{2} \alpha_2^* \alpha_2\right) \cdots \left(1 - \frac{1}{2} \alpha_N^* \alpha_N\right) (1 - \alpha_2 \hat{\psi}_2^\dagger) \cdots (1 - \alpha_N \hat{\psi}_N^\dagger) |e\rangle \end{aligned}$$

thus

$$\hat{\psi}_1 |\alpha\rangle = \alpha_1 |\alpha\rangle$$

Similarly,

$$\langle \alpha| = \langle e| e^{-\frac{1}{2} \alpha_n^* \alpha_n - \hat{\psi}_n \alpha_n^*}, \quad \langle \alpha| \hat{\psi}_n^\dagger = \langle \alpha| \alpha_n^*$$

We can prove that

$$\langle \alpha| \alpha\rangle = 1$$

where we've chosen the coefficient  $e^{-\frac{1}{2} \alpha_n^* \alpha_n}$ , such that  $|\alpha\rangle$  is normalized.

More generally, for any two fermionic coherent states  $|\alpha\rangle, |\beta\rangle$  we have

$$\langle \alpha | \beta \rangle = e^{-\frac{1}{2}\alpha_n^* \alpha_n - \frac{1}{2}\beta_n^* \beta_n + \alpha_n^* \beta_n}$$

thus

$$|\alpha\rangle = \sum_{j_1, \dots, j_N \in \{0,1\}} e^{-\frac{1}{2}\alpha_n^* \alpha_n} (-\alpha_1)^{j_1} \dots (-\alpha_N)^{j_N} \hat{\psi}_N^\dagger{}^{j_N} \dots \hat{\psi}_1^\dagger{}^{j_1} |e\rangle$$

Or introducing

$$|j_1, \dots, j_N\rangle = \hat{\psi}_N^\dagger{}^{j_N} \dots \hat{\psi}_1^\dagger{}^{j_1} |e\rangle$$

with

$$|\alpha\rangle = \sum_{j_1, \dots, j_N \in \{0,1\}} e^{-\frac{1}{2}\alpha_n^* \alpha_n} (-\alpha_1)^{j_1} \dots (-\alpha_N)^{j_N} |j_1, \dots, j_N\rangle$$

and that

$$\int d\alpha^* d\alpha |\alpha\rangle \langle \alpha| = \sum_{j_1, \dots, j_N \in \{0,1\}} |j_1, \dots, j_N\rangle \langle j_1, \dots, j_N| = 1$$

where

$$d\alpha^* d\alpha \equiv \prod_{n=1}^N d\alpha_n^* d\alpha_n$$

Now we consider the propagator between two coherent states for infinitesimal time  $\varepsilon$ :

$$\langle \alpha | e^{-iH(\hat{\psi}_1^\dagger, \dots, \hat{\psi}_N^\dagger; \hat{\psi}_1, \dots, \hat{\psi}_N)\varepsilon} | \beta \rangle$$

with

$$e^{-i\hat{H}\varepsilon} = 1 - i\hat{H}\varepsilon + O(\varepsilon^2)$$

we have

$$\langle \alpha | e^{-iH(\hat{\psi}_1^\dagger, \dots, \hat{\psi}_N^\dagger; \hat{\psi}_1, \dots, \hat{\psi}_N)\varepsilon} | \beta \rangle = \langle \alpha | \beta \rangle - i\varepsilon \langle \alpha | H(\hat{\psi}^\dagger, \hat{\psi}) | \beta \rangle + O(\varepsilon^2) = \langle \alpha | \beta \rangle - i\varepsilon \langle \alpha | H(\alpha^*, \beta) | \beta \rangle + O(\varepsilon^2)$$

where we've used

$$\hat{\psi}_n | \beta \rangle = \beta_n | \beta \rangle. \quad \langle \alpha | \hat{\psi}_n^\dagger = \langle \alpha | \alpha_n^*$$

thus

$$\langle \alpha | e^{-i\hat{H}\varepsilon} | \beta \rangle = e^{-\frac{1}{2}\alpha_n^* \alpha_n - \frac{1}{2}\beta_n^* \beta_n + \alpha_n^* \beta_n - i\varepsilon H(\alpha^*; \beta) + O(\varepsilon^2)} = e^{i\varepsilon \left[ \frac{1}{2}\alpha_n^* \frac{\alpha_n - \beta_n}{\varepsilon} - \frac{1}{2} \frac{\alpha_n^* - \beta_n^*}{\varepsilon} \beta_n - H(\alpha^*; \beta) \right] + O(\varepsilon^2)}$$

where

$$\frac{i}{2}\alpha_n^* \frac{\alpha_n - \beta_n}{\varepsilon} - \frac{i}{2} \frac{\alpha_n^* - \beta_n^*}{\varepsilon} \beta_n = \frac{i}{2}\psi_n^* \dot{\psi}_n - \frac{i}{2}\dot{\psi}_n^* \psi_n$$

Now consider the general propagator:

$$\langle f | e^{-i\hat{H}(t-t')} | g \rangle = \int d\alpha^* d\alpha d\beta^* d\beta \langle f | \alpha \rangle \langle \alpha | e^{-i\hat{H}(t-t')} | \beta \rangle \langle \beta | g \rangle$$

Since

$$\langle \alpha | e^{-i\hat{H}(t-t')} | \alpha' \rangle = \lim_{M \rightarrow \infty} \int d\alpha^{(1)\dagger} d\alpha^{(1)} \dots d\alpha^{(M)\dagger} d\alpha^{(M)} e^{i(t-t^{(1)}) \left[ \frac{1}{2}\alpha^* \frac{\alpha - \alpha^{(1)}}{t-t^{(1)}} - \frac{1}{2} \frac{\alpha^* - \alpha^{(1)\dagger}}{t-t^{(1)}} \alpha^{(1)} - H(\alpha^*; \alpha^{(1)}, \beta^{(1)}) \right] + \dots}$$

where

$$H(\alpha^*; \alpha^{(1)}, \beta^{(1)}) = H\left(\frac{\alpha^* + \alpha^{(1)}}{2}; \beta^{(1)}\right)$$

Hence



$$\langle \alpha | e^{-i\hat{H}(t-t')} | \alpha' \rangle = \int_{\psi(t)=\alpha; \psi(t')=\alpha'} \mathcal{D}\psi^* \mathcal{D}\psi e^{iS}$$

where

$$S = \int_{t'}^t L(t'') dt'', \quad L(\psi^*; \psi, t) = \frac{i}{2} \psi_n^*(t) \dot{\psi}_n(t) - \frac{i}{2} \dot{\psi}_n^*(t) \psi_n(t)$$

and

$$\mathcal{D}\psi^* \mathcal{D}\psi \equiv \prod_{j=1}^M \prod_{n=1}^N d\psi_n^{(j)*} d\psi_n^{(j)}$$

Now consider the operator

$$\hat{F}_j(t_j) = F_j[\hat{\psi}_1^\dagger(t_j), \dots, \hat{\psi}_N^\dagger(t_j); \hat{\psi}_1(t_j), \dots, \hat{\psi}_N(t_j)] = e^{i\hat{H}t_j} F_j[\hat{\psi}_1^\dagger, \dots, \hat{\psi}_N^\dagger; \hat{\psi}_1, \dots, \hat{\psi}_N] e^{-i\hat{H}t_j}, \quad 1 \leq j \leq a$$

with

$$\langle 0 | \hat{F}_1(t_1) \cdots \hat{F}_a(t_a) | 0 \rangle = \lim_{\substack{\text{Im } t_- \rightarrow +\infty \\ \text{Im } t_+ \rightarrow -\infty}} \frac{\langle \alpha | e^{-i\hat{H}(t_+-t_1)} \hat{F}_1(0) e^{-i\hat{H}(t_1-t_2)} \hat{F}_2(0) \cdots e^{-i\hat{H}(t_a-t_-)} | \alpha \rangle}{\langle \alpha | e^{-i\hat{H}(t_+-t_-)} | \alpha \rangle}$$

hence

$$\langle 0 | \hat{F}_1(t_1) \cdots \hat{F}_a(t_a) | 0 \rangle = \frac{\int \mathcal{D}\psi^* \mathcal{D}\psi F_1[\psi^*(t_1); \psi(t_1)] \cdots F_a[\psi^*(t_a); \psi(t_a)] e^{iS}}{\int \mathcal{D}\psi^* \mathcal{D}\psi e^{iS}}, \quad S = \int_{t_-}^{t_+} L dt$$

which may have U-turns. But the time-ordered product

$$\langle 0 | \mathcal{T} \hat{F}_1(t_1) \cdots \hat{F}_a(t_a) | 0 \rangle = \frac{\int \mathcal{D}\psi^* \mathcal{D}\psi F_1[\psi^*(t_1); \psi(t_1)] \cdots F_a[\psi^*(t_a); \psi(t_a)] e^{iS}}{\int \mathcal{D}\psi^* \mathcal{D}\psi e^{iS}}$$

without U-turns.

The Lagrangian

$$L_1 = \frac{i}{2} \psi_n^* \dot{\psi}_n - \frac{i}{2} \dot{\psi}_n^* \psi_n - H(\psi^*; \psi)$$

with the action

$$S = \int_{t_-}^{t_+} dt L_1 = \int_{t_-}^{t_+} dt L_2 + \text{boundary terms}$$

with

$$L_2 = i\psi_n^* \dot{\psi}_n - H(\psi^*; \psi)$$

Now consider the Dirac theory:

$$\{\hat{\Psi}_a(t, \vec{x}), \hat{\Psi}_b(t, \vec{y})\} = 0, \quad \{\hat{\Psi}_a(t, \vec{x}), \hat{\Psi}_b^\dagger(t, \vec{y})\} = \delta_{ab} \delta(\vec{x} - \vec{y}), \quad a, b \in \{1, 2, 3, 4\}$$

The Hamiltonian

$$\hat{H} = \int d^3x \hat{\Psi}^\dagger i C^0 (C^i \partial_i + m) \hat{\Psi}$$

Discretize the space into lattice where each point has volume  $v$ . Then define

$$\hat{\psi}_{a,\vec{x}}(t) \equiv \sqrt{v} \hat{\Psi}_a(t, \vec{x})$$

with

$$\{\hat{\psi}_{a,\vec{x}}(t), \hat{\psi}_{b,\vec{y}}^\dagger(t)\} = \delta_{ab} \delta_{\vec{x},\vec{y}}$$

Then the Lagrangian

$$L = i\psi_{a,\vec{x}}^*(t)\dot{\psi}_{a,\vec{x}}(t) - H(\psi^*; \psi) = i \int d^3x \Psi^\dagger \partial_0 \Psi - \int d^3x \Psi^\dagger iC^0(C^i \partial_i + m)\Psi = \int d^3x \mathcal{L}$$

with

$$\mathcal{L} = -\bar{\Psi}(\not{\partial} + m)\Psi, \quad \bar{\Psi} = \Psi^\dagger iC^9$$

The action

$$S = \int dt L = \int d^4x \mathcal{L}$$

with the expectation value

$$\langle 0 | T \hat{F}_1(x_1) \cdots \hat{F}_n(x_n) | 0 \rangle = \frac{\int D\bar{\Psi} D\Psi F_1[\bar{\Psi}(x_1); \Psi(x_1)] \cdots F_n[\bar{\Psi}(x_n); \Psi(x_n)] e^{iS}}{\int D\bar{\Psi} D\Psi e^{iS}}$$

Now go to the EM-field. The commutators

$$[E_i(t, \vec{x}), E_j(t, \vec{y})] = 0, \quad [B_i(t, \vec{x}), B_j(t, \vec{y})] = 0, \quad [E_i(t, \vec{x}), B_j(t, \vec{y})] = -i\varepsilon_{ijk} \frac{\partial \delta(\vec{x} - \vec{y})}{\partial x^k}$$

and the Hamiltonian

$$H = \int d^3x \frac{E^2 + B^2}{2}$$

Take the discretize condition as

$$\vec{E}(t, \vec{x} + \vec{n}l) = \vec{E}(t, x), \quad \vec{B}(t, \vec{x} + \vec{n}l) = \vec{B}(t, x)$$

hence the fields

$$\vec{E}(\vec{x}) = \sum_{\vec{k}} \sum_{a=1}^3 E_{a,\vec{k}} \vec{u}_a(\hat{k}) e^{i\vec{k} \cdot \vec{x}}, \quad \vec{B}(\vec{x}) = \sum_{\vec{k}} \sum_{a=1}^3 B_{a,\vec{k}} \vec{u}_a(\hat{k}) e^{i\vec{k} \cdot \vec{x}}$$

where

$$\vec{k} = \frac{2\pi}{l} \vec{n}$$

and we take

$$\vec{u}_a \cdot \vec{u}_b = \delta_{ab}, \quad \vec{u}_1(\hat{k}) \times \vec{u}_2(\hat{k}) = \vec{u}_3(\hat{k}) \equiv \hat{k}$$

From  $\nabla \cdot \vec{B} = 0$  we have

$$i\vec{k} \cdot \sum_{a=1}^3 B_{a,\vec{k}} \vec{u}_a(\hat{k}) e^{i\vec{k} \cdot \vec{x}} = 0 \implies B_{3,\vec{k}} = 0$$

Define the space

$$\mathcal{K} = \{\vec{k} \neq 0 | \text{one of } \vec{k}, -\vec{k} \text{ is in } \mathcal{K}\}$$

and we redefine the fields

$$B_{1,\vec{k}} = \sqrt{\frac{|\vec{k}|}{2l^3}} (Q_{1,\vec{k}} + iQ_{2,\vec{k}}), \quad E_{1,\vec{k}} = -i\sqrt{\frac{|\vec{k}|}{2l^3}} (P_{1,\vec{k}} + iP_{2,\vec{k}})$$

$$B_{2,\vec{k}} = \sqrt{\frac{|\vec{k}|}{2l^3}} (Q_{3,\vec{k}} + iQ_{4,\vec{k}}), \quad E_{1,\vec{k}} = i\sqrt{\frac{|\vec{k}|}{2l^3}} (P_{3,\vec{k}} + iP_{4,\vec{k}})$$

with the commutators

$$[Q_{n,\vec{k}}, Q_{n',\vec{k}'}] = 0, \quad [P_{n,\vec{k}}, P_{n',\vec{k}'}] = 0, \quad [Q_{n,\vec{k}}, P_{n',\vec{k}'}] = i\delta_{nn'}\delta_{\vec{k},\vec{k}'}, \quad n, n' \in \{1, 2, 3, 4\}, \quad \vec{k}, \vec{k}' \in \mathcal{K}$$

and the Hamiltonian

$$H = \int d^3x \frac{E^2 + B^2}{2} = \sum_{\vec{k} \in \mathcal{K}} \sum_{n=1}^4 \left( \frac{|\vec{k}|}{2} P_{n,\vec{k}}^2 + \frac{|\vec{k}|}{2} Q_{n,\vec{k}}^2 \right)$$

With

$$m = \frac{1}{|\vec{k}|}, \quad \omega = |\vec{k}|$$

for each harmonic oscillator, we have the Lagrangian

$$L = \sum_{\vec{k} \in \mathcal{K}} \sum_{n=1}^4 \left( \frac{m}{2} \dot{Q}_{n,\vec{k}}^2 - \frac{m\omega^2}{2} Q_{n,\vec{k}}^2 \right)$$

But with

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \implies \dot{Q}_{n,\vec{k}} = |\vec{k}| P_{n,\vec{k}} = \frac{P_{n,k}}{m}$$

we have

$$L = \sum_{\vec{k} \in \mathcal{K}} \sum_{n=1}^4 \left( \frac{|\vec{k}|}{2} P_{n,\vec{k}}^2 - \frac{|\vec{k}|}{2} Q_{n,\vec{k}}^2 \right) = \int d^3x \frac{E^2 - B^2}{2}$$

And the action

$$S = \int L dt = \int \mathcal{L} d^4x, \quad \mathcal{L} = \frac{E^2 - B^2}{2} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Also from the Bianchi identity for  $F_{\mu\nu}$  we have

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

To make  $F_{\mu\nu}$  certain,

$$\partial_\mu A_\nu - \partial_\nu A_\mu = \partial_\mu A'_\nu - \partial_\nu A'_\mu \implies A'_\mu = A_\mu - \partial_\mu \alpha$$

which is the **gauge transformation**.

To use  $A$  in calculating the expectation value, we use the functional  $X[A]$ :

$$\langle 0 | T w_1(x_1) \cdots w_N(x_N) | 0 \rangle = \frac{\int \mathcal{D}A w_1(x_1) \cdots w_N(x_N) X[A]}{\int \mathcal{D}A X[A]}$$

The expectation value will be right if we choose

$$\int_{A \in C[F]} \mathcal{D}A X[A] = \eta e^{iS} = \eta e^{-\frac{i}{4} \int d^4x F_{\mu\nu} F^{\mu\nu}}$$

where  $C[F]$  is the class of  $A$  that produce same  $F$ .

Different choices of  $X[A]$  are called different **gauges** for the path integral.

Consider the FT of  $A$ :

$$A^\mu(x) = \int \frac{d^4k}{(2\pi)^4} \tilde{A}^\mu(k) e^{ik \cdot x}$$

with the gauge transformation

$$\tilde{A}^\mu(k) \rightarrow \tilde{A}^\mu(k) - i\tilde{\alpha}(k)k^\mu$$

All the  $\tilde{A}$ 's in the same gauge equivalent class produce the same field tensor  $F$  and the same action  $S$ .

Using Euclidean action

$$-S_E = iS$$

and do Wick rotation,

$$S_E = \frac{1}{2} \int \frac{d^4 k_E}{(2\pi)^4} k^2 |\tilde{A}_\perp(k)|^2$$

where

$$A^4 = A_4 = iA^0, \quad x^4 = x_4 = ix^0, \quad \tilde{A}_i(k) = \int A_i(x) e^{-ik \cdot x} d^4 x_E$$

and  $\tilde{A}_\perp(k)$  is the component of  $\tilde{A}(k)$  that is perpendicular to  $k$ .

But gauge transformation is

$$\tilde{A}(k) \rightarrow \tilde{A}(k) + c\hat{k}$$

in the 4-dimensional Euclidean space, and  $\tilde{A}_{//}(k)$  is changed.

The field tensor

$$\tilde{F}^{ij} \propto k^i \tilde{A}^j - k^j \tilde{A}^i$$

depends on  $\tilde{A}_\perp(k)$  only.

**Hard gauge fixing:** freeze the fluctuations of  $\tilde{A}_{//}(k)$  completely

**Soft gauge fixing:** tame the fluctuations of  $\tilde{A}_{//}(k)$

Hard gauge fixing: at each point in the  $k$  space, force  $\tilde{A}(k)$  to be on a 3-dimensional hyperplane that intersects with each "gauge-equivalent class" at one point only.

Well-known choices:

(1) Impose

$$n^\mu \tilde{A}_\mu(k) = 0$$

where  $n^\mu$  is constant 4-vector.

If  $n^\mu$  is spacelike: axial gauge

If  $n^\mu$  is lightlike: lightcase gauge

If  $n^\mu$  is timelike: temporal gauge (a simple choice is  $n^\mu = (1, 0, 0, 0)$  then  $A^0 \equiv \varphi = 0$ )

(2) Impose

$$k_\mu \tilde{A}^\mu(k) = 0 \Leftrightarrow \partial_\mu A^\mu(x) = 0$$

which is the Lorenz gauge, aka Landau gauge in quantum electrodynamics.

In the above choices,

$$X[A] = e^{iS}, \quad S = \int d^4 x \left( -\frac{1}{4} \right) F_{\mu\nu} F^{\mu\nu}$$

Soft gauge fixing: most well-known one is the  $R_\xi$  gauge:

$$S_E^{\text{tamed}} = \frac{1}{2} \int \frac{d^4 k_E}{(2\pi)^4} k^2 \left( |\tilde{A}_\perp(k)|^2 + \frac{|\tilde{A}_{//}(k)|^2}{\xi} \right)$$

where  $\xi$  is a positive constant.

If  $\xi \rightarrow +0$ , we get  $\tilde{A}_{//}(k) = 0 \implies k_\mu \tilde{A}^\mu(k) = 0$ , which is Landau gauge.

If  $\xi \rightarrow \infty$ , no gauge fixing,  $S_E^{\text{tamed}} = S_E$ , and path integral is ill-defined.

In the  $R_\xi$  gauge,  $\tilde{A}$  fluctuates with probability density

$$X[A] = e^{-S_E^{\text{tamed}}} = e^{iS^{\text{tamed}}}$$

thus

$$\langle \tilde{A}^i(k) \tilde{A}^j(k) \rangle = \left[ \frac{1}{k^2} \left( \delta^{ij} - \frac{k^i k^j}{k^2} \right) + \frac{\xi}{k^2} \frac{k^i k^j}{k^2} \right] (2\pi)^4 \delta_E(k + k^0), \quad 1 \leq i, j \leq 4$$

where  $\delta^{ij} - \frac{k^i k^j}{k^2}$  is the projection to the perpendicular hyperplane, and  $\frac{k^i k^j}{k^2}$  is the projection to the longitudinal (parallel) direction. Hence

$$\langle A^i(x) A^i(y) \rangle = \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{k^2} \left[ \delta^{ij} - (1 - \xi) \frac{k^i k^j}{k^2} \right] e^{ik \cdot (x-y)}$$

Wick rotating back to real lines,

$$\begin{aligned} \Delta^{\mu\nu}(x-y) &\equiv \langle 0 | T A^\mu(x) A^\nu(y) | 0 \rangle \\ &= \int \frac{-i d^4 k_E}{(2\pi)^4} \frac{1}{k^2 - i\varepsilon} \left[ g^{\mu\nu} - (1 - \xi) \frac{k^\mu k^\nu}{k^2} \right] e^{ik \cdot (x-y)} \\ &= \int \frac{-i d^4 k_E}{(2\pi)^4} \hat{\Delta}^{\mu\nu}(k) e^{ik \cdot (x-y)} \end{aligned}$$

where

$$\hat{\Delta}^{\mu\nu}(k) \equiv \frac{1}{k^2 - i\varepsilon} \left[ g^{\mu\nu} - (1 - \xi) \frac{k^\mu k^\nu}{k^2} \right]$$

For  $\xi = 0$  or Landau gauge,

$$\hat{\Delta}^{\mu\nu}(k) \equiv \frac{1}{k^2 - i\varepsilon} \left( g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right)$$

For  $\xi = 1$  or Feynman gauge,

$$\hat{\Delta}^{\mu\nu}(k) \equiv \frac{1}{k^2 - i\varepsilon} g^{\mu\nu}$$

In the Feynman gauge,

$$S_E^{\text{tamed}} = \frac{1}{2} \int \frac{d^4 k_E}{(2\pi)^4} k^2 (|\tilde{A}_\perp(k)|^2 + |\tilde{A}_{//}(k)|^2) = \frac{1}{2} \int \frac{d^4 k_E}{(2\pi)^4} k^2 |\tilde{A}(k)|^2$$

where fluctuations of  $\tilde{A}(k)$  are isotropic in the 4-dimensional Euclidean space.

Revisiting

$$\Delta^{\mu\nu\rho\sigma}(x-y) \equiv \langle 0 | T F^{\mu\nu}(x) F^{\rho\sigma}(y) | 0 \rangle$$

which is the gauge invariant propagator of the photon. Substituting  $F^{\mu\nu}(x) = \partial^\mu A^\nu - \partial^\nu A^\mu$  and using the  $\Delta^{\mu\nu}(x-y)$  above we have

$$\Delta^{\mu\nu\rho\sigma}(x-y) = \int \frac{-i d^4 k}{(2\pi)^4} \frac{g^{\mu\rho} k^\nu k^\sigma + g^{\nu\sigma} k^\mu k^\rho - g^{\nu\rho} k^\mu k^\sigma - g^{\mu\sigma} k^\nu k^\rho}{k^2 - i\varepsilon} e^{ik \cdot (x-y)}$$

which is the same as what we found before.

## From Dirac theory to quantum electrodynamics

The Lagrangian density of the Dirac field

$$\mathcal{L} = -\bar{\Psi}(C^\mu \partial_\mu + m)\Psi, \quad \Psi(x) = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \\ \Psi_3(x) \\ \Psi_4(x) \end{pmatrix}, \quad \bar{\Psi}(x) = \Psi^\dagger(x) i C^0$$

and  $\Psi_a, \Psi_a^*$  are Grassmann variables.

The Dirac theory has  $\mathcal{U}(1)$  symmetry: The Lagrangian and hence the action is invariant under the transformation

$$\begin{cases} \Psi(x) \rightarrow \Psi'(x) = e^{-i\alpha} \Psi(x) \\ \bar{\Psi}(x) \rightarrow \bar{\Psi}'(x) = e^{+i\alpha} \bar{\Psi}(x) \end{cases}$$

Now let's generalize to a LOCAL  $\mathcal{U}(1)$  transformation in which  $\alpha$  becomes dependent on  $x$ :

$$\Psi(x) = e^{i\alpha(x)} \Psi'(x), \quad \bar{\Psi}(x) = e^{-i\alpha(x)} \bar{\Psi}'(x)$$

then

$$\mathcal{L} = -\bar{\Psi}' e^{-i\alpha(x)} (C^\mu \partial_\mu + m) e^{i\alpha(x)} \Psi' = -\bar{\Psi}' \left[ C^\mu \left( \partial_\mu + i \frac{\partial \alpha}{\partial x^\mu} \right) + m \right] \Psi'$$

We introduce a classical field

$$a_\mu(x) \equiv -\partial_\mu \alpha(x)$$

then

$$\mathcal{L} = -\bar{\Psi}' \{ C^\mu [\partial_\mu - i a_\mu(x)] + m \} \Psi'$$

But  $a_\mu(x)$  doesn't have any dynamics of its own. In particular, although  $a_\mu(x)$  may be changed by making another local  $\mathcal{U}(1)$  transformation on the Dirac field,  $a_\mu(x)$  always satisfies the exact constraint

$$\partial_\mu a_\nu - \partial_\nu a_\mu = 0$$

Does  $a_\mu(x)$  really rigorously satisfy  $\partial_\mu a_\nu - \partial_\nu a_\mu = 0$ ?

Recall the conceptual leap from classical pendulum to quantum pendulum:

Classical: string strictly vertical

Quantum: string not exactly vertical, and "quantum fluctuates" slightly.

Bold conjecture: perhaps  $a_\mu(x)$  is NOT a classical field either! Then  $\partial_\mu a_\nu - \partial_\nu a_\mu$  quantum fluctuates!

Let's invent a Lagrangian density that controls the quantum fluctuations of  $\partial_\mu a_\nu - \partial_\nu a_\mu$ .

We have THEORETICAL FREEDOM to invent anything that has no internal contradictions.

A simple choice:

$$\mathcal{L}^{(a)} = -\frac{1}{4e^2} (\partial_\mu a_\nu - \partial_\nu a_\mu) (\partial^\mu a^\nu - \partial^\nu a^\mu)$$

where  $e^2$  is a small positive number.

The minus sign is chosen such that the coefficient in front of  $(\partial_0 a_i)^2$  is positive:  $\frac{1}{2e^2} (\partial_0 a_i)^2$

In fact, the above choice is the ONLY renormalizable and Lorentz invariant choice in 4-dimensional spacetime. If we include terms like

$$[(\partial_\mu a_\nu - \partial_\nu a_\mu) (\partial^\mu a^\nu - \partial^\nu a^\mu)]^2$$

the theory is no longer renormalizable.

Thus

$$\mathcal{L} = -\bar{\Psi}' \{ C^\mu [\partial_\mu - i a_\mu(x)] + m \} \Psi' - \frac{1}{4e^2} (\partial_\mu a_\nu - \partial_\nu a_\mu) (\partial^\mu a^\nu - \partial^\nu a^\mu)$$

We can show that the above  $\mathcal{L}$  is invariant under the transformation

$$\begin{cases} \Psi(x) \rightarrow \Psi'(x) = e^{-i\alpha} \Psi(x) \\ \bar{\Psi}(x) \rightarrow \bar{\Psi}'(x) = e^{+i\alpha} \bar{\Psi}(x) \\ a_\mu(x) \rightarrow a'_\mu(x) = a_\mu(x) - \partial_\mu \alpha(x) \end{cases}$$

for any  $\alpha(x)$ .

Taking a second look at

$$-\frac{1}{4e^2}(\partial_\mu a_\nu - \partial_\nu a_\mu)(\partial^\mu a^\nu - \partial^\nu a^\mu)$$

we recall the  $\mathcal{L}$  of the EM-field. Define

$$\frac{a_\mu(x)}{e} \equiv A_\mu(x)$$

and we rewrite  $\mathcal{L}$ :

$$\mathcal{L} = -\bar{\Psi} \{C^\mu [\partial_\mu - ieA_\mu(x)] + m\} \Psi - \frac{1}{4} \underbrace{(\partial_\mu A_\nu - \partial_\nu A_\mu)}_{F_{\mu\nu}} \underbrace{(\partial^\mu A^\nu - \partial^\nu A^\mu)}_{F^{\mu\nu}}$$

This theory is called Quantum Electrodynamics, or more precisely speaking, spinor quantum electrodynamics, because here the spinor (Dirac) field is coupled with the EM-field.

The local transformation is rewritten as

$$\begin{cases} \Psi(x) \rightarrow \Psi'(x) = e^{-ie\Gamma(x)} \Psi(x) \\ \bar{\Psi}(x) \rightarrow \bar{\Psi}'(x) = e^{+ie\Gamma(x)} \bar{\Psi}(x) \\ A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) - \partial_\mu \Gamma(x) \end{cases}, \quad e\Gamma \equiv \alpha$$

which is called LOCAL GAUGE TRANSFORMATION.

"local":  $\Gamma(x)$  depends on  $x$ .

We've generalized the gauge transformation to include not only the transformation of  $A_\mu(x)$  but also the transformation of the Dirac field, which interacts with the EM-field via the coupling term

$$-\bar{\Psi} C^\mu (-ieA_\mu) \Psi = J^\mu A_\mu$$

where

$$J^\mu \equiv ie\bar{\Psi} C^\mu \Psi$$

is the charge current 4-vector.

Derivation of Maxwell's eqs:

$$\mathcal{L} = -\bar{\Psi} \{C^\mu [\partial_\mu - ieA_\mu(x)] + m\} \Psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad S = \int d^4x \mathcal{L}$$

Euler-Lagrange equation:

$$\frac{\delta S}{\delta A_\mu(x)} = 0$$

But

$$\frac{\delta S}{\delta A_\mu(x)} = \frac{\partial \mathcal{L}}{\partial A_\mu(x)} - \partial_\nu \frac{\partial \mathcal{L}}{\partial [\partial_\nu A_\mu(x)]} = ie\bar{\Psi} C^\mu \Psi + \partial_\nu (\partial^\nu A^\mu - \partial^\mu A^\nu)$$

thus the E-L eq

$$J^\mu - \partial_\nu (-F^{\mu\nu}) = 0 \implies \partial_\nu F^{\mu\nu} = J^\mu$$

Also from  $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$  we get

$$\partial^\mu F^{\nu\rho} + \partial^\nu F^{\rho\mu} + \partial^\rho F^{\mu\nu} = 0$$

The charge density

$$J^0 \equiv \rho(x) = ie\bar{\Psi}(x) C^0 \Psi(x) = ie\Psi^\dagger iC^0 C^0 \Psi = e\Psi^\dagger \Psi$$

thus the electric charge is

$$e \left\{ \sum_{s=\pm} \int \widetilde{d}\vec{p} [b_s^\dagger(\vec{p}) b_s(\vec{p}) - d_s^\dagger(\vec{p}) d_s(\vec{p})] + Const \right\}$$

b-type particle has electric charge  $e < 0$ , like electron, muon, etc;

d-type particle has electric charge  $-e > 0$ , like positron, antimuon, etc.

The electric current density

$$J^i(x) = ie\Psi^\dagger iC^0 C^i \Psi = -e\Psi^\dagger C^0 C^i \Psi, \quad 1 \leq i \leq 3$$

The sign of  $e$  isn't essential, only  $e^2$  matters,

In the  $\mathcal{L}$ , if we rewrite the sign of  $e$  and reverse the directions of  $A^\mu, F^{\mu\nu}$ ,  $\mathcal{L}$  isn't changed.

If we reverse the electric charge of every particle in the universe and also reverse the directions of  $A^\mu, F^{\mu\nu}$ , i.e. reverse the scalar potential, the vector potential, the electric field and the magnetic field, the laws of physics do NOT change.

So in QED, people usually quote the fine-structure constant

$$\frac{e^2}{4\pi}$$

which does NOT depend on the sign of  $e$ .

## QED involving spinor field and/or scalar fields

The EM-field can be coupled with the Dirac field and the complex scalar field which both have  $\mathcal{U}(1)$  symmetry, but the EM-field can NOT be coupled with the Majorana field or the real scalar (KG) field.

So the Majorana particles and the real-scalar particles have NO electric charge.

Different Dirac fields and different complex scalar fields may have different electric charges.

Let  $\Psi_i(x)$  have electric charge  $\rho_i$ , and let  $\varphi_i(x)$  have electric charge  $q_i$ , then  $\mathcal{L}$  should be invariant under the local gauge transformation

$$\begin{cases} \varphi_i(x) \rightarrow \varphi'_i(x) = e^{-iq_i\Gamma(x)}\varphi_i(x) \\ \varphi_i^\dagger(x) \rightarrow \varphi'^\dagger_i(x) = e^{+iq_i\Gamma(x)}\varphi_i^\dagger(x) \\ \Psi_i(x) \rightarrow \Psi'_i(x) = e^{-ie_i\Gamma(x)}\Psi_i(x) \\ \bar{\Psi}_i(x) \rightarrow \bar{\Psi}'_i(x) = e^{+ie_i\Gamma(x)}\bar{\Psi}_i(x) \\ A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) - \partial_\mu\Gamma(x) \end{cases}$$

The corresponding  $\mathcal{L}$  is



$$\begin{aligned}
\mathcal{L} = & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_{i=1}^N \left\{ -[(\partial^\mu + iq_i A^\mu)\varphi_i^\dagger][(\partial_\mu - iq_i A_\mu)\varphi_i] - \mu_i^2 \varphi_i^\dagger \varphi_i \right\} \\
& - \frac{1}{6} \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N g_{ijk}^{(0)} \varphi_i \varphi_j \varphi_k - \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N g_{ijk}^{(1)} \varphi_i^\dagger \varphi_j \varphi_k \\
& - \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N g_{ijk}^{(2)} \varphi_i^\dagger \varphi_j^\dagger \varphi_k - \frac{1}{6} \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N g_{ijk}^{(3)} \varphi_i^\dagger \varphi_j^\dagger \varphi_k^\dagger \\
& - \frac{1}{24} \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \sum_{l=1}^N \lambda_{ijkl}^{(0)} \varphi_i \varphi_j \varphi_k \varphi_l - \frac{1}{6} \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \sum_{l=1}^N \lambda_{ijkl}^{(1)} \varphi_i^\dagger \varphi_j \varphi_k \varphi_l \\
& - \frac{1}{4} \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \sum_{l=1}^N \lambda_{ijkl}^{(2)} \varphi_i^\dagger \varphi_j^\dagger \varphi_k \varphi_l \\
& - \frac{1}{6} \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \sum_{l=1}^N \lambda_{ijkl}^{(3)} \varphi_i^\dagger \varphi_j^\dagger \varphi_k^\dagger \varphi_l - \frac{1}{24} \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \sum_{l=1}^N \lambda_{ijkl}^{(4)} \varphi_i^\dagger \varphi_j^\dagger \varphi_k^\dagger \varphi_l^\dagger \\
& - \sum_{i=1}^M \bar{\Psi}_i \left[ C^\mu (\partial_\mu - ie_i A_\mu) + m_i \right] \Psi_i.
\end{aligned}$$

The terms

$\partial_\mu - iq_i A_\mu$  (acting on the complex scalar field  $\varphi_i$ )

$\partial^\mu + iq_i A^\mu$  (acting on  $\varphi_i^\dagger$ )

$\partial_\mu - ie A_\mu$  (acting on the Dirac field  $\Psi_i$ )

are called covariant derivatives.

EM-field coupled with Dirac field: spinor QED

EM-field coupled with complex scalar field: scalar QED

## Discrete symmetries in QFT

Relativistic QFT:

$$\mathcal{L} = \mathcal{L}(f, \partial_\mu f, \dots), \quad f = \begin{pmatrix} f_1(x) \\ \vdots \\ f_N(x) \end{pmatrix}, \quad S = \int d^d x \mathcal{L}$$

where  $f_a(x)$  are commuting variables or Grassmann variables, and  $d$  is dimension of spacetime.

If  $S, Df$  are both invariant under

$$\begin{cases} f \rightarrow f' = G[f], \\ i \rightarrow G[i] \end{cases}, \quad i \equiv \sqrt{-1}$$

We say that  $G$  is a **symmetry** of the theory, and equivalently that the theory has symmetry  $G$ . And

$$G[i] = \pm i$$

For most symmetries,  $G[i] = i$ , but the time reversal needs  $G[i] = -i$ .

If  $G$  is a symmetry but we can't find a continuous sequence of symmetry transformations that connect the identity transformation to  $G$ , we say that  $G$  is a **discrete symmetry**.

Examples of continuous symmetries:

- spacial translation — — linear momentum conservation
- time translation — — energy conservation
- spatial rotations — — angular momentum conservation
- Lorentz boost
- $\mathcal{U}(1)$  symmetry in QED

Examples of discrete symmetries:

- $\mathcal{Z}(2)$  symmetry in  $\varphi^4$  theory:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{24}\varphi^4$$

which is invariant under  $\varphi \rightarrow -\varphi$ .

- parity (spatial inversion), but it turns out parity is NOT a symmetry in some QFTs.
- time reversal, but it may not be a symmetry for some QFTs.
- charge conjugation

In general, for a transformation  $G$ , there's an associated operator  $\mathcal{U}(G)$  which acts on the Hilbert space defined by the condition

$$\mathcal{U}(G)^{-1}\hat{f}\mathcal{U}(G) = G[\hat{f}]$$

Even if  $G$  isn't a symmetry, we may still define a  $\mathcal{U}(G)$  satisfying the condition above.

When there's no danger of confusion, we may simply suppress the hat in  $\hat{f}$ .

$\mathcal{U}(G)$  may or may not satisfy  $\mathcal{U}(G)H = H\mathcal{U}(G)$ .

Example: for proper orthochronous Lorentz transformation  $\Lambda$ ,

$$\mathcal{U}(\Lambda)^{-1}p^\mu\mathcal{U}(\Lambda) = \Lambda^\mu{}_\nu p^\nu$$

For spacial rotations,

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 \\ 0 & R_{ij} \end{pmatrix}$$

thus

$$\mathcal{U}(\Lambda)^{-1}p^0\mathcal{U}(\Lambda) = p^0 \implies \mathcal{U}(\Lambda)H = H\mathcal{U}(\Lambda)$$

from which we have angular momentum conservation.

But for a Lorentz boost,

$$\mathcal{U}(\Lambda)^{-1}p^0\mathcal{U}(\Lambda) \neq p^0 \implies \mathcal{U}(\Lambda)H \neq H\mathcal{U}(\Lambda)$$

For spatial translations and time translations,

$$\mathcal{U}(G)H = H\mathcal{U}(G)$$

in flat spacetime.

If  $G[i] = -i$  we get

$$\mathcal{U}(G)^{-1}i\mathcal{U}(G) = -i \implies i\mathcal{U}(G) = -\mathcal{U}(G)i$$

which means  $\mathcal{U}(G)$  is antilinear operator.

If  $G[i] = +i$  we get

$$\mathcal{U}(G)^{-1}i\mathcal{U}(G) = +i \implies i\mathcal{U}(G) = +\mathcal{U}(G)i$$

which means  $\mathcal{U}(G)$  is linear operator.

For linear operator  $\mathcal{U}$ ,

$$\mathcal{U}(c_1|\psi_1\rangle + c_2|\psi_2\rangle) = c_1\mathcal{U}|\psi_1\rangle + c_2\mathcal{U}|\psi_2\rangle$$

and for antilinear  $\mathcal{U}$ ,

$$\mathcal{U}(c_1|\psi_1\rangle + c_2|\psi_2\rangle) = c_1^*\mathcal{U}|\psi_1\rangle + c_2^*\mathcal{U}|\psi_2\rangle$$

Usually we choose  $\mathcal{U}(G)$  to be a unitary operator or antiunitary operator. Antiunitary means antilinear operator that preserves the norm of each vector in the Hilbert space.

The parity transformation (spacial inversion)

$$\mathcal{P}^\mu_\nu = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}_{\mu\nu}$$

with  $\mathcal{P}^2 = 1$ .

Quantum parity transformation  $\mathcal{U}(\mathcal{P})$  satisfies

$$\mathcal{U}(\mathcal{P})^{-1}\hat{f}\mathcal{U}(\mathcal{P}) = L(\mathcal{P})\hat{f}(\mathcal{P}^{-1}x)$$

where  $L(\mathcal{P})$  is  $N \times N$  square matrix.

We usually choose  $L(\mathcal{P})$  such that the theory has a symmetry under the transformation

$$\hat{f}(x) \rightarrow L(\mathcal{P})\hat{f}(\mathcal{P}^{-1}x)$$

But if such an  $L(\mathcal{P})$  can not be found, we choose  $L(\mathcal{P})$  such that the free-field theory (ignoring the non-quadratic terms in  $\hat{f}$  in  $\mathcal{L}$ ) has a symmetry under the transformation above.

We require every physical observable  $O$  to be transformed back to its old value under two consecutive spatial inversions

$$\mathcal{U}(\mathcal{P})^{-2}O\mathcal{U}(\mathcal{P})^2 = O$$

If  $\mathcal{U}(\mathcal{P})$  is a symmetry, then  $\mathcal{U}(\mathcal{P})H = H\mathcal{U}(\mathcal{P})$  and the parity  $\mathcal{U}(\mathcal{P})$  is conserved.

Consider parity transformation of the scalar field  $\varphi(x)$ ,

$$\mathcal{U}(\mathcal{P})^{-1}\varphi(x)\mathcal{U}(\mathcal{P}) = L(\mathcal{P})\varphi(\mathcal{P}^{-1}x) \implies \mathcal{U}(\mathcal{P})^{-2}\varphi(x)\mathcal{U}(\mathcal{P})^2 = L(\mathcal{P})^2\varphi(\mathcal{P}^{-1}x)$$

But the scalar field itself is an observable, hence

$$L(\mathcal{P})^2 = 1 \implies L(\mathcal{P}) = \pm 1$$

If  $\mathcal{U}(\mathcal{P})^{-1}\varphi(x)\mathcal{U}(\mathcal{P}) = +\varphi(\mathcal{P}x)$ , we say that  $\varphi(x)$  is a proper-scalar field.

If  $\mathcal{U}(\mathcal{P})^{-1}\varphi(x)\mathcal{U}(\mathcal{P}) = -\varphi(\mathcal{P}x)$ , we say that  $\varphi(x)$  is a pseudoscalar field.

Both transformations have the KG Lagrangian

$$\mathcal{L}_{\text{free}} = \sum_{i=1}^N \left( -\frac{1}{2}\partial_\mu\varphi_i\partial^\mu\varphi_i - \frac{1}{2}m_i^2\varphi_i^2 \right)$$

invariant.

But for certain interacting field theories, we may need to treat certain scalar fields as pseudoscalar fields in order for an interaction term in  $\mathcal{L}$  to be invariant under parity.

Now consider parity transformation of the Majorana field

$$\mathcal{L} = -\frac{i}{2}\varphi^T C^0 (C^\mu\partial_\mu + m)\varphi, \quad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix}$$

For the theory to be invariant under parity, i.e. the action and path integration measure to be invariant under parity, we require that any field configuration that makes the action stationary must still make it stationary after the parity transformation.

In other words, if  $\varphi(x)$  satisfies the Majorana eq

$$(C^\mu \partial_\mu + m)\varphi(x) = 0$$

then the transformed field  $\varphi'(x) = L(\mathcal{P})\varphi(\mathcal{P}x)$  must still satisfy the Majorana eq

$$(C^\mu \partial_\mu + m)\varphi'(x) = 0$$

But

$$\begin{aligned} (C^\mu \partial_\mu + m)\varphi'(x) &= (C^\mu \frac{\partial}{\partial x^\mu} + m)L(\mathcal{P})\varphi(\mathcal{P}x) \\ &= L(\mathcal{P}) \left[ L(\mathcal{P})^{-1} C^\mu L(\mathcal{P}) \frac{\partial}{\partial x^\mu} + m \right] \varphi(y = \mathcal{P}x) \\ &= L(\mathcal{P}) \left[ L(\mathcal{P})^{-1} C^0 L(\mathcal{P}) \frac{\partial}{\partial y^0} - L(\mathcal{P})^{-1} C^i L(\mathcal{P}) \frac{\partial}{\partial y^i} + m \right] \varphi(y) \end{aligned}$$

In order to guarantee the Majorana eq we need

$$\begin{cases} L(\mathcal{P})^{-1} C^0 L(\mathcal{P}) = C^0 \\ L(\mathcal{P})^{-1} C^i L(\mathcal{P}) = -C^i, \quad 1 \leq i \leq 3 \end{cases} \Leftrightarrow \begin{cases} C^0 L(\mathcal{P}) = L(\mathcal{P}) C^0 \\ C^i L(\mathcal{P}) = -L(\mathcal{P}) C^i, \quad 1 \leq i \leq 3 \end{cases}$$

Solving these equations we find

$$L(\mathcal{P}) = \sigma_p C^0$$

where  $\sigma_p$  is constant.

The action of transformed field hence can be calculated:

$$S' = \sigma_p^2 S$$

To guarantee  $S' = S$  we need

$$\sigma_p = \pm 1$$

with

$$\mathcal{U}(\mathcal{P})^{-1} \hat{\varphi}(x) \mathcal{U}(\mathcal{P}) = \sigma_p C^0 \hat{\varphi}(\mathcal{P}x)$$

Both  $\sigma_p = +1$  and  $\sigma_p = -1$  are valid choices because the Majorana field itself is NOT physical observable, and because any physical observable must be expressed in terms of bilinear forms for fermionic field operators, such as

$$\varphi_a O_{ab} \varphi_b$$

but

$$\mathcal{U}(\mathcal{P})^{-1} \varphi_a(x) O_{ab} \varphi_b(y) \mathcal{U}(\mathcal{P}) = \sigma_p^2 \varphi_a(x) O_{ab} \varphi_b(y)$$

is independent of  $\sigma_p$ .

Note that  $\det(\pm C^0) = +1$ , so the path integration measure  $D\varphi$  is also invariant under the parity transformation.

Also we can show

$$\mathcal{U}(\mathcal{P})^{-2} \varphi(x) \mathcal{U}(\mathcal{P})^2 = -\varphi(x)$$

So the Majorana field transforms to MINUS itself under two consecutive parity transformations.

Recall that the Majorana field also transforms to MINUS itself under a  $360^\circ$  spatial rotation  $\varphi(x) \rightarrow -\varphi(x)$ .

Let

$$\bar{\varphi}(x) \equiv \varphi(x)^T i C^0$$

then we can show that

$$\mathcal{U}(\mathcal{P})^{-1} \bar{\varphi}(x) \mathcal{U}(\mathcal{P}) = \bar{\varphi}(\mathcal{P}x) (-\sigma_p C^0)$$

hence ( $y = \mathcal{P}x$ )

$$\begin{cases} \mathcal{U}(\mathcal{P})^{-1}\bar{\varphi}(x)\varphi(x)\mathcal{U}(\mathcal{P}) = +\bar{\varphi}(y)\varphi(y) \\ \mathcal{U}(\mathcal{P})^{-1}\bar{\varphi}(x)C_5\varphi(x)\mathcal{U}(\mathcal{P}) = -\bar{\varphi}(y)C_5\varphi(y) \\ \mathcal{U}(\mathcal{P})^{-1}\bar{\varphi}(x)iC^\mu\varphi(x)\mathcal{U}(\mathcal{P}) = +\mathcal{P}^\mu_\nu\bar{\varphi}(y)iC^\nu\varphi(y) \\ \mathcal{U}(\mathcal{P})^{-1}\bar{\varphi}(x)C^\mu C_5\varphi(x)\mathcal{U}(\mathcal{P}) = -\mathcal{P}^\mu_\nu\bar{\varphi}(y)C^\nu C_5\varphi(y) \\ \mathcal{U}(\mathcal{P})^{-1}\bar{\varphi}(x)S^{\mu\nu}\varphi(x)\mathcal{U}(\mathcal{P}) = +\mathcal{P}^\mu_\rho\mathcal{P}^\nu_\sigma\bar{\varphi}(y)S^{\rho\sigma}\varphi(y) \end{cases}$$

We can also show that

$$\mathcal{U}(\mathcal{P})^{-1}\left[\bar{\varphi}(x)C^\mu\frac{\partial}{\partial x^\mu}\varphi(x)\right]\mathcal{U}(\mathcal{P}) = \bar{\varphi}(y)C^\mu\frac{\partial}{\partial y^\mu}\varphi(y)$$

If the Majorana field is coupled to a scalar field  $S(x)$  in the manner

$$\mathcal{L}_{\text{interaction}} = gS(x)\bar{\varphi}(x)C_5\varphi(x)$$

then in order for this term to be invariant under parity so that parity is conserved, we need

$$\mathcal{U}(\mathcal{P})^{-1}S(x)\mathcal{U}(\mathcal{P}) = -S(\mathcal{P}x)$$

namely,  $S(x)$  should be pseudoscalar field.

Thus each particle associated with the  $S(x)$  has “intrinsic” parity  $-1$ .

The total parity of such a particle is  $(-1)^{l+1}$ , where  $l$  is the orbital angular momentum quantum number.

To prove this, we only need the particle state

$$|s\rangle = \int d^3x \psi(|\vec{x}|)Y_{lm}(\hat{x})s(t, \vec{x})|0\rangle$$

and we can show that

$$\mathcal{U}(\mathcal{P})|s\rangle = (-1)^{l+1}|s\rangle$$

if we assume

$$\mathcal{U}(\mathcal{P})|0\rangle = |0\rangle$$

Now consider the parity transformation of the Dirac field

$$\Psi = \frac{1}{\sqrt{2}}(\varphi + i\zeta), \quad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix}, \quad \zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \end{pmatrix}$$

Using

$$\begin{cases} \mathcal{U}(\mathcal{P})^{-1}\varphi(x)\mathcal{U}(\mathcal{P}) = \sigma_p C^0\varphi(\mathcal{P}x) \\ \mathcal{U}(\mathcal{P})^{-1}\zeta(x)\mathcal{U}(\mathcal{P}) = \sigma_p C^0\zeta(\mathcal{P}x) \end{cases}$$

we have

$$\mathcal{U}(\mathcal{P})^{-1}\Psi(x)\mathcal{U}(\mathcal{P}) = \sigma_p C^0\Psi(\mathcal{P}x)$$

and for  $\bar{\Psi} \equiv \Psi^\dagger iC^0$ ,

$$\mathcal{U}(\mathcal{P})^{-1}\bar{\Psi}(x)\mathcal{U}(\mathcal{P}) = \bar{\Psi}(\mathcal{P}x)(-\sigma_p C^0)$$

hence ( $y \equiv \mathcal{P}x$ )

$$\begin{cases} \mathcal{U}(\mathcal{P})^{-1}\bar{\Psi}(x)\Psi(x)\mathcal{U}(\mathcal{P}) = +\bar{\Psi}(y)\Psi(y) \\ \mathcal{U}(\mathcal{P})^{-1}\bar{\Psi}(x)C_5\Psi(x)\mathcal{U}(\mathcal{P}) = -\bar{\Psi}(y)C_5\Psi(y) \\ \mathcal{U}(\mathcal{P})^{-1}\bar{\Psi}(x)iC^\mu\Psi(x)\mathcal{U}(\mathcal{P}) = +\mathcal{P}^\mu_\nu\bar{\Psi}(y)iC^\nu\Psi(y) \\ \mathcal{U}(\mathcal{P})^{-1}\bar{\Psi}(x)C^\mu C_5\Psi(x)\mathcal{U}(\mathcal{P}) = -\mathcal{P}^\mu_\nu\bar{\Psi}(y)C^\nu C_5\Psi(y) \\ \mathcal{U}(\mathcal{P})^{-1}\bar{\Psi}(x)S^{\mu\nu}\Psi(x)\mathcal{U}(\mathcal{P}) = +\mathcal{P}^\mu_\rho\mathcal{P}^\nu_\sigma\bar{\Psi}(y)S^{\rho\sigma}\Psi(y) \end{cases}$$

and

$$\mathcal{U}(\mathcal{P})^{-1} \bar{\Psi}(x) C^\mu \frac{\partial}{\partial x^\mu} \Psi(x) \mathcal{U}(\mathcal{P}) = \bar{\Psi}(y) C^\mu \frac{\partial}{\partial y^\mu} \Psi(y)$$

The Dirac field also transforms to MINUS itself under two parity transformations or a  $360^\circ$  spatial rotation.

If in a theory we have a scalar field  $S(x)$  coupled to Dirac field in the term

$$\mathcal{L}_{\text{interaction}} = g S(x) \bar{\Psi} C_5 \Psi(x)$$

then in order for parity to be conserved, we need  $S(x)$  to be a pseudoscalar field, and each particle associated with  $S(x)$  has spin 0 and intrinsic parity  $-1$ .

Now consider the parity transformation of the electromagnetic field.

Using

$$\mathcal{U}(\mathcal{P})^{-1} F^{\mu\nu}(x) \mathcal{U}(\mathcal{P}) = \mathcal{P}^\mu_\rho \mathcal{P}^\nu_\sigma F^{\rho\sigma}(\mathcal{P}x)$$

we can satisfy the above eq by assuming

$$\mathcal{U}(\mathcal{P})^{-1} A^\mu(x) \mathcal{U}(\mathcal{P}) = \mathcal{P}^\mu_\nu A^\nu(\mathcal{P}x)$$

hence the spinor QED Lagrangian is invariant under parity, except that

$$x \rightarrow \mathcal{P}x$$

thus spinor QED has parity symmetry and parity is conserved in spinor QED.

For time reversal,

$$F'(t) = \mathcal{U}(\mathcal{T})^{-1} F(t) \mathcal{U}(\mathcal{T}) = \sigma_F F(-t)$$

We require any physical observable  $O$  to satisfy

$$\mathcal{U}(\mathcal{T})^{-2} O \mathcal{U}(\mathcal{T})^2 = O$$

hence  $\sigma_F = \pm 1$  for any physical observable  $F(t)$ .

Using the Heisenberg eq

$$i\dot{F}(t) = [F(t), H]$$

we have

$$i\dot{F}'(t) = i \frac{d}{dt} \sigma_F F(-t) = -i \sigma_F \frac{d}{d(-t)} F(-t) = -\sigma_F [F(-t), H] = -[F'(t), H]$$

or

$$i \frac{d}{dt} [\mathcal{U}(\mathcal{T})^{-1} F(t) \mathcal{U}(\mathcal{T})] = -[\mathcal{U}(\mathcal{T})^{-1} F(t) \mathcal{U}(\mathcal{T}), H] \implies [-\mathcal{U}(\mathcal{T}) i \mathcal{U}(\mathcal{T})^{-1}] \frac{d}{dt} F(t) = +[F(t), \mathcal{U}(\mathcal{T}) H \mathcal{U}(\mathcal{T})^{-1}]$$

If we require the Hamiltonian to be invariant under time reversal, namely

$$\mathcal{U}(\mathcal{T}) H \mathcal{U}(\mathcal{T})^{-1} = H$$

we must have

$$-\mathcal{U}(\mathcal{T}) i \mathcal{U}(\mathcal{T})^{-1} = +i \implies \mathcal{U}(\mathcal{T}) i = -i \mathcal{U}(\mathcal{T})$$

which means  $\mathcal{U}(\mathcal{T})$  is antilinear operator.

More specifically,  $\mathcal{U}(\mathcal{T})$  is antiunitary:

$$\begin{cases} |\psi'_1\rangle = \mathcal{U}(\mathcal{T}) |\psi_1\rangle \\ |\psi'_2\rangle = \mathcal{U}(\mathcal{T}) |\psi_2\rangle \end{cases} \implies \langle \psi'_1 | \psi'_2 \rangle = \langle \psi_1 | \psi_2 \rangle^*$$

For an  $N$ -component field  $f_a(x)$  ( $1 \leq a \leq N$ ) satisfying  $f_a^\dagger(x) = f_a(x)$ ,

$$\mathcal{U}(\mathcal{T})^{-1} f_a(x) \mathcal{U}(\mathcal{T}) = L(\mathcal{T})_{ab} f_b(\mathcal{T}x), \quad \mathcal{T}^\mu{}_\nu = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}_{\mu\nu}$$

where  $L(\mathcal{T})$  is  $N \times N$  real matrix, chosen to make the theory invariant under time reversal if possible.

If a theory cannot have time reversal symmetry, we choose  $L(\mathcal{T})$  to make free-field theory invariant under time reversal.

If a theory is invariant under the reversal then

$$\mathcal{U}(\mathcal{T})H = H\mathcal{U}(\mathcal{T})$$

hence

$$H|\psi\rangle = E|\psi\rangle \implies H[\mathcal{U}(\mathcal{T})|\psi\rangle] = E[\mathcal{U}(\mathcal{T})|\psi\rangle]$$

and if  $|\psi\rangle$  is a nondegenerate energy eigenstate, then

$$\mathcal{U}(\mathcal{T})|\psi\rangle = e^{i\phi}|\psi\rangle, \quad \phi \in \mathbb{R}$$

Now consider the time reversal transformations of the scalar field:  $\varphi^\dagger(x) = \varphi(x)$

$$\mathcal{U}(\mathcal{T})^{-1}\varphi(x)\mathcal{U}(\mathcal{T}) = \sigma\varphi(\mathcal{T}x), \quad \mathcal{U}(\mathcal{T})^{-2}\varphi(x)\mathcal{U}(\mathcal{T})^2 = \sigma^*\sigma\varphi(x)$$

hence

$$|\sigma|^2 = 1$$

If we require the transformed field to be hermitian, we get

$$\sigma = \pm 1$$

If  $\sigma = +1$ : even- $T$  scalar field

If  $\sigma = -1$ : odd- $T$  scalar field

In some theories involving the scalar field  $S(x)$  coupled to other fields, we may need to assume that  $S(x)$  is an odd- $T$  scalar field, in order for some interaction term in  $\mathcal{L}$  to be invariant under time reversal.

Now consider the time reversal transformations of the Majorana field:

$$\mathcal{U}(\mathcal{T})^{-1}\varphi(x)\mathcal{U}(\mathcal{T}) = L(\mathcal{T})\varphi(\mathcal{T}x) \equiv \varphi'(x)$$

where  $L(\mathcal{T})$  is  $4 \times 4$  real matrix.

In order for

$$\mathcal{L} = -\frac{i}{2}\varphi^T C^0 (C^\mu \partial_\mu + m)\varphi$$

to be invariant under time reversal, we must require that any field configuration that makes the action stationary must still make it stationary after time reversal.

So if

$$(C^\mu \partial_\mu + m)\varphi(x) = 0$$

we require that

$$(C^\mu \partial_\mu + m)\varphi'(x) = 0$$

But ( $y \equiv \mathcal{T}x$ )

$$\begin{aligned} (C^\mu \partial_\mu + m)\varphi'(x) &= (C^\mu \frac{\partial}{\partial x^\mu} + m)L(\mathcal{T})\varphi(\mathcal{T}x) \\ &= L(\mathcal{T}) \left[ L(\mathcal{T})^{-1} C^\mu L(\mathcal{T}) \frac{\partial}{\partial x^\mu} + m \right] \varphi(y) \\ &= L(\mathcal{T}) \left[ -L(\mathcal{T})^{-1} C^0 L(\mathcal{T}) \frac{\partial}{\partial y^0} + L(\mathcal{T})^{-1} C^i L(\mathcal{T}) \frac{\partial}{\partial y^i} + m \right] \varphi(y) \end{aligned}$$

To guarantee that this is zero we need

$$\begin{cases} L(\mathcal{T})^{-1}C^0L(\mathcal{T}) = -C^0 \\ L(\mathcal{T})^{-1}C^iL(\mathcal{T}) = +C^i \end{cases} \implies L(\mathcal{T}) = \sigma_T C^0 C_5$$

Requiring  $\mathcal{L}$  to be invariant under time reversal, under which

$$\varphi(x) \rightarrow \sigma_T C^0 C_5 \varphi(\mathcal{T}x), \quad i \rightarrow -i$$

we get

$$\sigma_T^2 = 1 \implies \sigma_T = \pm 1$$

or

$$\mathcal{U}(\mathcal{T})^{-1} \varphi(x) \mathcal{U}(\mathcal{T}) = \sigma_T C^0 C_5 \varphi(\mathcal{T}x)$$

Both  $\sigma_T = \pm 1$  are valid choices, since physical observable  $O$  can be expressed in terms of bilinear forms of fermionic field operators, and

$$\mathcal{U}(\mathcal{T})^{-1} O \mathcal{U}(\mathcal{T})$$

is independent of  $\sigma_T$ .

Since

$$\det(\pm C^0 C_5) = \pm 1$$

we know that  $D\varphi$  is invariant under time reversal.

And we can show that

$$\begin{cases} \mathcal{U}(\mathcal{T})^{-2} \varphi(x) \mathcal{U}(\mathcal{T})^2 = -\varphi(x) \\ \mathcal{U}(\mathcal{T})^{-1} \bar{\varphi}(x) \mathcal{U}(\mathcal{T}) = \bar{\varphi}(y) (-\sigma_T C^0 C_5) \\ \mathcal{U}(\mathcal{T})^{-1} \bar{\varphi}(x) \varphi(x) \mathcal{U}(\mathcal{T}) = +\bar{\varphi}(y) \varphi(y) \\ \mathcal{U}(\mathcal{T})^{-1} \bar{\varphi}(x) C_5 \varphi(x) \mathcal{U}(\mathcal{T}) = -\bar{\varphi}(y) C_5 \varphi(y) \\ \mathcal{U}(\mathcal{T})^{-1} \bar{\varphi}(x) i C^\mu \varphi(x) \mathcal{U}(\mathcal{T}) = -\mathcal{T}^\mu{}_\nu \bar{\varphi}(y) i C^\nu \varphi(y) \\ \mathcal{U}(\mathcal{T})^{-1} \bar{\varphi}(x) C^\mu C_5 \varphi(x) \mathcal{U}(\mathcal{T}) = -\mathcal{T}^\mu{}_\nu \bar{\varphi}(y) C^\nu C_5 \varphi(y) \\ \mathcal{U}(\mathcal{T})^{-1} \bar{\varphi}(x) S^{\mu\nu} \varphi(x) \mathcal{U}(\mathcal{T}) = -\mathcal{T}^\mu{}_\rho \mathcal{T}^\nu{}_\sigma \bar{\varphi}(y) S^{\rho\sigma} \varphi(y) \end{cases}$$

and

$$\mathcal{U}(\mathcal{T})^{-1} \bar{\varphi}(x) C^\mu \frac{\partial}{\partial x^\mu} \varphi(x) \mathcal{U}(\mathcal{T}) = \bar{\varphi}(y) C^\mu \frac{\partial}{\partial y^\mu} \varphi(y)$$

If in a theory there is a coupling term

$$\mathcal{L}_{\text{interaction}} = g S(x) \bar{\varphi}(x) C_5 \varphi(x)$$

where  $S(x)$  is scalar field.

Then in order for  $\mathcal{L}_{\text{interaction}}$  to be invariant under time reversal, we need to regard  $S(x)$  as odd- $\mathcal{T}$  scalar field.

Now consider the time reversal of Dirac field:

$$\Psi = \frac{1}{\sqrt{2}} (\varphi + i\zeta)$$

with

$$\begin{cases} \mathcal{U}(\mathcal{T})^{-1} \varphi(x) \mathcal{U}(\mathcal{T}) = \sigma_\varphi C^0 C_5 \varphi(\mathcal{T}x) \\ \mathcal{U}(\mathcal{T})^{-1} \zeta(x) \mathcal{U}(\mathcal{T}) = \sigma_\zeta C^0 C_5 \zeta(\mathcal{T}x) \end{cases}$$

and the electric charge

$$eQ = e \int d^3x \Psi^\dagger(x) \Psi(x) = ie \int d^3x \varphi^T(x) \zeta(x) + \text{Const}$$

We require the electric charge to be invariant under time reversal. But



$$\mathcal{U}(\mathcal{T})^{-1}eQ\mathcal{U}(\mathcal{T}) = -i\sigma_\varphi\sigma_\zeta e \int d^3x \varphi^T(\mathcal{T}x)\zeta(\mathcal{T}x) + Const$$

If we let

$$\mathcal{U}(\mathcal{T})^{-1}eQ\mathcal{U}(\mathcal{T}) = eQ \implies \sigma_\varphi\sigma_\zeta = -1 \implies \sigma_\zeta = -\sigma_\varphi \in \{-1, +1\}$$

hence we have

$$\mathcal{U}(\mathcal{T})^{-1}\Psi(x)\mathcal{U}(\mathcal{T}) = \sigma_\varphi C^0 C_5 \Psi(\mathcal{T}x)$$

and

$$\begin{cases} \mathcal{U}(\mathcal{T})^{-1}\bar{\Psi}(x)\mathcal{U}(\mathcal{T}) = \bar{\Psi}(y)(-\sigma_\varphi C^0 C_5) \\ \mathcal{U}(\mathcal{T})^{-1}\bar{\Psi}(x)\Psi(x)\mathcal{U}(\mathcal{T}) = +\bar{\Psi}(y)\Psi(y) \\ \mathcal{U}(\mathcal{T})^{-1}\bar{\Psi}(x)C_5\Psi(x)\mathcal{U}(\mathcal{T}) = -\bar{\Psi}(y)C_5\Psi(y) \\ \mathcal{U}(\mathcal{T})^{-1}\bar{\Psi}(x)iC^\mu\Psi(x)\mathcal{U}(\mathcal{T}) = -\mathcal{T}^\mu{}_\nu\bar{\Psi}(y)iC^\nu\Psi(y) \\ \mathcal{U}(\mathcal{T})^{-1}\bar{\Psi}(x)C^\mu C_5\Psi(x)\mathcal{U}(\mathcal{T}) = -\mathcal{T}^\mu{}_\nu\bar{\Psi}(y)C^\nu C_5\Psi(y) \\ \mathcal{U}(\mathcal{T})^{-1}\bar{\Psi}(x)S^{\mu\nu}\Psi(x)\mathcal{U}(\mathcal{T}) = -\mathcal{T}^\mu{}_\rho\mathcal{T}^\nu{}_\sigma\bar{\Psi}(y)S^{\rho\sigma}\Psi(y) \end{cases}$$

From the 4th eq we see that under time reversal, the charge density doesn't change, but current density reverses direction.

And

$$\begin{aligned} \mathcal{U}(\mathcal{T})^{-1}\bar{\Psi}(x)C^\mu\frac{\partial}{\partial x^\mu}\Psi(x)\mathcal{U}(\mathcal{T}) &= \bar{\Psi}(y)C^\mu\frac{\partial}{\partial y^\mu}\Psi(y) \\ \mathcal{U}(\mathcal{T})^{-2}\Psi(x)\mathcal{U}(\mathcal{T})^2 &= -\Psi(x) \end{aligned}$$

If

$$\mathcal{L}_{\text{interaction}} = gS(x)\bar{\varphi}(x)C_5\varphi(x)$$

then for this to be invariant under time reversal, we need  $S(x)$  to be odd- $T$  scalar field.

Now consider time reversal of complex scalar field:

$$\varphi(x) = \frac{1}{\sqrt{2}}[\varphi_1(x) + i\varphi_2(x)]$$

with

$$\begin{cases} \mathcal{U}(\mathcal{T})^{-1}\varphi_1(x)\mathcal{U}(\mathcal{T}) = \sigma_1\varphi_1(\mathcal{T}x) \\ \mathcal{U}(\mathcal{T})^{-1}\varphi_2(x)\mathcal{U}(\mathcal{T}) = -\sigma_1\varphi_2(\mathcal{T}x) \end{cases}$$

so that electric charge density is invariant under time reversal. Hence

$$\begin{cases} \mathcal{U}(\mathcal{T})^{-1}\varphi(x)\mathcal{U}(\mathcal{T}) = \sigma_1\varphi(\mathcal{T}x) \\ \mathcal{U}(\mathcal{T})^{-1}\varphi^\dagger(x)\mathcal{U}(\mathcal{T}) = \sigma_1\varphi^\dagger(\mathcal{T}x) \end{cases}$$

Now consider time reversal of EM-field:

$$\mathcal{U}(\mathcal{T})^{-1}A^\mu(x)\mathcal{U}(\mathcal{T}) = \eta\mathcal{T}^\mu{}_\nu A^\nu(\mathcal{T}x)$$

We choose  $\eta$  such that

$$\mathcal{L}_{\text{interaction}} = A_\mu(x)J^\mu(x)$$

is invariant under time reversal.

In Dirac and complex scalar field theories, we have

$$\mathcal{U}(\mathcal{T})^{-1}J^\mu(x)\mathcal{U}(\mathcal{T}) = -\mathcal{T}^\mu{}_\nu J^\nu(\mathcal{T}x)$$

thus  $\eta = -1$ . Hence

$$\begin{aligned} \mathcal{U}(\mathcal{T})^{-1}A^\mu(x)\mathcal{U}(\mathcal{T}) &= -\mathcal{T}^\mu{}_\nu A^\nu(\mathcal{T}x) \\ \mathcal{U}(\mathcal{T})^{-1}F^{\mu\nu}(x)\mathcal{U}(\mathcal{T}) &= -\mathcal{T}^\mu{}_\rho\mathcal{T}^\nu{}_\sigma F^{\rho\sigma}(\mathcal{T}x) \end{aligned}$$

$$\begin{cases} \mathcal{U}(\mathcal{T})^{-1} \vec{E}(x) \mathcal{U}(\mathcal{T}) = +\vec{E}(\mathcal{T}x) \\ \mathcal{U}(\mathcal{T})^{-1} \vec{B}(x) \mathcal{U}(\mathcal{T}) = -\vec{B}(\mathcal{T}x) \end{cases}$$

Therefore spinor QED is invariant under time reversal.

Scalar QED is also invariant under time reversal if the scalar coupling terms are invariant under time reversal.

For charge conjugation,

in Dirac theory  $\Psi = \frac{1}{\sqrt{2}}(\varphi + i\zeta)$  and complex scalar field theory  $S(x) = \frac{1}{\sqrt{2}}(S_1 + iS_2)$ , according to the  $\mathcal{U}(1)$  symmetry, we have a symmetry under reflection about a line in the complex plane:

$$(\varphi, \zeta) \rightarrow (\varphi, -\zeta), \quad (S_1, S_2) \rightarrow (S_1, -S_2)$$

which is called **charge conjugation**.

It maps any state with additive charges (such as electric charge, lepton number, baryon number, etc) to a state with opposite charges.

$\mathcal{U}(C)$ : charge conjugation, linear operator

We require for any physical observable  $O$ ,

$$\mathcal{U}(C)^{-2} O \mathcal{U}(C)^2 = O$$

If a theory is invariant under charge conjugation, i.e.

$$\mathcal{U}(C)H = H\mathcal{U}(C)$$

thus  $\mathcal{U}(C)$  is conserved.

Now consider the charge conjugation of complex scalar field:

$$S(x) = \frac{1}{\sqrt{2}}[S_1(x) + iS_2(x)], \quad S_1^\dagger = S_1, \quad S_2^\dagger = S_2$$

with

$$\begin{cases} \mathcal{U}(C)^{-1} S_1(x) \mathcal{U}(C) = S_1(x) \\ \mathcal{U}(C)^{-1} S_2(x) \mathcal{U}(C) = -S_2(x) \end{cases} \implies \begin{cases} \mathcal{U}(C)^{-1} S(x) \mathcal{U}(C) = S^\dagger(x) \\ \mathcal{U}(C)^{-1} S^\dagger(x) \mathcal{U}(C) = S(x) \end{cases}$$

If

$$\mathcal{L} = -(\partial_\mu \varphi^*)(\partial^\mu \varphi) - m^2 \varphi^* \varphi - \frac{\lambda}{4} (\varphi^* \varphi)^2$$

then  $\mathcal{L}$  is invariant under charge conjugation.

And the electric charge-current 4-vector

$$J^\mu = q[-i\varphi_s^\dagger \partial^\mu \varphi_s + i(\partial^\mu \varphi_s^\dagger) \varphi_s]$$

with

$$\mathcal{U}(C)^{-1} J^\mu \mathcal{U}(C) = -J^\mu + \text{Const}$$

Now consider the charge conjugation of real scalar field:

$$\mathcal{U}(C)^{-1} \varphi(x) \mathcal{U}(C) = \eta \varphi(x), \quad \eta \in \{+1, -1\}$$

Usually we choose  $\eta$  such that some interaction in  $\mathcal{L}$  is invariant under charge conjugation.

$\eta$  is the  $C$ -parity (not usual spatial parity) of the particle associated with the field.

For the Majorana field,

$$\mathcal{U}(C)^{-1} \varphi(x) \mathcal{U}(C) = R\varphi(x) \equiv \varphi'(x)$$

where  $R$  is  $4 \times 4$  real matrix. And

$$(C^\mu \partial_\mu + m)R\varphi = R(R^{-1}C^\mu R \partial_\mu + m)\varphi$$

If  $\varphi$  satisfies Majorana eq, in order for  $R\varphi$  to also satisfy Majorana eq, we need

$$R^{-1}C^\mu R = C^\mu \implies R = r1$$

To keep  $\mathcal{L}$  invariant under charge conjugation we must choose

$$r \in \{+1, -1\}$$

hence

$$\mathcal{U}(C)^{-1}\varphi(x)\mathcal{U}(C) = \sigma_C\varphi(x), \quad \sigma_C \in \{+1, -1\}$$

Both  $\sigma_C = \pm 1$  are valid choices since any  $O$  can be expressed in terms of bilinear forms of the fermionic field operators, and thus

$$\mathcal{U}(C)^{-1}O\mathcal{U}(C)$$

is independent of  $\sigma_C$ .

Since

$$\det \begin{pmatrix} \pm 1 & & & \\ & \pm 1 & & \\ & & \pm 1 & \\ & & & \pm 1 \end{pmatrix} = 1$$

thus  $D\varphi$  is also invariant under charge conjugation.

And

$$\mathcal{U}(C)|0\rangle = |0\rangle \implies \mathcal{U}(C)\varphi(x)|0\rangle = \sigma_C\varphi(x)\mathcal{U}(C)|0\rangle = \sigma_C\varphi(x)|0\rangle$$

hence the Majorana particle is an eigenstate of  $\mathcal{U}(C)$ , with eigenvalue  $\sigma_C$ . And the Majorana particle is its own antiparticle.

And we have

$$\begin{cases} \mathcal{U}(C)^{-1}\bar{\varphi}(x)\varphi(x)\mathcal{U}(C) = \bar{\varphi}(x)\varphi(x) \\ \mathcal{U}(C)^{-1}\bar{\varphi}(x)C_5\varphi(x)\mathcal{U}(C) = \bar{\varphi}(x)C_5\varphi(x) \\ \mathcal{U}(C)^{-1}\bar{\varphi}(x)iC^\mu\varphi(x)\mathcal{U}(C) = \bar{\varphi}(x)iC^\mu\varphi(x) \\ \mathcal{U}(C)^{-1}\bar{\varphi}(x)C^\mu C_5\varphi(x)\mathcal{U}(C) = \bar{\varphi}(x)C^\mu C_5\varphi(x) \\ \mathcal{U}(C)^{-1}\bar{\varphi}(x)S^{\mu\nu}\varphi(x)\mathcal{U}(C) = \bar{\varphi}(x)S^{\mu\nu}\varphi(x) \end{cases}$$

and

$$\mathcal{U}(C)^{-1}\bar{\varphi}C^\mu\partial_\mu\varphi\mathcal{U}(C) = \bar{\varphi}C^\mu\partial_\mu\varphi$$

Now consider the charge conjugation transformation of the Dirac field:

$$\Psi(x) = \frac{1}{\sqrt{2}}(\varphi(x) + i\zeta(x)), \quad \mathcal{U}(C)^{-1}\varphi(x)\mathcal{U}(C) = \varphi(x), \quad \mathcal{U}(C)^{-1}\zeta(x)\mathcal{U}(C) = \varphi(x)$$

we have

$$\begin{cases} \mathcal{U}(C)^{-1}\Psi_a(x)\mathcal{U}(C) = \Psi_a^\dagger(x) \\ \mathcal{U}(C)^{-1}\Psi_a^\dagger(x)\mathcal{U}(C) = \Psi_a(x) \end{cases}, \quad 1 \leq a \leq 4 \implies \begin{cases} \mathcal{U}(C)^{-1}\Psi(x)\mathcal{U}(C) = -iC^0\bar{\Psi}^\dagger(x) \\ \mathcal{U}(C)^{-1}\bar{\Psi}(x)\mathcal{U}(C) = \Psi^\dagger(x)iC^0 \end{cases}$$

In path integration, under charge conjugation

$$\begin{cases} \Psi_a \rightarrow \Psi_a^* \\ \Psi_a^* \rightarrow \Psi_a \end{cases}$$

For any  $4 \times 4$  matrix  $L$ , we have

$$\mathcal{U}(C)^{-1}\Psi^\dagger(x)L\Psi(x)\mathcal{U}(C) = \Psi^\dagger(-L^\dagger)\Psi(x) + (\text{tr } L)\delta(\vec{0})$$

thus

$$\mathcal{U}(C)^{-1}\bar{\Psi}\Psi\mathcal{U}(C) = +\bar{\Psi}\Psi, \quad \mathcal{U}(C)^{-1}\bar{\Psi}C_5\Psi\mathcal{U}(C) = +\bar{\Psi}C_5\Psi$$

$$\mathcal{U}(C)^{-1}\bar{\Psi}iC^\mu\Psi\mathcal{U}(C) = -\bar{\Psi}iC^\mu\Psi + 4\delta_\nu^\mu\delta(\vec{0})$$

$$\mathcal{U}(C)^{-1}\bar{\Psi}C^\mu C_5\Psi\mathcal{U}(C) = +\bar{\Psi}C^\mu C_5\Psi, \quad \mathcal{U}(C)^{-1}\bar{\Psi}S^{\mu\nu}\Psi\mathcal{U}(C) = -\bar{\Psi}S^{\mu\nu}\Psi$$

In path integration,  $\Psi, \bar{\Psi}$  are Grassmann variables, and thus under charge conjugation,

$$\bar{\Psi}iC^\mu\Psi \rightarrow -\bar{\Psi}iC^\mu\Psi$$

From

$$\mathcal{U}(C)^{-1}\bar{\Psi}iC^\mu\Psi\mathcal{U}(C) = -\bar{\Psi}iC^\mu\Psi + 4\delta_\nu^\mu\delta(\vec{0})$$

we know that the charge current 4-vector

$$J^\mu = e\bar{\Psi}iC^\mu\Psi$$

changes sign under charge conjugation (because particle  $\rightarrow$  antiparticle with opposite charge)

From

$$\mathcal{U}(C)^{-1}\bar{\Psi}C^\mu\partial_\mu\Psi\mathcal{U}(C) = \bar{\Psi}C^\mu\partial_\mu\Psi + \partial_\mu(\dots)$$

we know that Dirac action is invariant under charge conjugation, since

$$\mathcal{L} = -\bar{\Psi}C^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi$$

Now consider the charge conjugation transformation of the EM-field.

In QED,  $\mathcal{L}, J^\mu$  is coupled to  $A^\mu$  in the term  $J^\mu A_\mu$ , but  $\mathcal{U}(C)^{-1}J^\mu\mathcal{U}(C) = -J^\mu + \text{const.}$

We choose

$$\mathcal{U}(C)^{-1}A^\mu(x)\mathcal{U}(C) = -A^\mu(x)$$

in order for the term  $J^\mu A_\mu$  to be invariant under charge conjugation. Thus

$$\mathcal{U}(C)^{-1}F^{\mu\nu}\mathcal{U}(C) = -F^{\mu\nu}$$

Since single-photon state = spacetime smeared version of  $F^{\mu\nu}$  acting on  $|0\rangle$ , we know that each photon has  $C$ -parity  $-1$ , and a quantum state containing  $N$  photons but nothing else has  $C$ -parity  $(-1)^N$ .

We can check that  $\mathcal{L}_{QED}$  is invariant under charge conjugation (provided that the sum of the scalar self-interaction terms are invariant under charge conjugation).

Hence  $\mathcal{U}(C)$  is conserved in QED.

Let  $CPT \equiv \mathcal{U}(C)\mathcal{U}(P)\mathcal{U}(T)$ .

For Majorana field, using previous results we can show that

$$(CPT)^{-1}(\bar{\varphi}(x)\varphi(x))CPT = +\bar{\varphi}(-x)\varphi(-x)$$

$$(CPT)^{-1}\bar{\varphi}(x)C_5\varphi(x)CPT = +\bar{\varphi}(-x)C_5\varphi(-x)$$

$$(CPT)^{-1}\bar{\varphi}(x)C^\mu C_5\varphi(x)CPT = -\bar{\varphi}(-x)C^\mu C_5\varphi(-x)$$

For a single Dirac field,

$$(CPT)^{-1}\bar{\Psi}(x)\Psi(x)CPT = +\bar{\Psi}(-x)\Psi(-x)$$

$$(CPT)^{-1}\bar{\Psi}(x)C_5\Psi(x)CPT = +\bar{\Psi}(-x)C_5\Psi(-x)$$

$$(CPT)^{-1}\bar{\Psi}(x)iC^\mu\Psi(x)CPT = -\bar{\Psi}(-x)iC^\mu\Psi(-x) + 4\delta_\nu^\mu\delta(\vec{0})$$

$$(CPT)^{-1}\bar{\Psi}(x)C^\mu C_5\Psi(x)CPT = -\bar{\Psi}(-x)C^\mu C_5\Psi(-x)$$

$$(CPT)^{-1}\bar{\Psi}(x)S^{\mu\nu}\Psi(x)CPT = +\bar{\Psi}(-x)S^{\mu\nu}\Psi(-x)$$

In path integral,  $\bar{\Psi}(x)iC^\mu\Psi(x) \rightarrow -\bar{\Psi}(x)iC^\mu\Psi(x)$  under CPT. And

$$(CPT)^{-1}A^\mu(x)CPT = -A^\mu, \quad (CPT)^{-1}F^{\mu\nu}(x)CPT = +F^{\mu\nu}(-x)$$

All the above results conform to a single rule: a local real quantity (at event  $x$ ) carrying  $n$  vector indices (and no uncontracted indices) is transformed to  $(-1)^n$  times its original value (evaluated at  $-x$ ) after CPT transformation.

Any additional  $\partial_\mu$  contributing a factor of  $(-1)$  to CPT.

For scalar and other vector fields, one can also choose the phase factors in the  $C, P, T$  such that any hermitian local quantity having  $n$  vector indices is transformed to  $(-1)^n$  times its original value (at  $-x$ ) after CPT.

Since  $\mathcal{L}$  is Lorentz invariant in any relativistic QFT, every term in  $\mathcal{L}$  contains no uncontracted indices, and must transform to its value at  $(-x)$  under  $CPT$ .

So  $\mathcal{L}(x) \rightarrow \mathcal{L}(-x)$  under  $CPT$ ,  $S = \int d^4x \mathcal{L}$  is invariant under  $CPT$ . This is the **CPT theorem**.

CPT theorem is valid even for nonrenormalizable theories (relativistic effective field theories).

Parity symmetry is violated in some relativistic QFT's, and time reversal and charge conjugation are also violated in some relativistic QFT's.

But  $CPT$  is still a perfect symmetry in all these relativistic QFT's.

$$CPT \implies \mathcal{J}_{A+B \rightarrow C+D} = \mathcal{J}_{\bar{C}+\bar{D} \rightarrow \bar{A}+\bar{B}}(\text{spacetime reversed process of the antiparticles})$$

CPT — Every particle and its antiparticle have exactly the same rest mass.

## Spinor Quantum Electrodynamics: Perturbative Approach

(Useful in understanding electrons, positrons and the EM field)

$$\mathcal{L} = -\bar{\psi}\{C^\mu[\partial_\mu - ieA_\mu(x)] + m\}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \underbrace{-\bar{\psi}(C^\mu\partial_\mu + m)\psi}_{\mathcal{L}_{\text{Dirac}}} - \underbrace{\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\mathcal{L}_{\text{EM}}} + \underbrace{A_\mu\bar{\psi}ieC^\mu\psi}_{\mathcal{L}_{\text{interaction}}}$$

Single-particle states in free Dirac theory: ignoring interactions we have

$$\psi(x) = \sum_{s=\pm} \int \widetilde{d\vec{p}} [b_s(\vec{p})u_s(\vec{p})e^{ipx} + d_s^\dagger(\vec{p})v_s(\vec{p})e^{-ipx}], \quad v_s(\vec{p}) \equiv u_s^*(\vec{p})$$

Define single-particle states:

b-type particle:  $|p, s, +\rangle \equiv b_s^\dagger(\vec{p})|0\rangle$ , d-type particle:  $|p, s, -\rangle \equiv d_s^\dagger(\vec{p})|0\rangle$

satisfying

$$Q|p, s, \pm\rangle = \pm|p, s, \pm\rangle, \quad Q = \int d^3x \psi^\dagger \psi - \text{const} = \sum_s \int \widetilde{d\vec{p}} [b_s^\dagger(\vec{p})b_s(\vec{p}) - d_s^\dagger(\vec{p})d_s(\vec{p})] = \frac{\text{electric charge}}{e}$$

we can show that

$$\begin{aligned} \langle 0|\psi(x)|0\rangle &= 0, & \langle p, s, +|\psi(x)|0\rangle &= 0, & \langle p, s, -|\psi(x)|0\rangle &= v_s(\vec{p})e^{-ipx} \\ \langle p, s, +|\bar{\psi}(x)|0\rangle &= \bar{u}_s(\vec{p})e^{-ipx}, & \langle p, s, -|\bar{\psi}(x)|0\rangle &= 0 \end{aligned}$$

In the Interacting Theory, because of charge conservation, and because  $[Q, \psi(x)] = -\psi(x)$  remains valid, and because  $Q|p, s, \pm\rangle = \pm|p, s, \pm\rangle$  remains valid, (we still have isolated spin- $\frac{1}{2}$  particles with well-defined charges), thus the eqs (1)(2)(5) remain valid.

$$\langle 0|\underbrace{\psi(x)}_{\text{charge } -1}|0\rangle = 0, \quad \langle p, s, +|\underbrace{\psi(x)}_{\text{charge } -1}|0\rangle = 0, \quad \langle p, s, -|\underbrace{\bar{\psi}(x)}_{\text{charge } +1}|0\rangle = 0,$$

From eqs (3)(4),

$$\langle p, s, -|\bar{v}_s(\vec{p})\psi(x)|0\rangle = -2me^{-ipx}, \quad \langle p, s, +|\bar{\psi}(x)u_s(\vec{p})|0\rangle = +2me^{-ipx}$$

thus

$$\bar{v}_s(\vec{p})\tilde{\psi}(-k)|0\rangle \approx 2\pi(-2m)\delta(k^2 + m^2)\theta(k^0)|p, s, -\rangle, \quad \tilde{\psi}(-k)u_s(\vec{p})|0\rangle \approx 2\pi(+2m)\delta(k^2 + m^2)\theta(k^0)|p, s, +\rangle$$

if  $k$  is close to  $p$ .

In QED, we may define the amplitudes of  $\psi(x)$  such that the two eqs above are still satisfied.

Using the general formula (Scattering.pdf, (3)(4)(22)), we can find the scattering amplitudes for the processes with Dirac particles as incoming and outgoing particles.

Example:

$$b + b \rightarrow b + b, \quad (p_1, s_1, +) + (p_2, s_2, +) \rightarrow (p'_1, s'_1, +) + (p'_2, s'_2, +)$$

$$\text{in-state: } |i\rangle = |p_1 s_1 +; p_2 s_2 +\rangle \quad \text{out-state: } |f\rangle = |p'_1 s'_1 +; p'_2 s'_2 +\rangle$$

Then

$$\begin{aligned} \langle f|i\rangle &= i\mathcal{J}_{(p'_1 s'_1 +; p'_2 s'_2 +) \leftarrow (p_1 s_1 +; p_2 s_2 +)} (2\pi)^4 \delta(p_1 + p_2 - p'_1 - p'_2) \\ &= \frac{1}{(2m)^4} \int d^4 x_1 \int d^4 x_2 \int d^4 x'_1 \int d^4 x'_2 e^{ip_1 x_1 + ip_2 x_2 - ip'_1 x'_1 - ip'_2 x'_2} \\ &\quad (m^2 - \partial_1^2)(m^2 - \partial_2^2)(m^2 - \partial_1'^2)(m^2 - \partial_2'^2) \\ &\quad \langle 0|T(\bar{\psi}(x'_2)u_{s'_2}(\vec{p}'_2))^\dagger (\bar{\psi}(x'_1)u_{s'_1}(\vec{p}'_1))^\dagger \bar{\psi}(x_1)u_{s_1}(\vec{p}_1)\bar{\psi}(x_2)u_{s_2}(\vec{p}_2)|0\rangle \end{aligned}$$

Now we calculate  $\langle f|i\rangle$  in spinor QED with  $\mathcal{L}$

$$\mathcal{L} = -\bar{\psi}(\not{\partial} + m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu \bar{\psi}ieC^\mu\psi$$

If  $e = 0$ ,

$$\langle 0|T\psi(x)\bar{\psi}(x)|0\rangle \equiv \frac{\int D\bar{\psi}D\psi \psi(x)\bar{\psi}(x) e^{-i\int d^4 x \bar{\psi}(\not{\partial} + m)\psi}}{\int D\bar{\psi}D\psi e^{-i\int d^4 x \bar{\psi}(\not{\partial} + m)\psi}} \equiv S(x)$$

which is the free propagator of the Dirac particles. And

$$\langle 0|T\psi_a(x)\bar{\psi}_b(x)|0\rangle_{e=0} = S_{ab}(x), \quad 1 \leq a, b \leq 4$$

Using the formula in Grassmann.pdf(51a)(52) we have

$$-i(\not{\partial} + m)S(x) = -\delta(x) \implies S(x) = \int \frac{-id^4 k}{(2\pi)^4} \frac{m1 - i\not{k}}{k^2 + m^2 - i\varepsilon} e^{ikx} = \int \frac{-id^4 k}{(2\pi)^4} \hat{S}(k) e^{ikx}$$

where

$$\hat{S}(k) = \frac{m1 - i\not{k}}{k^2 + m^2 - i\varepsilon}$$

We've found the photon propagator in  $R_\xi$  gauge before,

$$\langle 0|TA^\mu(x)A^\nu(x)|0\rangle_{e=0} = \frac{\int DA e^{iS_{EM}^{\text{tamed}}} A^\mu(x)A^\nu(0)}{\int DA e^{iS_{EM}^{\text{tamed}}}}$$

where

$$S_{EM}^{\text{tamed}} = \int d^4 x \left[ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\text{gauge fixing term}) \right]$$

thus

$$\langle 0|TA^\mu(x)A^\nu(x)|0\rangle_{e=0} = \int \frac{-id^4 k}{(2\pi)^4} \hat{\Delta}^{\mu\nu} e^{ikx} \equiv \Delta^{\mu\nu}(x), \quad \hat{\Delta}^{\mu\nu}(k) \equiv \frac{1}{k^2 - i\varepsilon} \left[ g^{\mu\nu} - (1 - \xi) \frac{k^\mu k^\nu}{k^2} \right]$$

Note that  $(\bar{\psi}u)^\dagger = (\psi^\dagger iC^0 u) = u^\dagger (iC^0)^\dagger \psi$ , But  $C^0 = \begin{pmatrix} & & -1 \\ & -1 & \\ +1 & & \end{pmatrix}$ , so  $(iC^0)^\dagger = iC^0$ , and

$(\bar{\psi}u)^\dagger = u^\dagger iC^0 \psi = \bar{u}\psi$ . Thus

$$\begin{aligned}
& \langle 0 | T(\bar{\psi}(x'_2)u_{s'_2}(\vec{p}'_2))^\dagger (\bar{\psi}(x'_1)u_{s'_1}(\vec{p}'_1))^\dagger \bar{\psi}(x_1)u_{s_1}(\vec{p}_1) \bar{\psi}(x_2)u_{s_2}(\vec{p}_2) | 0 \rangle \\
&= \langle 0 | T(\bar{u}_{s'_2}(\vec{p}'_2)\psi(x'_2))(\bar{u}_{s'_1}(\vec{p}'_1)\psi(x'_1))\bar{\psi}(x_1)u_{s_1}(\vec{p}_1)\bar{\psi}(x_2)u_{s_2}(\vec{p}_2) | 0 \rangle \\
&= \langle 0 | T\bar{u}_{s'_2 a'_2}(\vec{p}'_2)\psi_{a'_2}(x'_2)\bar{u}_{s'_1 a'_1}(\vec{p}'_1)\psi_{a'_1}(x'_1)\bar{\psi}_{a_1}(x_1)u_{s_1 a_1}(\vec{p}_1)\bar{\psi}_{a_2}(x_2)u_{s_2 a_2}(\vec{p}_2) | 0 \rangle \\
&= \bar{u}_{s'_2 a'_2}(\vec{p}'_2)\bar{u}_{s'_1 a'_1}(\vec{p}'_1)\bar{u}_{s_1 a_1}(\vec{p}_1)\bar{u}_{s_2 a_2}(\vec{p}_2) \langle 0 | T\psi_{a'_2}(x'_2)\psi_{a'_1}(x'_1)\bar{\psi}_{a_1}(x_1)\bar{\psi}_{a_2}(x_2) | 0 \rangle
\end{aligned}$$

So to find  $\langle f|i \rangle$ , we need to calculate  $\langle 0 | T\psi_{a'_2}(x'_2)\psi_{a'_1}(x'_1)\bar{\psi}_{a_1}(x_1)\bar{\psi}_{a_2}(x_2) | 0 \rangle$ .

Assuming that  $e$  is small, we can calculate this perturbatively.

Path integral formula:

$$\langle 0 | T\psi_{a'_2}(x'_2)\psi_{a'_1}(x'_1)\bar{\psi}_{a_1}(x_1)\bar{\psi}_{a_2}(x_2) | 0 \rangle = \frac{\int D\bar{\psi}D\psi DA \psi_{a'_2}(x'_2)\psi_{a'_1}(x'_1)\bar{\psi}_{a_1}(x_1)\bar{\psi}_{a_2}(x_2) e^{iS}}{\int D\bar{\psi}D\psi DA e^{iS}}$$

where

$$S \equiv \underbrace{-\int d^4x \bar{\psi}(\not{\partial} + m)\psi - \int d^4x \frac{1}{4} F_{\mu\nu} F^{\mu\nu}}_{S_0} + \underbrace{\int d^4x A_\mu \bar{\psi}_a i e C_{ab}^\mu \psi_b}_{S_1}$$

Treating  $S_1$  as perturbation, we have diagrammatic expansion:

$$\begin{aligned}
\langle 0 | T\psi_{a'_2}(x'_2)\psi_{a'_1}(x'_1)\bar{\psi}_{a_1}(x_1)\bar{\psi}_{a_2}(x_2) | 0 \rangle &= \langle 0 | T\psi_{a'_2}(x'_2)\bar{\psi}_{a_2}(x_2) | 0 \rangle \langle 0 | T\psi_{a'_1}(x'_1)\bar{\psi}_{a_1}(x_1) | 0 \rangle \\
&\quad - \langle 0 | T\psi_{a'_2}(x'_2)\bar{\psi}_{a_1}(x_1) | 0 \rangle \langle 0 | T\psi_{a'_1}(x'_1)\bar{\psi}_{a_2}(x_2) | 0 \rangle \\
&\quad + \underbrace{\langle 0 | T\psi_{a'_2}(x'_2)\psi_{a'_1}(x'_1)\bar{\psi}_{a_1}(x_1)\bar{\psi}_{a_2}(x_2) | 0 \rangle}_{\substack{\psi_{2'} \quad \psi'_1 \quad \bar{\psi}_1 \quad \bar{\psi}_2}} \text{connected}
\end{aligned}$$

The first two terms correspond to processes without scattering.

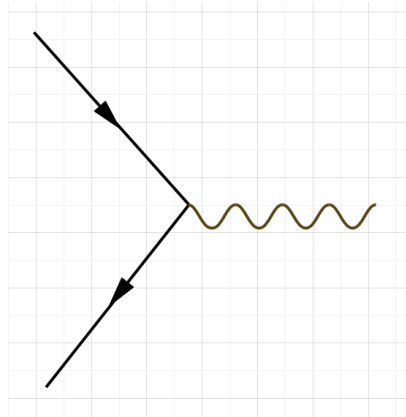
If we consider a scattering process with  $p_i \neq p_j$  ( $1 \leq i \leq 2, 1 \leq j \leq 2$ ), then only the connected diagrams contribute to the LSZ reduction formula for the scattering amplitude.

$$\langle 0 | T\psi_{2'}\psi_{1'}\bar{\psi}_1\bar{\psi}_2 | 0 \rangle = \frac{\frac{\int D\bar{\psi}D\psi DA \psi_{2'}\psi_{1'}\bar{\psi}_1\bar{\psi}_2 e^{iS_0+iS_1}}{\int D\bar{\psi}D\psi DA e^{iS_0}}}{\frac{\int D\bar{\psi}D\psi DA e^{iS_0+iS_1}}{\int D\bar{\psi}D\psi DA e^{iS_0}}}$$

with

$$S_1 = \int d^4y A_\mu(y) \bar{\psi}_b(y) i e C_{bc}^\mu \psi_c(y)$$

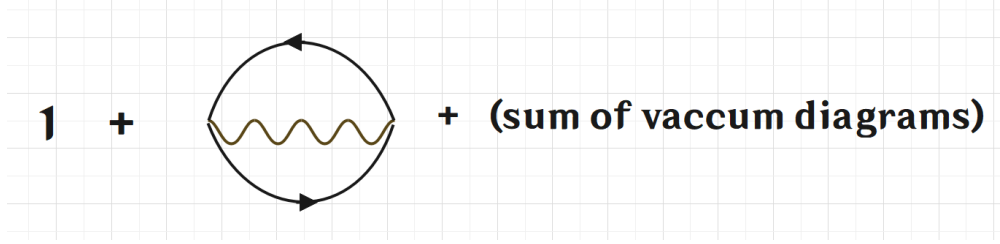
whose diagram is



The denominator

$$\frac{\int D\bar{\psi}D\psi DA e^{iS_0+iS_1}}{\int D\bar{\psi}D\psi DA e^{iS_0}} = \frac{\int D\bar{\psi}D\psi DA \left[ 1 + iS_1 + \frac{(iS_1)^2}{2!} \right] e^{iS_0}}{\int D\bar{\psi}D\psi DA e^{iS_0}}$$

which is



As usual, when we calculate the correlation functions like  $\langle 0 | T \psi_{2'} \psi_{1'} \bar{\psi}_1 \bar{\psi}_2 | 0 \rangle$ , the denominator  $\frac{\int D\bar{\psi} D\psi D A e^{iS_0 + iS_1}}{\int D\bar{\psi} D\psi D A e^{iS_0}}$  cancels the vacuum bubbles in the numerator  $\frac{\int D\bar{\psi} D\psi D A T \psi_{2'} \psi_{1'} \bar{\psi}_1 \bar{\psi}_2 e^{iS_0 + iS_1}}{\int D\bar{\psi} D\psi D A e^{iS_0}}$ .

Let

$$\langle TO \rangle_0 \equiv \frac{\int D\bar{\psi} D\psi D A O e^{iS_0}}{\int D\bar{\psi} D\psi D A e^{iS_0}}$$

then

$$\begin{aligned} & \langle 0 | T \psi_{2'} \psi_{1'} \bar{\psi}_1 \bar{\psi}_2 | 0 \rangle \\ &= \langle T \psi_{2'} \psi_{1'} \bar{\psi}_1 \bar{\psi}_2 \left[ 1 + iS_1 + \frac{(iS_1)^2}{2!} + \frac{(iS_1)^3}{3!} + \dots \right] \rangle_0 \\ & \text{(with the diagrams containing vacuum bubbles dropped)} \\ &= \langle T \psi_{2'} \psi_{1'} \bar{\psi}_1 \bar{\psi}_2 \rangle_0 + \langle T \psi_{2'} \psi_{1'} \bar{\psi}_1 \bar{\psi}_2 i \int d^4 y_1 A_{\mu_1}(y_1) \bar{\psi}_{b_1}(y_1) i e C_{b_1 c_1}^{\mu_1} \rangle_0 \\ &+ \frac{1}{2!} \langle T \psi_{2'} \psi_{1'} \bar{\psi}_1 \bar{\psi}_2 \left( i \int d^4 y_1 A_{\mu_1} \bar{\psi} i e C \psi \right) \left( i \int d^4 y_2 A_{\mu_2} \bar{\psi} i e C \psi \right) \rangle_0 \\ &+ \frac{1}{3!} \langle T \psi_{2'} \psi_{1'} \bar{\psi}_1 \bar{\psi}_2 \left( i \int d^4 y_1 A_{\mu_1} \bar{\psi} i e C \psi \right) \left( i \int d^4 y_2 A_{\mu_2} \bar{\psi} i e C \psi \right) \left( i \int d^4 y_3 A_{\mu_3} \bar{\psi} i e C \psi \right) \rangle_0 \\ &+ \frac{1}{4!} \langle T \psi_{2'} \psi_{1'} \bar{\psi}_1 \bar{\psi}_2 \left( i \int d^4 y_1 A_{\mu_1} \bar{\psi} i e C \psi \right) \left( i \int d^4 y_2 A_{\mu_2} \bar{\psi} i e C \psi \right) \left( i \int d^4 y_3 A_{\mu_3} \bar{\psi} i e C \psi \right) \left( i \int d^4 y_4 A_{\mu_4} \bar{\psi} i e C \psi \right) \rangle_0 \\ &+ \dots \text{(with the diagrams containing vacuum bubbles dropped)} \end{aligned}$$

Terms with ODD powers of  $e$  do not contribute to  $\langle 0 | T \psi \psi \bar{\psi} \bar{\psi} | 0 \rangle$ , because we cannot contract all the  $A$ 's.

Leading order with scattering:  $O(e^2)$ :

$$\frac{2}{2!} \psi_{2'} \psi_{1'} \bar{\psi}_1 \bar{\psi}_2 i \int d^4 y_1 A_{\mu_1}(y_1) \bar{\psi}_{b_1}(y_1) i e C_{b_1 c_1}^{\mu_1} \psi_{c_1}(y_1) i \int d^4 y_2 A_{\mu_2}(y_2) \bar{\psi}_{b_2}(y_2) i e C_{b_2 c_2}^{\mu_2} \psi_{c_2}(y_2)$$

by contracting,

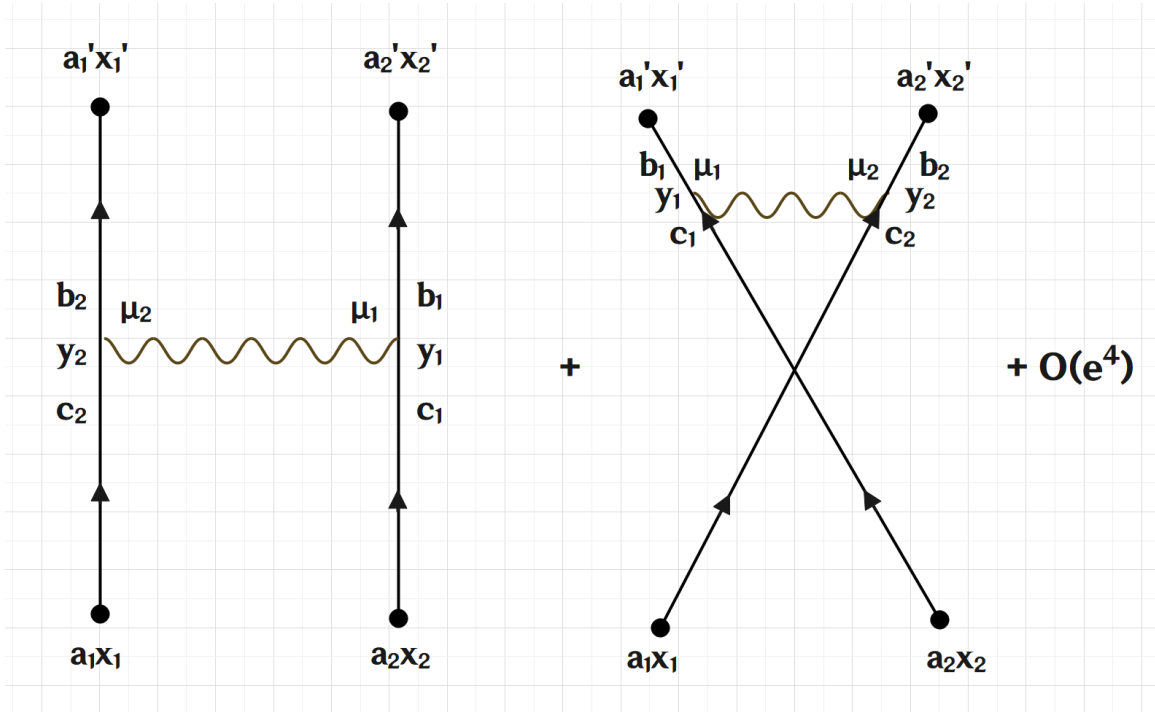


$$\begin{aligned}
& \frac{2}{2!} \psi_{2'} \psi_{1'} \bar{\psi}_1 \bar{\psi}_2 i \int d^4 y_1 A_{\mu_1}(y_1) \bar{\psi}_{b_1}(y_1) i e C_{b_1 c_1}^{\mu_1} \psi_{c_1}(y_1) i \int d^4 y_2 A_{\mu_2}(y_2) \bar{\psi}_{b_2}(y_2) i e C_{b_2 c_2}^{\mu_2} \psi_{c_2}(y_2) \\
& + \\
& \frac{2}{2!} \psi_{2'} \psi_{1'} \bar{\psi}_1 \bar{\psi}_2 i \int d^4 y_1 A_{\mu_1}(y_1) \bar{\psi}_{b_1}(y_1) i e C_{b_1 c_1}^{\mu_1} \psi_{c_1}(y_1) i \int d^4 y_2 A_{\mu_2}(y_2) \bar{\psi}_{b_2}(y_2) i e C_{b_2 c_2}^{\mu_2} \psi_{c_2}(y_2) \\
& + O(e^4)
\end{aligned}$$

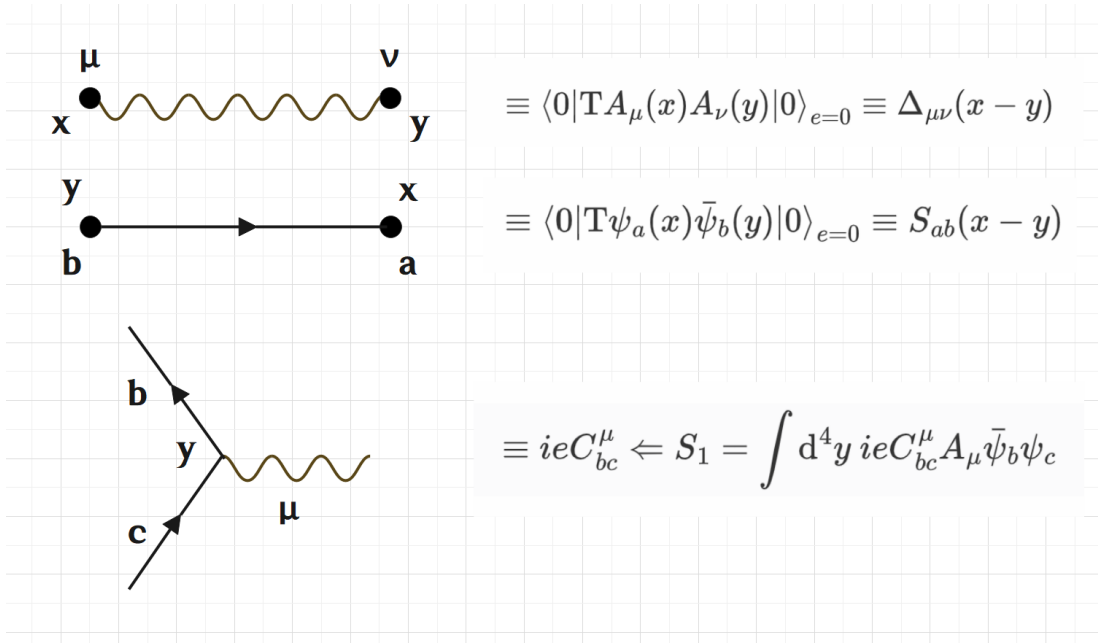
which is

$$\begin{aligned}
& = + \int d^4 y_1 d^4 y_2 i e C_{b_1 c_1}^{\mu_1} i e C_{b_2 c_2}^{\mu_2} \Delta_{\mu_1 \mu_2}(y_1 - y_2) S_{c_1 a_2}(y_1 - x_2) S_{c_2 a_1}(y_2 - x_1) S_{a'_2 b_1}(x'_2 - y_1) S_{a'_1 b_2}(x'_1 - y_2) \\
& - \int d^4 y_1 d^4 y_2 i e C_{b_1 c_1}^{\mu_1} i e C_{b_2 c_2}^{\mu_2} \Delta_{\mu_1 \mu_2}(y_1 - y_2) S_{c_1 a_2}(y_1 - x_2) S_{c_2 a_1}(y_2 - x_1) S_{a'_1 b_1}(x'_1 - y_1) S_{a'_2 b_2}(x'_2 - y_2) \\
& + O(e^4)
\end{aligned}$$

In diagrams,



Feynman rules for spinor QED in real spacetime:



integration measure  $\int d^4y_i$  for each internal vertex at  $y_i$ .

Substituting the eq above into LSZ formula and using

$$\not{p}u_s(\vec{p}) = imu_s(\vec{p}), \quad \bar{u}_s(\vec{p})\not{p} = im\bar{u}_s(\vec{p})$$

we get

$$\begin{aligned}
& i\mathcal{T}_{(p'_1s'_1+;p'_2s'_2+)\leftarrow(p_1s_1+;p_2s_2+)}(2\pi)^4\delta(p_1+p_2-p'_1-p'_2) \\
&= \frac{1}{(2m)^4}\bar{u}_{s'_2a'_2}(\vec{p}'_2)\bar{u}_{s'_1a'_1}(\vec{p}'_1)u_{s_1a_1}(\vec{p}_1)u_{s_2a_2}(\vec{p}_2)\int d^4x_1d^4x_2d^4x'_1d^4x'_2e^{ip_1x_1+ip_2x_2-ip'_1x'_1-ip'_2x'_2} \\
& (m^2-\partial_1^2)(m^2-\partial_2^2)(m^2-\partial_1'^2)(m^2-\partial_2'^2) \\
& \left[ \int d^4y_1d^4y_2ieC_{b_1c_1}^{\mu_1}ieC_{b_2c_2}^{\mu_2}\Delta_{\mu_1\mu_2}(y_1-y_2)S_{c_1a_2}(y_1-x_2)S_{c_2a_1}(y_2-x_1)S_{a'_2b_1}(x'_2-y_1)S_{a'_1b_2}(x'_1-y_2) \right. \\
& - \int d^4y_1d^4y_2ieC_{b_1c_1}^{\mu_1}ieC_{b_2c_2}^{\mu_2}\Delta_{\mu_1\mu_2}(y_1-y_2)S_{c_1a_2}(y_1-x_2)S_{c_2a_1}(y_2-x_1)S_{a'_1b_1}(x'_1-y_1)S_{a'_2b_2}(x'_2-y_2) \\
& \left. + O(e^4) \right] \\
&= \frac{1}{(2m)^4}\int d^4x_1d^4x_2d^4x'_1d^4x'_2e^{ip_1x_1+ip_2x_2-ip'_1x'_1-ip'_2x'_2}(m^2-\partial_1^2)(m^2-\partial_2^2)(m^2-\partial_1'^2)(m^2-\partial_2'^2) \\
& \left[ \int d^4y_1d^4y_2\bar{u}_{s'_2}(\vec{p}'_2)S(x'_2-y_1)ieC^{\mu_1}S(y_1-x_2)u_{s_2}(\vec{p}_2) \right. \\
& \bar{u}_{s'_1}(\vec{p}'_1)S(x'_1-y_2)ieC^{\mu_2}S(y_2-x_1)u_{s_1}(\vec{p}_1)\Delta_{\mu_1\mu_2}(y_1-y_2) \\
& - \int d^4y_1d^4y_2\bar{u}_{s'_2}(\vec{p}'_2)S(x'_2-y_1)ieC^{\mu_2}S(y_2-x_1)u_{s_1}(\vec{p}_1) \\
& \bar{u}_{s'_1}(\vec{p}'_1)S(x'_1-y_1)ieC^{\mu_1}S(y_1-x_2)u_{s_2}(\vec{p}_2)\Delta_{\mu_1\mu_2}(y_1-y_2) \\
& \left. + O(e^4) \right]
\end{aligned}$$

But

$$(m^2-\partial_1^2)S(y_2-x_1) = (m^2-\partial_1^2)\int \frac{-id^4k}{(2\pi)^4}\frac{m-i\not{k}}{k^2+m^2-i\varepsilon}e^{ik(y_2-x_1)} = \int \frac{-id^4k}{(2\pi)^4}(m-i\not{k})e^{ik(y_2-x_1)}$$

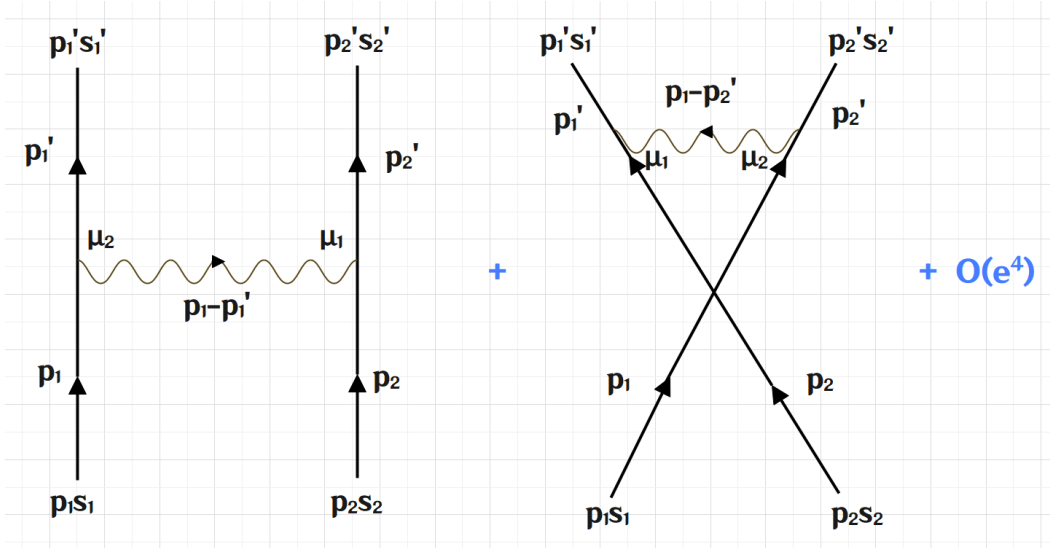
and similarly for other  $S$ 's. Hence

$$\begin{aligned}
& i\mathcal{T}_{(p'_1s'_1+p'_2s'_2+)\leftarrow(p_1s_1+p_2s_2+)}(2\pi)^4\delta(p_1+p_2-p'_1-p'_2) \\
&= \frac{1}{(2m)^4} \int d^4x_1 d^4x_2 d^4x'_1 d^4x'_2 d^4y_1 d^4y_2 e^{ip_1x_1+ip_2x_2-ip'_1x'_1-ip'_2x'_2} \frac{(-id^4k_1)}{(2\pi)^4} \frac{(-id^4k_2)}{(2\pi)^4} \frac{(-id^4k'_1)}{(2\pi)^4} \frac{(-id^4k'_2)}{(2\pi)^4} \\
& e^{ik'_2(x'_2-y_1)+ik_2(y_1-x_2)+ik'_1(x'_1-y_2)+ik_1(y_2-x_1)} \\
& \bar{u}_{s'_2}(\vec{p}'_2)(m-ik'_2)ieC^{\mu_1}(m-ik_2)u_{s_2}(\vec{p}_2)\bar{u}_{s'_1}(\vec{p}'_1)(m-ik'_1)ieC^{\mu_2}(m-ik_1)u_{s_1}(\vec{p}_1) \\
& \frac{-id^4l}{(2\pi)^4}\hat{\Delta}_{\mu_1\mu_2}(l)e^{il(y_1-y_2)} \\
&= \int \frac{-id^4l}{(2\pi)^4} i(2\pi)^4\delta(p_2-p'_2+l)i(2\pi)^4\delta(-p'_1+p_1-l)\bar{u}_{s'_1}(\vec{p}'_1)ieC^{\mu_2}u_{s_1}(\vec{p}_1)\bar{u}_{s'_2}(\vec{p}'_2)ieC^{\mu_1}u_{s_2}(\vec{p}_2)\hat{\Delta}_{\mu_1\mu_2}(l) \\
& - \int \frac{-id^4l}{(2\pi)^4} i(2\pi)^4\delta(-p'_1+p_2+l)i(2\pi)^4\delta(-p'_2+p_1-l)\bar{u}_{s'_1}(\vec{p}'_1)ieC^{\mu_1}u_{s_2}(\vec{p}_2)\bar{u}_{s'_2}(\vec{p}'_2)ieC^{\mu_2}u_{s_1}(\vec{p}_1)\hat{\Delta}_{\mu_1\mu_2}(l) \\
& + O(e^4) \\
&= i(2\pi)^4\delta(p_1+p_2-p'_1-p'_2)\left[\hat{\Delta}_{\mu_1\mu_2}(p_1-p'_1)(\bar{u}_{s'_1}(\vec{p}'_1)ieC^{\mu_2}u_{s_1}(\vec{p}_1))(\bar{u}_{s'_2}(\vec{p}'_2)ieC^{\mu_1}u_{s_2}(\vec{p}_2))\right. \\
& \left.- \hat{\Delta}_{\mu_1\mu_2}(p_1-p'_2)(\bar{u}_{s'_1}(\vec{p}'_1)ieC^{\mu_1}u_{s_2}(\vec{p}_2))(\bar{u}_{s'_2}(\vec{p}'_2)ieC^{\mu_2}u_{s_1}(\vec{p}_1)) + O(e^4)\right]
\end{aligned}$$

hence we get (with  $p_1 + p_2 = p'_1 + p'_2$ )

$$\begin{aligned}
\mathcal{T}_{(p'_1s'_1+p'_2s'_2+)\leftarrow(p_1s_1+p_2s_2+)} &= \hat{\Delta}_{\mu_1\mu_2}(p_1-p'_1)(\bar{u}_{s'_1}(\vec{p}'_1)ieC^{\mu_2}u_{s_1}(\vec{p}_1))(\bar{u}_{s'_2}(\vec{p}'_2)ieC^{\mu_1}u_{s_2}(\vec{p}_2)) \\
&- \hat{\Delta}_{\mu_1\mu_2}(p_1-p'_2)(\bar{u}_{s'_1}(\vec{p}'_1)ieC^{\mu_1}u_{s_2}(\vec{p}_2))(\bar{u}_{s'_2}(\vec{p}'_2)ieC^{\mu_2}u_{s_1}(\vec{p}_1)) + O(e^4)
\end{aligned}$$

whose diagram is



**Feynman rules in the momentum space:**

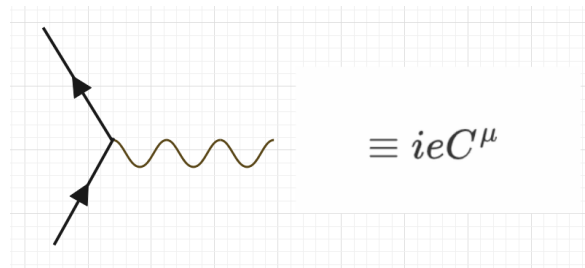
(1) internal fermion line:

$$\equiv \hat{S}(p) = \frac{m - i\not{p}}{p^2 + m^2 - i\varepsilon}$$

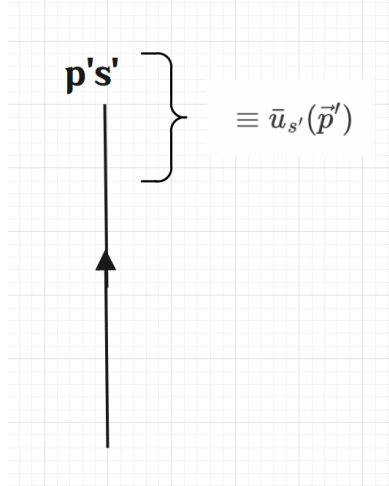
(2) internal photon line:

$$\equiv \hat{\Delta}_{\mu\nu}(l) = \frac{1}{l^2 - i\varepsilon} \left[ g_{\mu\nu} - (1 - \xi) \frac{l_\mu l_\nu}{l^2} \right]$$

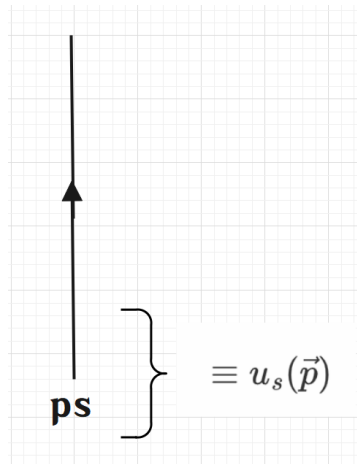
(3) 3-point vertex:



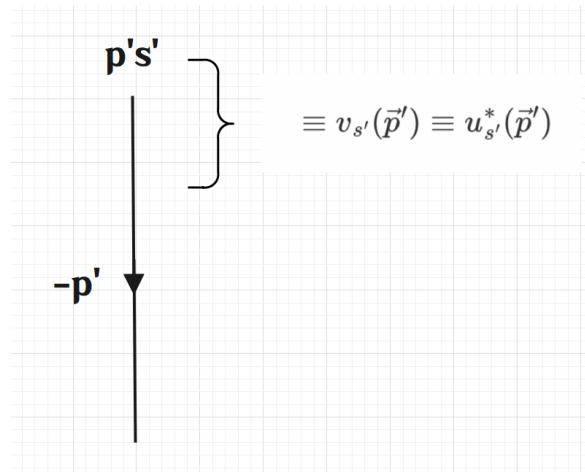
(4) outgoing b-type particle (such as electron) with 4-momentum  $p'$  in spin state  $s'$ : (external)



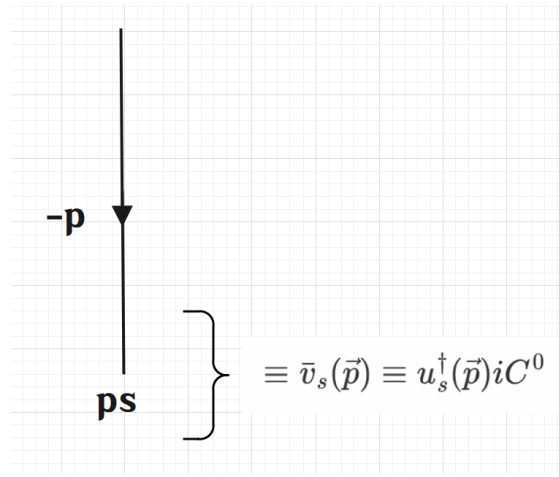
(5) incoming b-type particle (such as electron) with 4-momentum  $p$  in spin state  $s$ : (internal)



(6) outgoing d-type particle (such as positron) with 4-momentum  $p'$  in spin state  $s'$ : (internal)



(7) incoming d-type particle (such as positron) with 4-momentum  $p$  in spin state  $s$ : (internal)

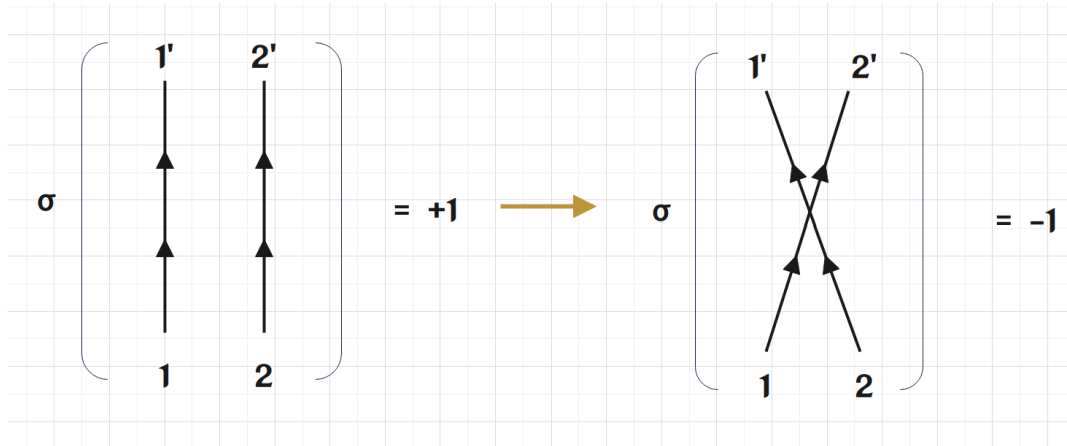


(8) spinor matrix multiplication along each continuous solid line (in the written order on paper opposite to the arrows on the fermion lines, starting from a  $\bar{u}$  or  $\bar{v}$  at the leftmost positron, ending at a  $u$  or  $v$  at the rightmost positron.) But for each closed fermion loop, there's no  $\bar{u}, \bar{v}, u, v$ .

(9) conserve 4-momentum at each vertex

(10) integrate over independent internal momenta  $l_i$  using the measure  $\prod_{i=1}^{N_I} \frac{-i d^4 l_i}{(2\pi)^4}$ , where  $N_I$  is the number of independent loops.

(11) Each diagram contains an overall factor  $\sigma = \pm 1$ , where  $\sigma$  is the signature of externally connected fermion paths. For example,



(12) Each closed fermion loop contains an extra factor of  $-1$ . Take the trace of the matrix multiplication for each closed fermion loop.

(13) Feynman rules for incoming and outgoing photons (to be introduced later)

(14) The T-matrix element  $\mathcal{T}$  is equal to the sum of all the topologically different diagrams (containing no vacuum bubbles) for the process.

(15)

$$\langle \text{final} | \text{initial} \rangle = i(2\pi)^4 \delta[(\text{total incoming 4-momentum} - (\text{total outgoing 4-momentum}))] \cdot \mathcal{T}$$

## LSZ reduction involving photons in spinor QED

First consider the noninteracting EM field:

In  $R_\xi$ -gauge:  $\langle 0 | A^\mu(x) | 0 \rangle = 0$ , with single-photon states  $|k, \lambda\rangle = a_\lambda^\dagger(\vec{k})|0\rangle$ , where

$$a_\lambda(\vec{k}) = \vec{e}_\lambda(\hat{k}) \int d^3x e^{-ikx} [\vec{E}(x) + i\lambda \vec{B}(x)], \quad \vec{e}_\lambda(\hat{k}) = \frac{1}{\sqrt{2}}(\vec{u}_1 - i\lambda \vec{u}_2), \quad \lambda = \pm 1$$

Using the formula

$$\langle 0|TA^\mu(x)A^\nu(y)|0\rangle = \int \frac{-i d^4 q}{(2\pi)^4} \hat{\Delta}^{\mu\nu}(q) e^{iq(x-y)}, \quad \hat{\Delta}^{\mu\nu}(q) = \frac{1}{q^2 - i\varepsilon} \left[ g^{\mu\nu} - (1 - \xi) \frac{q^\mu q^\nu}{q^2} \right]$$

we get

$$\langle k, \lambda | A^\mu(x) | 0 \rangle = \varepsilon_\lambda^\mu(\hat{k}) e^{-ikx}, \quad \varepsilon_\lambda^\mu(\hat{k}) = (0, i\vec{e}_\lambda(\hat{k}))$$

Also define

$$\varepsilon_{\lambda\mu}(\hat{k}) \equiv g_{\mu\nu} \varepsilon_\lambda^\nu(\hat{k}) = (0, i\vec{e}_\lambda(\hat{k}))$$

they satisfy

$$\varepsilon_\lambda^{\mu*}(\hat{k}) \varepsilon_{\lambda'\mu}(\hat{k}) = \delta_{\lambda\lambda'}, \quad \varepsilon_\lambda^\mu(\hat{k}) k_\mu = 0$$

hence

$$\langle k, \lambda' | \varepsilon_\lambda^{\mu*}(\hat{k}) A_\mu(x) | 0 \rangle = \delta_{\lambda\lambda'} e^{-ikx}$$

Then study the interacting theory (EM-field coupled with Dirac fields and/or complex scalar fields):  $\langle 0 | A^\mu(x) | 0 \rangle$  guaranteed by charge conjugation symmetry and nondegeneracy of vacuum.

Since  $\mathcal{U}(C)H = H\mathcal{U}(C)$ , we have  $H(\mathcal{U}(C)|0\rangle) = \mathcal{U}(C)H|0\rangle = E_0(\mathcal{U}(C)|0\rangle)$ .

If vacuum  $|0\rangle$  is nondegenerate, then it's an eigenstate of  $\mathcal{U}(C)$ , with  $|\text{eigenvalue}| = 1$ .

Thus

$$\langle 0 | A^\mu(x) | 0 \rangle = \langle 0 | \mathcal{U}(C)^\dagger A^\mu(x) \mathcal{U}(C) | 0 \rangle = -\langle 0 | A^\mu(x) | 0 \rangle \implies \langle 0 | A^\mu(x) | 0 \rangle = 0$$

Single-photon states are still normalized as

$$\langle k, \lambda | k', \lambda' \rangle = 2|\vec{k}|(2\pi)^3 \delta(\vec{k} - \vec{k}') \delta_{\lambda\lambda'}$$

We may choose the overall amplitude of  $A^\mu(x)$  such that  $\langle k, \lambda' | \varepsilon_\lambda^{\mu*}(\hat{k}) A_\mu(x) | 0 \rangle = \delta_{\lambda\lambda'} e^{-ikx}$  is still satisfied, which is called on-shell scheme for the amplitude of  $A^\mu(x)$ .

We may then use the spacetime smeared version of  $\varepsilon_\lambda^{\mu*}(\hat{k}) A_\mu(x)$  to generate properly-normalized single-photon states from the vacuum.

Example:

$$\text{electron}(p_1 s_1 +) + \text{positron}(p_2 s_2 -) \rightarrow \gamma_{(\text{photon})}(k'_1 \lambda'_1) + \gamma_{(\text{photon})}(k'_2 \lambda'_2)$$

we have

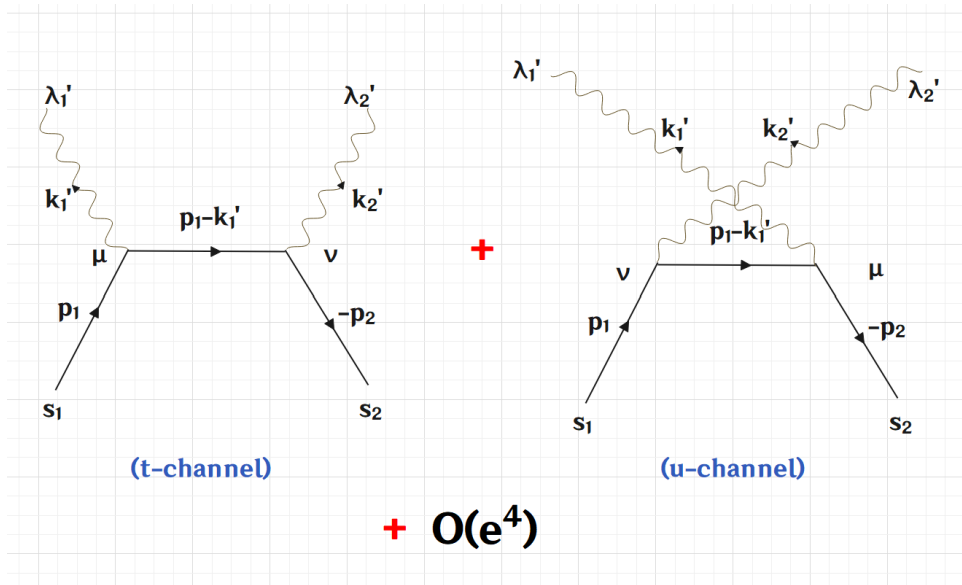
$$\begin{aligned} i\mathcal{T}(2\pi)^4 \delta(p_1 + p_2 - k'_1 - k'_2) &= \frac{1}{(2m)(-2m)} \int d^4 x_1 d^4 x_2 d^4 x'_1 d^4 x'_2 e^{ip_1 x_1 + ip_2 x_2 - ik'_1 x'_1 - ik'_2 x'_2} \\ &\quad (m^2 - \partial_1^2)(m^2 - \partial_2^2)(-\partial_{1'}^2)(-\partial_{2'}^2) \\ &\quad \langle 0 | T(\varepsilon_{\lambda'_2}^\nu(\hat{k}'_2) A_\nu(x'_2))^\dagger (\varepsilon_{\lambda'_1}^\mu(\hat{k}'_1) A_\mu(x'_1))^\dagger (\bar{\psi}(x_1) u_{s_1}(\vec{p}_1)) (\bar{v}_{s_2}(\vec{p}_2) \psi(x_2)) | 0 \rangle \end{aligned}$$

where  $v_s(\vec{p}) \equiv u_s^*(\vec{p})$ .

Doing the perturbative calculation similar to what we did for the  $b + b \rightarrow b + b$  process, using the spinor QED Lagrangian, after spending **six pages** doing the algebra,

$$\begin{aligned} \mathcal{T} &= \varepsilon_{\lambda'_1\mu}(\hat{k}'_1) \varepsilon_{\lambda'_2\nu}(\hat{k}'_2) \bar{v}_{s_2}(\vec{p}_2) i e C^\nu \hat{S}(p_1 - k'_1) i e C^\mu u_{s_1}(\vec{p}_1) \\ &\quad + \varepsilon_{\lambda'_1\mu}(\hat{k}'_1) \varepsilon_{\lambda'_2\nu}(\hat{k}'_2) \bar{v}_{s_2}(\vec{p}_2) i e C^\mu \hat{S}(p_1 - k'_2) i e C^\nu u_{s_1}(\vec{p}_1) + O(e^4) \end{aligned}$$

whose diagram is

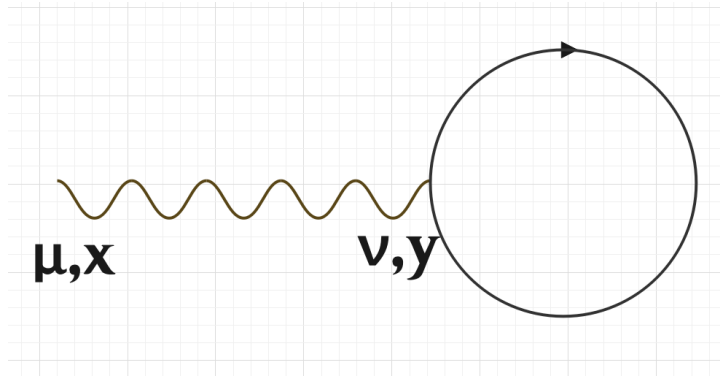


Given  $\mathcal{T}$ , one can calculate the scattering cross section. The rule is the same as before.

For the scattering cross sections involving spinful particles (such as Dirac particles and/or photons):

- Sum over final spin states (if we don't measure the spin states of outgoing particles);
- Average over initial spin states (if the spin states of the incoming particles are prepared randomly)

For  $\langle 0|A_\mu(x)|0\rangle$ , the loop diagram is



which

$$\begin{aligned}
 &\propto \int d^4y \Delta_{\mu\nu}(x-y) \text{tr}[C^\nu S(y-y)] \\
 &\propto \int d^4y \Delta_{\mu\nu}(x-y) \int \frac{-id^4q}{(2\pi)^4} \frac{\text{tr}[C^\nu(m-i\cancel{q})]}{q^2+m^2-i\epsilon} \\
 &= \int d^4y \Delta_{\mu\nu}(x-y) \int \frac{-id^4q}{(2\pi)^4} \frac{-4iq^\nu}{q^2+m^2-i\epsilon} \\
 &= 0
 \end{aligned}$$

which is consistent with charge conjugation symmetry.

Closed fermion loop comes from

$$\int d^4y_1 ieC_{b_1c_1}^{\mu_1}(y_1)A_{\mu_1}(y_1)\bar{\psi}_{b_1}(y_1)\psi_{c_1}(y_1) \int d^4y_2 ieC_{b_2c_2}^{\mu_2}(y_2)A_{\mu_2}(y_2)\bar{\psi}_{b_2}(y_2)\psi_{c_2}(y_2) \cdots \int d^4y_n ieC_{b_nc_n}^{\mu_n}(y_n)A_{\mu_n}(y_n)\bar{\psi}_{b_n}(y_n)\psi_{c_n}(y_n)$$

When contracting  $\bar{\psi}_{b_1}(y_1)$  with  $\psi_{c_n}(y_n)$ , we pick up a minus sign. Hence for each closed fermion loop in the diagram, there's an extra factor of  $-1$ .

## Higher order perturbations in spinor QED

We need to consider loop diagrams

Some of the loop diagrams are UV divergent. But only three basic types of diagrams remain UV divergent after we take into account the charge conjugation symmetry and gauge invariance:

1. photon self energy
2. fermion self energy
3. 3-point vertex function with 2 fermion lines and 1 photon line

Remarkably, they're related to the terms in  $\mathcal{L}$ .

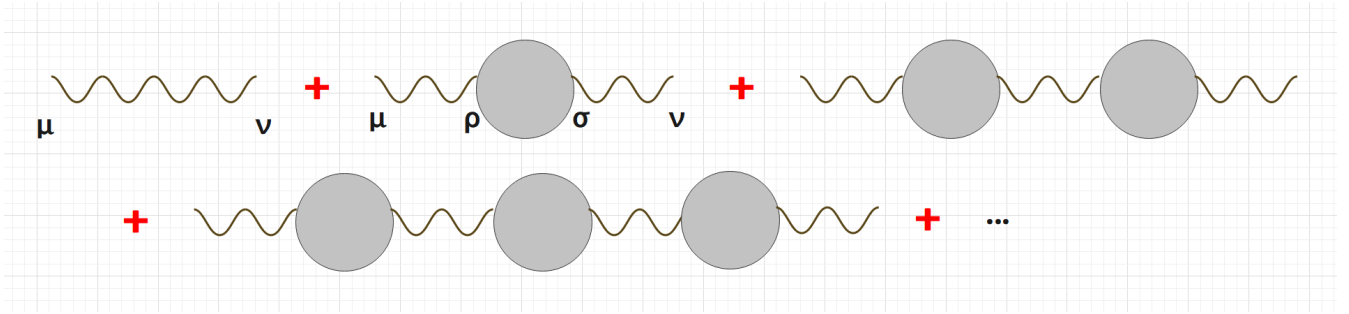
All the degrees of freedom in the UV divergent integrals can be fixed, with only two physical inputs for the theory:  $e$ ,  $m$ .

So the spinor QED is renormalizable.

But we're unable to predict the value of the elementary charge  $e$ , even though  $e$  is a dimensionless number in natural units ( $c = \hbar = \varepsilon_0 = \mu_0 = 1$ )

### Photon self energy

To study the photon self energy, we first study the exact photon propagator:  $\hat{\Delta}_{\mu\nu}(k)$



which equals (Dyson series)

$$\hat{\Delta}_{\mu\nu}(k) + \hat{\Delta}_{\mu\rho}(k)\Pi^{\rho\sigma}(k)\hat{\Delta}_{\sigma\nu}(k) + \hat{\Delta}_{\mu\rho_1}(k)\Pi^{\rho_1\sigma_1}(k)\hat{\Delta}_{\sigma_1\rho_2}(k)\Pi^{\rho_2\sigma_2}(k)\hat{\Delta}_{\sigma_2\nu}(k) + \dots$$

where

$$\hat{\Delta}_{\mu\nu}(k) = \frac{1}{k^2 - i\varepsilon} \left[ g_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2} \right]$$

Observable amplitudes should not depend on  $\xi$ . This suggests that  $\Pi^{\mu\nu}(k)$  should be transverse

$$k_\mu \Pi^{\mu\nu}(k) = k_\nu \Pi^{\mu\nu}(k) = 0$$

This can be proved using the Ward identity for the charge current.

Since the theory is Lorentz invariant, we expect that for any proper orthochronous Lorentz transformation  $\Lambda^\mu_\nu$  that leaves  $k$  invariant, it must also leave  $\Pi^{\mu\nu}(k)$  invariant:

$$\text{If } \Lambda^\mu_\nu k^\nu = k^\mu, \quad \text{then } \Lambda^\mu_\rho \Lambda^\nu_\sigma \Pi^{\rho\sigma}(k) = \Pi^{\mu\nu}(k)$$

One can show that this implies

$$\Pi^{\mu\nu}(k) = f_1(k)g^{\mu\nu} + f_2(k)k^\mu k^\nu$$

Lorentz invariance of the theory requires  $f_1, f_2$  depend on  $k^2$  only. Hence

$$\Pi^{\mu\nu}(k) = g_1(k^2)g^{\mu\nu} - g_2(k^2)k^\mu k^\nu$$

Substituting into transverse condition we get

$$g_1(k^2)k^\nu - g_2(k^2)k^2 k^\nu = 0 \implies g_1(k^2) = k^2 g_2(k^2)$$

Thus

$$\Pi^{\mu\nu}(j) = g_2(k^2)(k^2 g^{\mu\nu} - k^\mu k^\nu) \equiv \Pi(k^2)(k^2 g^{\mu\nu} - k^\mu k^\nu)$$

Let



$$P^{\mu\nu}(k) \equiv g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}$$

which is the projection matrix onto the 3D hypersurface orthogonal to  $k$ . Then

$$\Pi^{\mu\nu}(k) = k^2 \Pi(k^2) P^{\mu\nu}(k), \quad \hat{\Delta}_{\mu\nu} = \frac{1}{k^2 - i\varepsilon} \left[ P_{\mu\nu}(k) + \xi \frac{k_\mu k_\nu}{k^2} \right]$$

and the exact propagator

$$\hat{\Delta}_{\mu\nu}(k) = \frac{P_{\mu\nu}(k)}{k^2[1 - \Pi(k^2)] - i\varepsilon} + \xi \frac{k_\mu k_\nu}{k^2 - i\varepsilon}$$

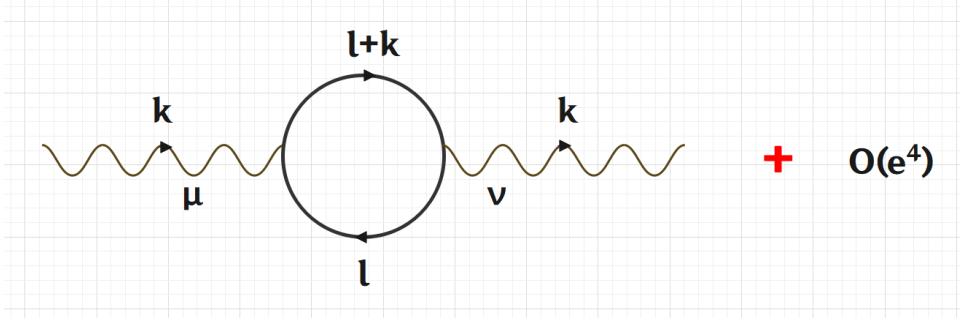
The  $\xi$ -dependent term is physically irrelevant, and can be set to zero by the choice  $\xi = 0$  corresponding to the Landau gauge.

The remaining term has a pole at  $k^2 = 0$  with residue  $\frac{P_{\mu\nu}(k)}{1 - \Pi(0)}$ ,

In the on-shell scheme for the amplitude of  $A_\mu$ , we should have

$$\frac{P_{\mu\nu}(k)}{1 - \Pi(0)} = P_{\mu\nu}(k) \implies \Pi(0) = 0$$

Let's now turn to the calculation of  $\Pi^{\mu\nu}(k)$ . The diagram is



which equals

$$(-1) \int \frac{-i d^4 l}{(2\pi)^4} \text{Tr}[\hat{S}(l+k) i e C^\mu \hat{S}(l) i e C^\nu] + O(e^4) = \frac{e^2}{(2\pi)^4} \int (-i d^4 l) \text{Tr}[\hat{S}(l+k) C^\mu \hat{S}(l) C^\nu] + O(e^4)$$

Let  $k_4 \equiv k^4 \equiv i k^0$ ,  $l_4 \equiv l^4 \equiv i l^0$ ,  $C_4 \equiv C^4 \equiv i C^0$  we have

$$\frac{1}{2} \{C^i, C^j\} = \delta^{ij}, \quad \text{Tr}(C^i C^j) = 4 \delta^{ij} \quad 1 \leq i, j \leq 4$$

Perform simultaneous Wick rotation of  $k^0, l^0$  by  $90^\circ$  CCW.

Also let

$$\Pi^{4i}(k) \equiv i \Pi^{0i}(k), \quad \Pi^{i4}(k) \equiv i \Pi^{i0}(k), \quad 1 \leq i \leq 3, \quad \Pi^{44}(k) \equiv i^2 \Pi^{00}(k)$$

We get

$$\Pi^{ij}(k) = \frac{e^2}{(2\pi)^4} \int d^4 l_E \text{Tr} \left[ \frac{m - i(l+k)}{(l+k)^2 + m^2} C^i \frac{m - i l}{l^2 + m^2} C^j \right] + O(e^4), \quad 1 \leq i, j \leq 4$$

where

$$l = -l^0 C^0 + l^1 C^1 + l^2 C^2 + l^3 C^3 = l^1 C^1 + l^2 C^2 + l^3 C^3 + l^4 C^4$$

and similarly for  $k$ . Doing Feynman parametrization we get

$$\int d^4 l_E \text{Tr} \left[ \frac{m - i(l+k)}{(l+k)^2 + m^2} C^i \frac{m - i l}{l^2 + m^2} C^j \right] = \int_0^1 dx \int d^4 l_E \frac{4 \{ [l(l+k) + m^2] \delta^{ij} - (l^i + k^i) l^j - (l^j + k^j) l^i \}}{[(l+xk)^2 + D]^2}$$

where  $D \equiv x(1-x)k^2 + m^2$ .

Setting  $l = q - xk$  and dropping terms linear in  $q$  in the numerator in the integrand (since they integrate to zero), we find

$$\begin{aligned} & \int d^4 l_E \text{Tr} \left[ \frac{m - i(l + k)}{(l + k)^2 + m^2} C^i \frac{m - i l}{l^2 + m^2} C^j \right] \\ &= 4 \int_0^1 dx \int d^4 q_E \frac{[q^2 - x(1-x)k^2 + m^2] \delta^{ij} - 2q^i q^j + 2x(1-x)k^i k^j}{(q^2 + D)^2} \end{aligned}$$

Using formula

$$\int d^4 q_E q^i q^j f(q^2) = \frac{1}{4} \delta^{ij} \int d^4 q_E q^2 f(q^2)$$

we get

$$\begin{aligned} & \int d^4 l_E \text{Tr} \left[ \frac{m - i(l + k)}{(l + k)^2 + m^2} C^i \frac{m - i l}{l^2 + m^2} C^j \right] \\ &= 4 \int_0^1 dx \int_0^\infty 2\pi^2 q^3 dq \frac{[\frac{q^2}{2} - x(1-x)k^2 + m^2] \delta^{ij} + 2x(1-x)k^i k^j}{(q^2 + D)^2} \end{aligned}$$

Then truncate

$$\int_0^\infty = \int_0^Q + \int_Q^\infty$$

we have

$$\begin{aligned} \int d^4 l_E \text{Tr} \left[ \frac{m - i(l + k)}{(l + k)^2 + m^2} C^i \frac{m - i l}{l^2 + m^2} C^j \right] &= \int_0^1 dx \left\{ [2\pi^2 Q^2 + 2\pi^2 D(1 - 4 \ln Q) + 4\pi^2 D \ln D] \delta^{ij} \right. \\ &\quad + [2x(1-x)k^i k^j + (m^2 - x(1-x)k^2) \delta^{ij}] [4\pi^2(-1 + 2 \ln Q) - 4\pi^2 \ln D] \\ &\quad \left. + O\left(\frac{1}{Q}\right) + \int_Q^\infty (\dots) \right\} \end{aligned}$$

All the dependencies on  $Q$  should be canceled, leaving us with a result having three degrees of freedom:

$$\begin{aligned} & \int d^4 l_E \text{Tr} \left[ \frac{m - i(l + k)}{(l + k)^2 + m^2} C^i \frac{m - i l}{l^2 + m^2} C^j \right] \\ &= 4\pi^2 \delta^{ij} \int_0^1 dx [x(1-x)k^2 + m^2] \ln \frac{D}{m^2} \\ &\quad - 4\pi^2 \int_0^1 dx \{ 2x(1-x)k^i k^j + [m^2 - x(1-x)k^2] \delta^{ij} \} \ln \frac{D}{m^2} + (c_1 + c_2 k^2) \delta^{ij} + c_3 k^i k^j \\ &= 8\pi^2 (k^2 \delta^{ij} - k^i k^j) \int_0^1 dx x(1-x) \ln \frac{D}{m^2} + (c_1 + c_2 k^2) \delta^{ij} + c_3 k^i k^j \end{aligned}$$

Hence

$$\Pi^{ij}(k) = \frac{e^2}{2\pi^2} \left[ (k^2 \delta^{ij} - k^i k^j) \int_0^1 dx x(1-x) \ln \frac{D}{m^2} + (b_1 + b_2 k^2) \delta^{ij} + b_3 k^i k^j \right] + O(e^4)$$

Coming back to Minkowski spacetime we have

$$\Pi^{\mu\nu}(k) = \frac{e^2}{2\pi^2} \left[ (k^2 g^{\mu\nu} - k^\mu k^\nu) \int_0^1 dx x(1-x) \ln \frac{D}{m^2} + (b_1 + b_2 k^2) g^{\mu\nu} + b_3 k^\mu k^\nu \right] + O(e^4)$$

Imposing the condition  $k_\mu \Pi^{\mu\nu}(k) = 0$  we get

$$b_1 = 0, \quad b_3 = -b_2$$

thus

$$\Pi^{\mu\nu}(k) = \frac{e^2}{2\pi^2} (k^2 g^{\mu\nu} - k^\mu k^\nu) \left[ b_2 + \int_0^1 dx x(1-x) \ln \frac{x(1-x)k^2 + m^2}{m^2} \right] + O(e^4) = (k^2 g^{\mu\nu} - k^\mu k^\nu) \Pi(k^2)$$

hence

$$\Pi(k^2) = \frac{e^2}{2\pi^2} \left[ b_2 + \int_0^1 dx x(1-x) \ln \frac{x(1-x)k^2 + m^2}{m^2} \right] + O(e^4)$$

so

$$\Pi(0) = \frac{e^2}{2\pi^2} b_2 + O(e^4) = 0 \implies b_2 = 0$$

then

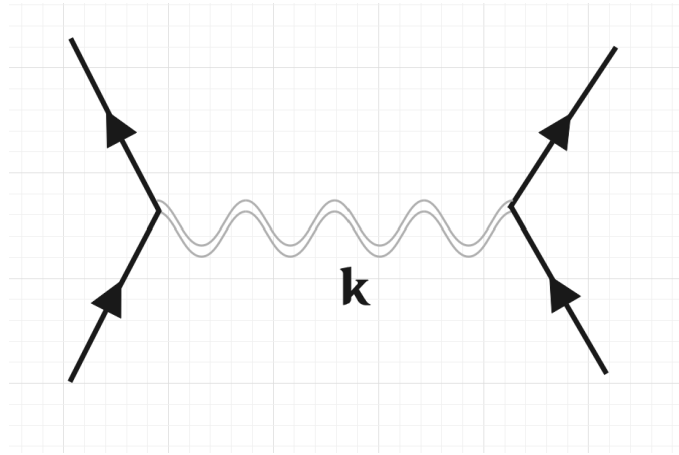
$$\Pi(k^2) = \frac{e^2}{2\pi^2} \int_0^1 dx x(1-x) \ln \frac{x(1-x)k^2 + m^2}{m^2} + O(e^4)$$

The integral can be done analytically.

When  $k^2 \gg m^2$ ,

$$\Pi(k^2) \approx \frac{e^2}{2\pi^2} \left( \frac{1}{6} \ln \frac{k^2}{m^2} - \frac{5}{18} \right) = \frac{e^2}{6\pi^2} \left( \ln \frac{k}{m} - \frac{5}{6} \right)$$

For high-energy collisions of two charged particles (with virtual photon momentum  $k \gg m$ ), the diagram is



which

$$\propto \frac{e^2}{k^2[1 - \Pi(k^2)]}$$

Effective electric charge  $\tilde{e}(k)$  satisfies

$$\tilde{e}(k)^2 \approx \frac{e^2}{1 - \Pi(k^2)} \approx \frac{e^2}{1 - \frac{e^2}{6\pi^2} \left( -\frac{5}{6} + \ln \frac{k}{m} \right)} > e^2$$

So the effective charge for high-energy collisions is slightly larger than the elementary charge.

This is consistent with the so-called running coupling constant

$$\frac{d\tilde{e}}{d \ln k} \approx \frac{\tilde{e}^3}{12\pi^2}$$

The effective strength of electromagnetic interaction grows at shorter length scales


$$l \sim \frac{1}{k} \ll \frac{1}{m}$$

i.e.  $l \ll$  Compton wavelength of electron. Thus higher energies ( $k \gg m$ ).

## Spinor QED

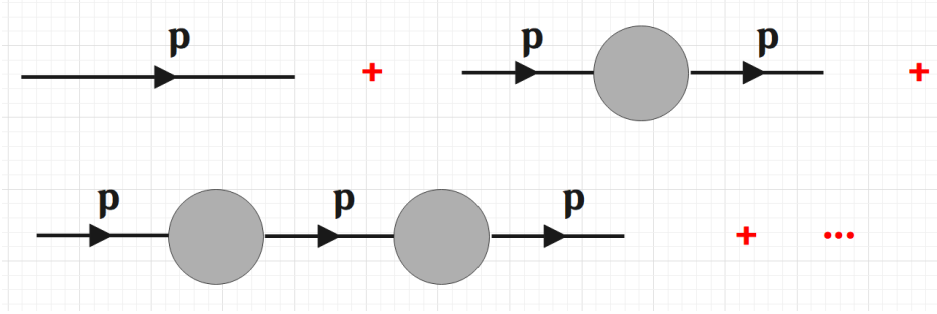
$$\mathcal{L} = -\bar{\psi}(\not{\partial} + m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu\bar{\psi}ieC^\mu\psi$$

Free fermion propagator:



$$\equiv \hat{S}(p) = \frac{m - i\not{p}}{p^2 + m^2 - i\varepsilon}$$

Exact fermion propagator:



which is

$$\begin{aligned}\hat{S}(p) &= \hat{S}(p) + \hat{S}(p)\Sigma(\not{p})\hat{S}(p) + \hat{S}(p)\Sigma(\not{p})\hat{S}(p)\Sigma(\not{p})\hat{S}(p) + \dots \\ &= \hat{S}(p)[1 - \Sigma(\not{p})\hat{S}(p)]^{-1} \\ &= [(1 - \Sigma(\not{p})\hat{S}(p))\hat{S}(p)^{-1}]^{-1} \\ &= (\hat{S}(p)^{-1} - \Sigma(\not{p}))^{-1} \\ &= \frac{1}{i\not{p} + m - \Sigma(\not{p}) - i\varepsilon}\end{aligned}$$

where  $\Sigma(\not{p})$  is the fermion self energy.

On-shell scheme for  $\psi$ :  $\hat{S}(p)$ , when expressed as a function of  $\not{p}$ , has a pole at  $\not{p} = im$ , with residue  $\frac{1}{i}$ . Hence

$$\Sigma(im) = 0, \quad \Sigma'(im) = 0$$

Lehmann-Kallen form of the exact fermion propagator:

$$\hat{S}(p) = \frac{1}{i\not{p} + m - i\varepsilon} + \int_{m_{th}^2}^{\infty} ds \frac{\rho(s)}{i\not{p} + \sqrt{s} - i\varepsilon}$$

$m_{th}$ : minimum center-of-mass energy of a composite system consisting of a fermion together with one or more photons.

But the photon has zero rest mass! Hence  $m_{th} = m$ .

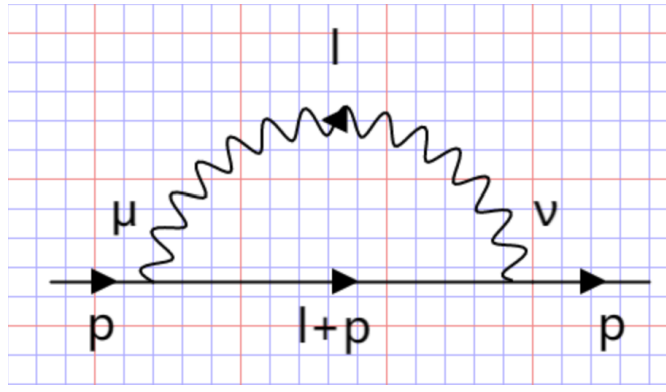
So the pole in the first term is not well-isolated from the branch cut at  $i\not{p} \leq -m$ , and the residue of the pole is ill-defined. Physically, this is related to the fact that the detector cannot distinguish a 1-fermion state from those states in which there's this fermion together with one or more very-low-energy photons, as long as the total energy of those soft photons is less than the detector's energy sensitivity  $E_0$ .

To be able to use the LSZ formula (which requires the 1-particle state to be energetically separated from the 2-particle and more-particle states), we assign the photon a tiny mass  $m_\gamma$ , satisfying  $m_\gamma \ll E_0$ . This is called **infrared cutoff**.

Then  $m_{th} = m + m_\gamma$ , and now the pole in  $\hat{S}(p)$  is now separated from the branch cut.

When calculating experimentally measurable cross sections, we include the contribution to the cross section from those processes in which there're one or more soft photons whose total energy  $< E_0$ , and then send  $m_\gamma \rightarrow 0$ .

1-loop contribution to  $\Sigma$ :



with

$$\Sigma(\not{p}) = \int \frac{-id^4l}{(2\pi)^4} ieC^\nu \hat{S}(p+l) ieC^\mu \hat{\Delta}_{\mu\nu}(l) + O(e^4) = \Sigma_{1\text{loop}}(\not{p}) + O(e^4)$$

(For photon, the sequence of  $\mu, \nu$  doesn't matter)

It's simplest to work in the Feynman gauge ( $\xi = 1$ )

$$\hat{\Delta}_{\mu\nu}(l) = \frac{g_{\mu\nu}}{l^2 - i\varepsilon}$$

Assigning a small mass  $m_\gamma$  to the photon we may assume

$$\hat{\Delta}_{\mu\nu}(l) = \frac{g_{\mu\nu}}{l^2 + m_\gamma^2 - i\varepsilon}$$

hence

$$\Sigma_{1\text{loop}}(\not{p}) = -e^2 \int \frac{-id^4l}{(2\pi)^4} C^\nu \frac{m - i(\not{p} + \not{l})}{(p+l)^2 + m^2 - i\varepsilon} C^\mu \frac{g_{\mu\nu}}{l^2 + m_\gamma^2 - i\varepsilon}$$

After  $90^\circ$ -Wick rotation CCW on the complex frequency plane we get

$$\Sigma_{1\text{loop}}(\not{p}) = -e^2 \int \frac{d^4l_E}{(2\pi)^4} C^i \frac{m - i(\not{p} + \not{l})}{(p+l)^2 + m^2} C^i \frac{1}{l^2 + m_\gamma^2}$$

Feynman parametrization:

$$\Sigma_{1\text{loop}}(\not{p}) = -\frac{e^2}{(2\pi)^4} \int_0^1 dx \int d^4l_E \frac{4m + 2i(\not{l} + \not{p})}{[(l+xp)^2 + D]^2}$$

where  $D \equiv x(1-x)p^2 + xm^2 + (1-x)m_\gamma^2$ . Set  $l+xp = q$  we have

$$\Sigma_{1\text{loop}}(\not{p}) = -\frac{e^2}{(2\pi)^4} \int_0^1 dx \int d^4q_E \frac{4m + 2i[\not{q} + (1-x)\not{p}]}{(q^2 + D)^2} = -\frac{e^2}{(2\pi)^4} \int_0^1 dx \int d^4q_E \frac{4m + 2i(1-x)\not{p}}{(q^2 + D)^2}$$

Integral over  $q$  is logarithmically divergent. At large  $q$ , the integrand =  $\frac{4m + 2i(1-x)\not{p}}{q^4} + O(q^{-6})$ .

So the contribution to the integral from  $q > Q$  is of the form

$$c_2 + c_1 i\not{p} + (\text{terms independent on } Q)$$

Contribution to the integral from  $q < Q$  can be calculated analytically and expanded at large  $Q$ .

Adding the two contributions, the terms dependent on  $Q$  are canceled. Hence

$$\Sigma_{1\text{loop}}(\not{p}) = +\frac{e^2}{8\pi^2} \left\{ \int_0^1 dx [2m + (1-x)i\not{p}] \ln \frac{D}{D_0} + c_1 i\not{p} + c_2 \right\}, \quad D_0 = D|_{p^2=-m^2} = x^2 m^2 + (1-x)m_\gamma^2$$

Since  $p^2 = \not{p}^2$  in  $D$ ,  $\Sigma_{1\text{loop}}(\not{p})$  is indeed a function of  $\not{p}$  only.

Calculating  $\Sigma_{1\text{loop}}(\not{p})$  analytically and imposing the on-shell conditions  $\Sigma(im) = 0, \Sigma'(im) = 0$  we get

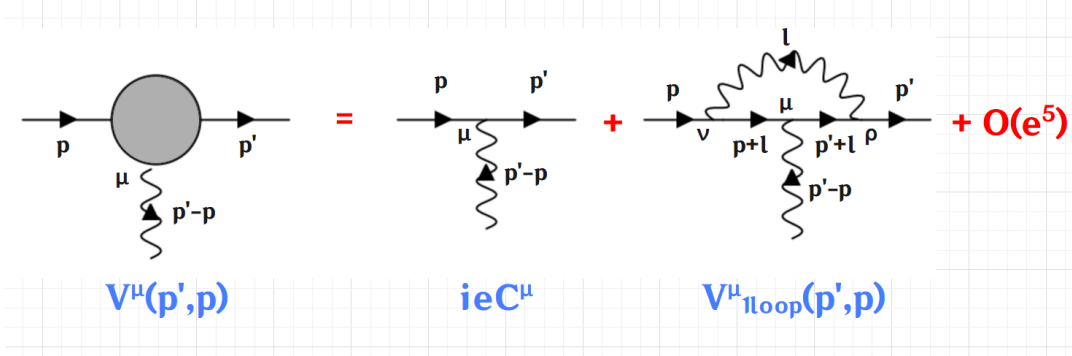
$$\begin{cases} c_1 = 1 - 2 \ln \frac{m}{m_\gamma} + O\left(\frac{m_\gamma}{m}\right) \\ c_2 = mc_1 \end{cases}$$

Thus

$$\Sigma_{1\text{loop}}(\not{p}) = \frac{e^2}{8\pi^2} \left\{ \int_0^1 dx [2m + (1-x)i\not{p}] \ln \frac{D}{D_0} + c_1(i\not{p} + m) \right\} + O(e^4)$$

3-point vertex function:  $V^\mu(p', p)$ : sum of 1PI diagrams with two fermion legs and one photon leg.

In diagram,



The loop part

$$\begin{aligned} V_{1\text{loop}}^\mu(p', p) &= \int \frac{-id^4l}{(2\pi)^4} ieC^\rho \hat{S}(p' + l) ieC^\mu \hat{S}(p + l) ieC^\nu \hat{\Delta}_{\nu\rho}(l) \\ &= -ie^3 \int \frac{-id^4l}{(2\pi)^4} C^\rho \frac{m - i(\not{p}' + \not{l})}{(p' + l)^2 + m^2 - i\varepsilon} C^\mu \frac{m - i(\not{p} + \not{l})}{(p + l)^2 + m^2 - i\varepsilon} C^\nu \frac{g_{\nu\rho}}{l^2 + m_\gamma^2 - i\varepsilon} \end{aligned}$$

After  $90^\circ$  CCW Wick rotation of  $p^0, l^0$  we get

$$V_{1\text{loop}}^i(p', p) = -ie^3 \int \frac{d^4l_E}{(2\pi)^4} C^j \frac{m - i(\not{p}' + \not{l})}{(p' + l)^2 + m^2} C^i \frac{m - i(\not{p} + \not{l})}{(p + l)^2 + m^2} C^j \frac{1}{l^2 + m_\gamma^2}$$

where  $V_{1\text{loop}}^4(p', p) = iV_{1\text{loop}}^0(p', p)$ .

After Feynman parametrization and defining new variable  $q = l + x_1 p + x_2 p'$  we get

$$V_{1\text{loop}}^i(p', p) = -ie^3 \int dF_3 \int \frac{d^4q_E}{(2\pi)^4} (q^2 + D)^{-3} [4q_i \not{q} - 2q^2 C^i + M^i(x_1, x_2, p, p')]$$

where

$$\int dF_3 \equiv 2 \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \delta(x_1 + x_2 + x_3 - 1)$$

$$D \equiv x_1(1 - x_1)p^2 + x_2(1 - x_2)p'^2 - 2x_1x_2pp' + (x_1 + x_2)m^2 + x_3m_\gamma^2$$

$$\begin{aligned} M^i(x_1, x_2, p, p') &\equiv -2(1 - x_1)(1 - x_2)\not{p}C^i\not{p}' - 2x_1x_2\not{p}C^i\not{p}' - 2C^i[m^2 - x_1(1 - x_1)p^2 - x_2(1 - x_2)p'^2] \\ &\quad + 4(2x_1x_2 - x_1 - x_2 + 1)(p^i\not{p}' + p'^i\not{p} - p \cdot p' C^i) \\ &\quad + 4im(2x_1 - 1)p^i + 4im(2x_2 - 1)p'^i - 4x_1(1 - x_1)p^i\not{p}' - 4x_2(1 - x_2)p'^i\not{p} \end{aligned}$$

The integral over  $q$  is logarithmically divergent at large  $q$ .

Let  $Q$  be a large number then

$$\text{integrand} = q^{-6}(4q^i \not{q} - 2q^2 C^i) + O(q^{-6}), \quad q > Q$$

and

$$\begin{aligned}
\int dF_3 \int_{q>Q} \frac{d^4 q_E}{(2\pi)^4} q^{-6} (4q^i q^j - 2q^2 C^i) &= \int dF_3 \int_{q>Q} \frac{d^4 q_E}{(2\pi)^4} \frac{4q^i q^j C^j - 2q^2 C^i}{q^6} \\
&= \int dF_3 \int_{q>Q} \frac{d^4 q_E}{(2\pi)^4} \frac{4q^{i^2} C^i - 2q^2 C^i}{q^6} \\
&= \int dF_3 \int_{q>Q} \frac{d^4 q_E}{(2\pi)^4} \frac{q^2 C^i - 2q^2 C^i}{q^6} \\
&= \left( \int dF_3 \int_{q>Q} \frac{d^4 q_E}{(2\pi)^4} \frac{-1}{q^4} \right) C^i \\
&= (c_1 + (\text{term dependent on } Q)) C^i
\end{aligned}$$

Adding the contribution to the integral from  $q < Q$  we get

$$V_{\text{loop}}^i(p', p) = \frac{-ie^3}{16\pi} \int dF_3 \left[ \frac{M^i}{2D} + C^i \left( \zeta + \ln \frac{D}{D_0} \right) \right]$$

where  $\zeta$  is free parameter, and  $D_0 = D|_{p'=p, p^2=-m^2} = (1 - x_3)^2 m^2 + x_3 m_\gamma^2$ .

To determine  $\zeta$ , we need to sharpen our definition of  $e$ .

In classical electrodynamics,  $\frac{e^2}{4\pi}$  is defined as the coefficient in Coulomb's law  $f = \frac{e^2}{4\pi} \frac{1}{r^2}$  for two electrons, where  $r$  is the distance between two electrons. This Coulomb's law formula is valid only when  $r \gg \frac{1}{m}$  (namely when the distance is much larger than the Compton wavelength of the electron), or else quantum effects will be significant. This corresponds to the electron-electron scattering process in which the momentum  $q$  of the exchanged virtual photon is small:  $q \sim \frac{1}{r} \ll m$ , and both electrons remain on-shell before and after the scattering.

So we define  $e$  such that

$$\bar{u}_{s'}(\vec{p}') V^\mu(p', p) u_s(\vec{p}) = \bar{u}_{s'}(\vec{p}') i e C^\mu u_s(\vec{p}), \quad \text{if } p' = p, p^2 = -m^2$$

Hence

$$\bar{u}_{s'}(\vec{p}) [i e C^\mu + V_{\text{loop}}^\mu(p, p) + O(e^5)] u_s(\vec{p}) = \bar{u}_{s'}(\vec{p}) i e C^\mu u_s(\vec{p}), \quad \text{if } p^2 = -m^2$$

which means

$$\bar{u}_{s'}(\vec{p}) V_{\text{loop}}^\mu u_s(\vec{p}) = 0, \quad \text{if } p^2 = -m^2$$

At  $p' = p, p^2 = -m^2$  we get

$$V_{\text{loop}}^\mu = -\frac{ie^3}{16\pi^2} \left[ (\zeta - 3) C^\mu + \frac{8i}{m} p^\mu (1 + \ln \varepsilon) + \frac{2}{m^2} (-3 - 2 \ln \varepsilon) p^\mu \not{p} + O(\varepsilon) \right], \quad \varepsilon \equiv \frac{m_\gamma}{m} \ll 1$$

Thus

$$\bar{u}_{s'}(\vec{p}) V_{\text{loop}}^\mu u_s(\vec{p}) = -\frac{ie^3}{16\pi^2} 2i p^\mu \delta_{s's} (-\zeta + 5 + 4 \ln \varepsilon + O(\varepsilon)) = 0$$

Hence we get

$$\zeta = 5 + 4 \ln \varepsilon + O(\varepsilon) = 5 - 4 \ln \frac{m}{m_\gamma} + O\left(\frac{m_\gamma}{m}\right)$$

### On shell form of the 3-point vertex function

Given  $V^\mu(p', p)$ , we can extract some physics from it.

In the scattering of an electron with some other charged particles, we need  $\bar{u}_{s'}(\vec{p}') V^\mu(p', p) u_s(\vec{p})$  at  $p'^2 = -m^2, p^2 = -m^2, p^0 > 0, p^0 > 0$ .

Using the Gordon identity

$$\bar{u}_{s'}(\vec{p}') (p^\mu + p'^\mu) u_s(\vec{p}) = \bar{u}_{s'}(\vec{p}') (2m i C^\mu + 2i S^{\mu\nu} q_\nu) u_s(\vec{p}), \quad S^{\mu\nu} \equiv -\frac{i}{4} [C^\mu, C^\nu], \quad q = p' - p$$

and other properties of  $u$ , we can show that when  $p', p$  are both on-shell,

$$\bar{u}_{s'}(\vec{p}') V^\mu(p', p) u_s(\vec{p}) = e \bar{u}_{s'}(\vec{p}') \left[ F_1(q^2) i C^\mu - \frac{i}{m} F_2(q^2) S^{\mu\nu} q_\nu \right] u_s(\vec{p})$$

where  $F_1(q^2), F_2(q^2)$  are functions dependent on  $q^2$  only., and called form factors,

$$F_1(q^2) = 1 - \frac{e^2}{16\pi^2} \int dF_3 \left[ \frac{(x_3 + x_1 x_2) \frac{q^2}{m^2} - (1 - 4x_3 + x_3^2)}{x_1 x_2 \frac{q^2}{m^2} + (1 - x_3)^2 + x_3 \varepsilon^2} + \frac{1 - 4x_3 + x_3^2}{(1 - x_3)^2 + x_3 \varepsilon^2} \right. \\ \left. + \ln \left( 1 + \frac{x_1 x_2}{(1 - x_3)^2} \frac{q^2}{m^2} \right) \right] + O(e^4) \\ F_2(q^2) = \frac{e^2}{8\pi^2} \int dF_3 \frac{x_3(1 - x_3)}{x_1 x_2 \frac{q^2}{m^2} + (1 - x_3)^2} + O(e^4)$$

where we've ignored  $O(\varepsilon)$  contributions.

Doing the integral,

$$F_2(q^2) = \frac{e^2}{8\pi^2 \frac{q}{m} \sqrt{1 + \frac{q^2}{4m^2}}} \ln \left( 1 + \frac{q^2}{2m^2} + \frac{q}{m} \sqrt{1 + \frac{q^2}{4m^2}} \right) + O(e^4)$$

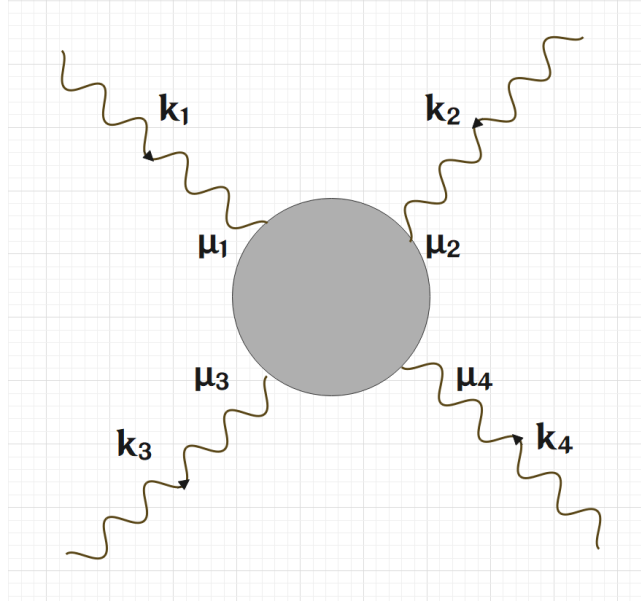
Define fine-structure constant  $\alpha \equiv \frac{e^2}{4\pi}$ , we can show that

$$\begin{cases} F_1(0) = 1 + O(\alpha^2) \\ F_2(0) = \frac{\alpha}{2\pi} + O(\alpha^2) \end{cases}$$

These results will be used when we calculate the magnetic moment of the electron.

#### 4-photon vertex function

The function  $V^{\mu_1 \mu_2 \mu_3 \mu_4}(k_1, k_2, k_3, k_4)$ , in diagram,



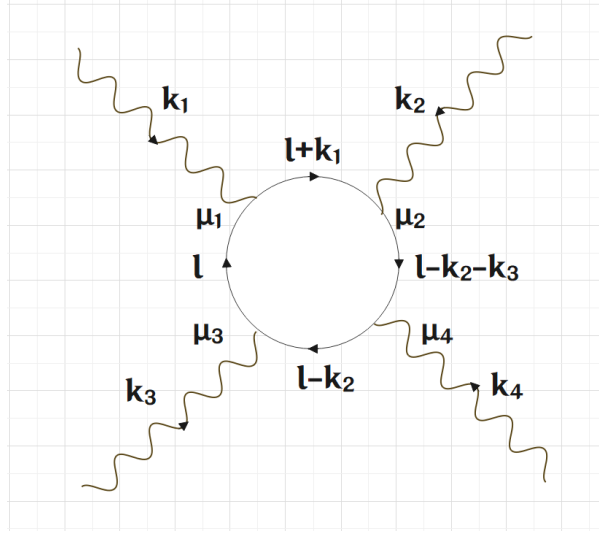
with  $k_1 + k_2 + k_3 + k_4 \equiv 0$ , and is sum of 1PI diagrams with 4 wavy-line legs.

It equals

$$f^{\mu_1 \mu_2 \mu_3 \mu_4}(k_1, k_2, k_3, k_4) + f^{\mu_1 \mu_3 \mu_4 \mu_2}(k_1, k_3, k_4, k_2) + f^{\mu_1 \mu_4 \mu_2 \mu_3}(k_1, k_4, k_2, k_3) \\ + f^{\mu_1 \mu_4 \mu_3 \mu_2}(k_1, k_4, k_3, k_2) + f^{\mu_1 \mu_2 \mu_4 \mu_3}(k_1, k_2, k_4, k_3) + f^{\mu_1 \mu_3 \mu_2 \mu_4}(k_1, k_3, k_2, k_4) + O(e^6) \\ \equiv g^{\mu_1 \mu_2 \mu_3 \mu_4}(k_1, k_2, k_3, k_4) + O(e^6)$$



where the diagram of  $f^{\mu_1\mu_2\mu_3\mu_4}(k_1, k_2, k_3, k_4)$  is



which equals

$$- \int \frac{-id^4l}{(2\pi)^4} \text{Tr} \left[ ieC^{\mu_1} \hat{S}(l) ieC^{\mu_2} \hat{S}(l-k_2) ieC^{\mu_3} \hat{S}(l-k_2-k_3) ieC^{\mu_4} \hat{S}(l+k_1) \right]$$

After Wick rotation we get

$$f^{i_1 i_2 i_3 i_4}(k_1, k_2, k_3, k_4) = -e^4 \int \frac{d^4l_E}{(2\pi)^4} \text{Tr} \left[ C^{i_1} \frac{m - i\hat{l}}{l^2 + m^2} C^{i_2} \frac{m - i\hat{l} + i\hat{k}_2}{(l-k_2)^2 + m^2} \right. \\ \left. C^{i_3} \frac{m - i\hat{l} + i\hat{k}_2 + i\hat{k}_3}{(l-k_2-k_3)^2 + m^2} C^{i_4} \frac{m - i\hat{l} - i\hat{k}_1}{(l+k_1)^2 + m^2} \right]$$

which is logarithmically divergent at large  $l$ .

But the most divergent term in the integrand is of the form

$$\frac{\text{Tr}(C^{i_1} \not{l} C^{i_2} \not{l} C^{i_3} \not{l} C^{i_4} \not{l})}{l^8} \equiv K^{i_1 i_2 i_3 i_4}(l)$$

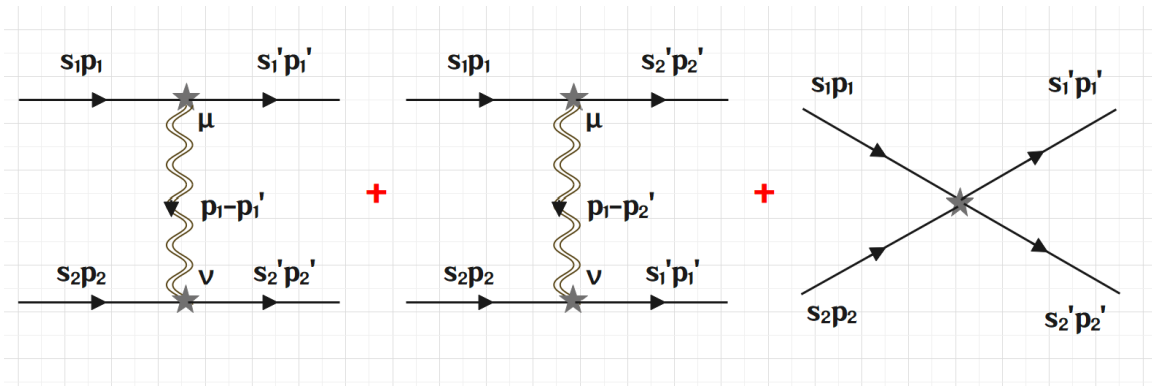
we can show that

$$\int d^3\hat{l} [K^{i_1 i_2 i_3 i_4}(l) + K^{i_1 i_3 i_4 i_2}(l) + K^{i_1 i_4 i_2 i_3}(l) + K^{i_1 i_4 i_3 i_2}(l) + K^{i_1 i_2 i_4 i_3}(l) + K^{i_1 i_3 i_2 i_4}(l)] = 0$$

thus  $g^{\mu_1\mu_2\mu_3\mu_4}(k_1, k_2, k_3, k_4)$  is convergent.

T-matrix element for any process: sum of all the tree-level diagrams made of exact vertex functions and exact propagators.

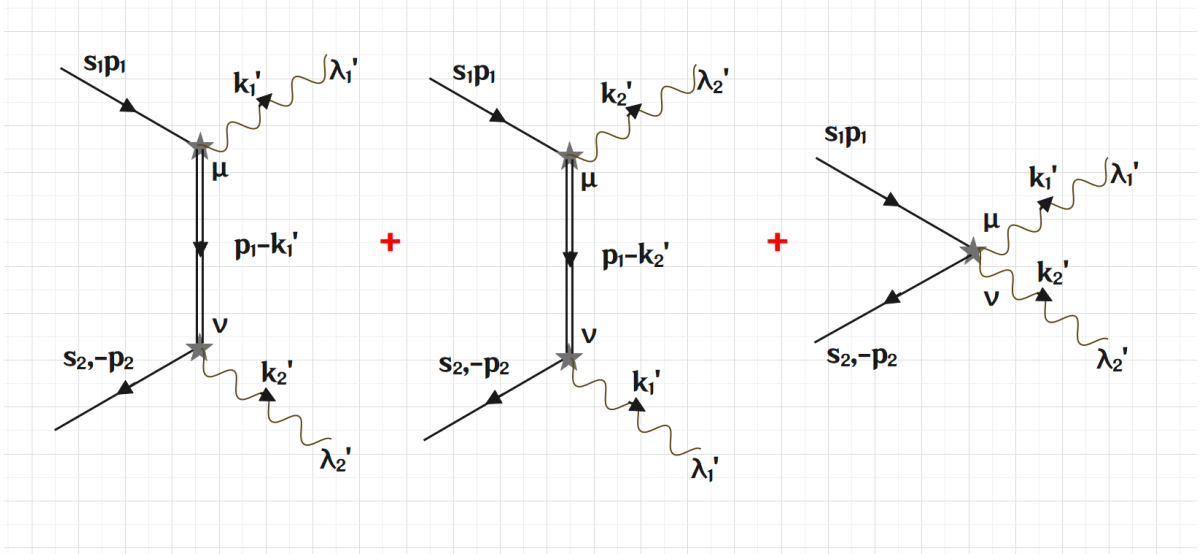
Example 1:  $b + b \rightarrow b + b$ , the T-matrix  $\mathcal{T}$  equals



which equals

$$[\bar{u}_{s'_1}(\vec{p}'_1)V^\mu(p'_1, p_1)u_{s_1}(\vec{p}_1)][\bar{u}_{s'_2}(\vec{p}'_2)V^\mu(p'_2, p_2)u_{s_2}(\vec{p}_2)]\hat{\Delta}_{\mu\nu}(p_1 - p'_1) - (s'_1 p'_1 \leftrightarrow s'_2 p'_2) \\ + \bar{u}_{s'_1 a'_1}(\vec{p}'_1)\bar{u}_{s'_2 a'_2}(\vec{p}'_2)u_{s_1 a_1}(\vec{p}_1)u_{s_2 a_2}(\vec{p}_2)V_{a'_1 a'_2 a_1 a_2}(p'_1 p'_2 p_1 p_2)$$

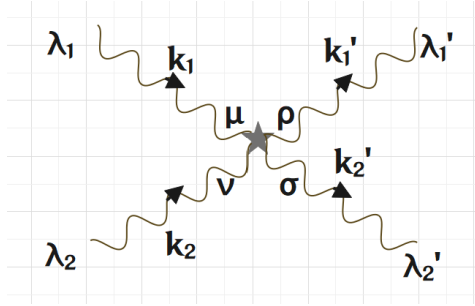
Example 2:  $b + d \rightarrow \gamma + \gamma$



which equals

$$\bar{v}_{s_2}(\vec{p}_2)V^\nu(-p_2, p_1 - k_1)\hat{S}(p_1 - k_1)V^\mu(p_1 - k_1, p_1)u_{s_1}(\vec{p}_1)\varepsilon_{\lambda'_1\mu}(\hat{k}'_1)\varepsilon_{\lambda'_2\nu}(\hat{k}'_2) + (k'_1\lambda'_1 \leftrightarrow k'_2\lambda'_2) \\ + \bar{v}_{s_2 a_2}(\vec{p}_2)u_{s_1 a_1}(\vec{p}_1)V_{a_1 a_2}^{\mu\nu}(k'_1, k'_2, p_1, -p_2)\varepsilon_{\lambda'_1\mu}(\hat{k}'_1)\varepsilon_{\lambda'_2\nu}(\hat{k}'_2)$$

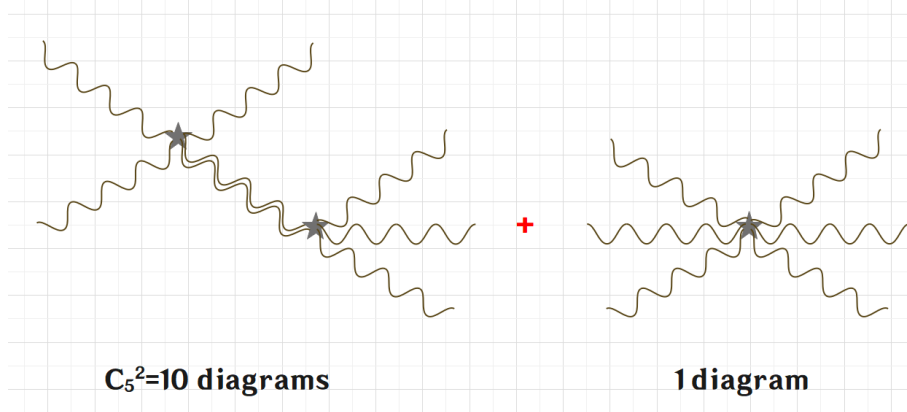
Example 3:  $\gamma + \gamma \rightarrow \gamma + \gamma$



which equals

$$\varepsilon_{\lambda_1\mu}^*(\hat{k}_1)\varepsilon_{\lambda_2\nu}^*(\hat{k}_2)\varepsilon_{\lambda'_1\rho}(\hat{k}'_1)\varepsilon_{\lambda'_2\sigma}(\hat{k}'_2)V^{\mu\nu\rho\sigma}(k_1, k_2, -k'_1, -k'_2)$$

Example 4:  $\gamma_1 + \gamma_2 \rightarrow \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6$



In spinor QED:

$$\mathcal{L} = -\bar{\psi}(\not{\partial} + m)\psi - \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) + A_\mu \bar{\psi}ieC^\mu\psi$$

Consider the dimension

$$[m\bar{\psi}\psi] = 4 \implies [\psi] = [\bar{\psi}] = \frac{3}{2}, \quad [A] = 1$$

If we introduce an extra term of the form  $g\bar{\psi}^{\frac{E_e}{2}}\psi^{\frac{E_e}{2}}A^{E_\gamma}$  into the Lagrangian density, then

$$[g] = 4 - \frac{3}{2}E_e - E_\gamma$$

hence

$$D = 4 - \frac{3}{2}E_e - E_\gamma = [g] = [g] - V[e]$$

noting that  $[e] = 0$ .