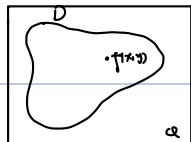


$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(\xi_i) \Delta x_i$$



$\{(x_1, \dots, x_n) \mid a_i \leq x_i \leq b_i\}$ 直方体.

$$Vol = \prod_{i=1}^n (b_i - a_i)$$

$D \subset \mathbb{R}^n$ 有界

$D \subset \mathbb{R}^n$ 直方体

用矩形分割直方体 $P = \{P_i\}_{i=1}^{np}$

$$P_i := \{(x_1, \dots, x_n) \mid a_j^i \leq x_j \leq b_j^i, j=1, 2, \dots, n\}$$

$$\|P\| := \max \{b_j^i - a_j^i, j=1, 2, \dots, n\}$$

$$\lim_{\|P\| \rightarrow 0} \sum_{i=1}^{np} f(\xi_i) Vol(P_i)$$

P 细

定义: $L(D, P) := \sum_{i=1}^{np} f(\xi_i) Vol(P_i)$

$$L(D) := \lim_{\|P\| \rightarrow 0} L(D, P) = \sup P L(D, P)$$

$$U(D, P) := \sum_{i=1}^{np} f(\eta_i) Vol(P_i)$$

$$U(D) := \lim_{\|P\| \rightarrow 0} U(D, P) = \inf P U(D, P)$$

Jordan 可测集

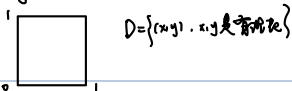
定义: 设 $D \subset \mathbb{R}^n, n \geq 2$ 为有界集, 若 $L(D) = U(D)$, 则称 D 为 Jordan 可测集

$$\int_D d\sigma = L(D) = U(D)$$

$$\int_D d\sigma, n=2 \quad \int_D d\sigma, n=3$$

"用直方体逼近" 逼近集合.

注: Jordan 不可测



Jordan 可测: 边界可积

计算 "面积" 的面积 D Jordan 可测

$f: D \rightarrow \mathbb{R}$ "函数"

$$\tilde{f}(x_1, x_2, \dots, x_n) = \begin{cases} f(x_1, \dots, x_n) & (x_1, \dots, x_n) \in D \\ 0 & (x_1, \dots, x_n) \in \mathbb{Q}^n \setminus D \end{cases}$$

$$\int_D f d\sigma = \int_{\mathbb{Q}^n} \tilde{f} d\sigma$$

重积分

$$\lim_{\|P\| \rightarrow 0} \sum_{i=1}^{np} f(\xi_i) Vol(P_i)$$

$$\int_D f d\sigma = D \text{ 的面积}$$

D 上取 1, 其值为 1

设 $D \subset \mathbb{R}^n$ 有界, D Jordan 可测 \Leftrightarrow 边界可积. $f: D \rightarrow \mathbb{R}$,

$$\tilde{f} = \begin{cases} f(x_1, \dots, x_n) & (x_1, \dots, x_n) \in D \\ 0 & (x_1, \dots, x_n) \in \mathbb{Q}^n \setminus D \end{cases}$$

设 D 为 n 个直方体并 $P = \{P_i\}_{i=1}^{np}$

若 $\sum_{i=1}^{np} f(\xi_i) Vol(P_i)$ 其中 $\xi_i \in P_i$ 任意

若 $\lim_{\|P\| \rightarrow 0} \sum_{i=1}^{np} f(\xi_i) Vol(P_i)$ 存在, 则称 f 在 D 上可积. 记其值为 $\int_D f d\sigma$

注: f 在 D 上可积 $\Leftrightarrow \tilde{f}$ 在 \mathbb{Q}^n 上可积

① f 在 D 上有界

$$\Leftrightarrow \text{ex. } f(x, y) = \begin{cases} 1 & x, y \in \mathbb{Q} \cap [a, b] \\ 0 & \text{其他} \end{cases}$$

$$\Leftrightarrow f \text{ 在 } D \text{ 上可积} \Leftrightarrow \lim_{\|P\| \rightarrow 0} \sum_{i=1}^{np} f(\xi_i) Vol(P_i) = 0$$

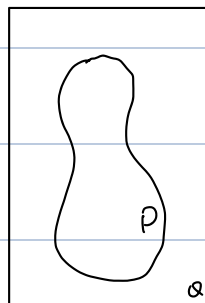
② f 在 D 上可积

$$\sup \{f(x) \mid x \in P_i\} - \inf \{f(x) \mid x \in P_i\} \rightarrow 0$$

③ f 在 D 上连续, 则 f 在 D 上可积

④ f 在 D 上可积 \Leftrightarrow 不连续点 Lebesgue 测度为 0.

$$\int_D f d\sigma = \int_{\mathbb{Q}^n} \tilde{f} d\sigma$$



$$S_D = \iint_D 1 d\sigma = \iint_D 1 d\sigma \quad b(x, y) = \begin{cases} 1 & (x, y) \in D \\ 0 & (x, y) \notin D \end{cases}$$

积分计算 Fubini 定理 (重积分 \rightarrow 多次积分) 积分非空

Thm. 设 f 在 $D = [a, b] \times [c, d]$ 上连续, 则

$$\int_a^b \left(\int_c^d f(x, y) dy \right) dx = \iint_D f(x, y) d\sigma = \int_c^d \left(\int_a^b f(x, y) dx \right) dy$$

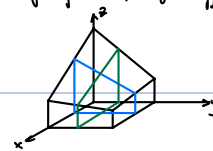
$$\iint_D f(x, y) d\sigma \leq \sum_{i=1}^n \sum_{j=1}^m f(\xi_i, \eta_j) \Delta x_i \Delta y_j$$

$$I(x) = \int_c^d f(x, y) dy \quad \int_a^b I(x) dx \leq \sum_{i=1}^n I(\xi_i) \Delta x_i$$

$$I(y) = \int_a^b f(x, y) dx \quad \int_c^d I(y) dy \leq \sum_{j=1}^m I(\eta_j) \Delta y_j$$

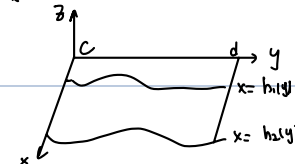
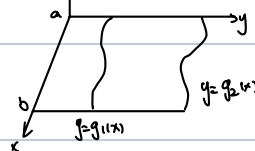
eg. $f(x, y) = x - y$ $\iint_D f(x, y) d\sigma$ 以 f 为底, D 为底的柱体的体积

$$D = \{(x, y) \mid x, y \in [0, 1]\}$$



两种情形

$$\iint_D f(x, y) d\sigma = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

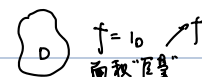


$$\iint_D f(x, y) d\sigma = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

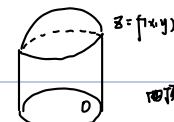
多重积分 $\Leftrightarrow \lim_{\|P\| \rightarrow 0} \sum_{i=1}^{np} f(\xi_i) Vol(P_i)$

$\int_D f(x_1, \dots, x_n) d\sigma$ 直方体有限个逼近 $D \subset \mathbb{R}^n$ Jordan 可测集

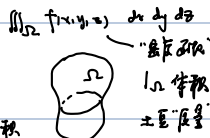
$$\iint_D f(x, y) d\sigma$$



$f = 1_0$ 面积 "度量"



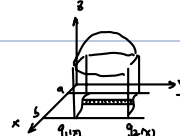
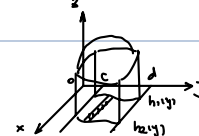
圆柱体的体积



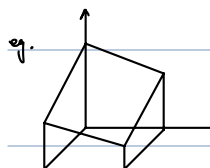
"度量" 度量

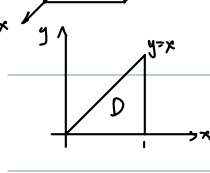
Fubini 定理

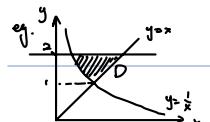
$$\int_a^b \left(\int_{h_1(y)}^{h_2(y)} f(x, y) dx \right) dy = \int \left(\int_{h_1(y)}^{h_2(y)} f(x, y) dx \right) dy$$



$$\iint_D f(x, y) d\sigma = \int_a^b \left(\int_{h_1(y)}^{h_2(y)} f(x, y) dx \right) dy = \int \left(\int_{h_1(y)}^{h_2(y)} f(x, y) dx \right) dy$$

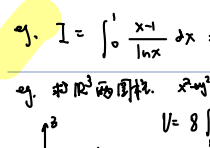
eg.  $\iint_D (4-x-y) dx dy$
 $0 \leq x \leq 2$
 $0 \leq y \leq 1$
 $\int_0^1 \int_0^2 (4-x-y) dx dy = \int_0^1 (4y - \frac{1}{2}y^2) dy = 7 - 2 = 5$

 $\iint_D (4-x-y) dx dy$
 $= \int_0^1 \int_0^x (4-x-y) dy dx$
 $= \int_0^1 (4x - \frac{1}{2}x^2) dx$
 $= (2x^2 - \frac{1}{6}x^3) \Big|_0^1 = \frac{11}{6}$

eg.  $\int_1^2 \int_{y/2}^y f(x,y) dx dy$
 $\int_1^2 (\int_{y/2}^y \frac{\sin x}{x} dx) dy = \int_1^2 (\int_{x/2}^x \frac{\sin x}{x} dx) dx = 1$

eg. $I = \int_0^1 \frac{x-1}{\ln x} dx = \int_0^1 \int_0^x x^{-y} dy dx = \int_0^1 \frac{x^{-y}}{y} dy = \int_0^1 \frac{1}{y} dy = \ln 2$

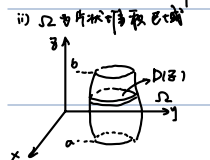
eg. 求 \mathbb{R}^2 内两圆环. $x^2+y^2 \leq a^2$ 和 $x^2+y^2 \leq a^2$ 相交部分之体积

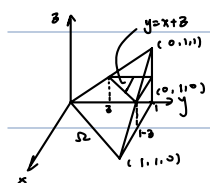
 $V = 8 \iint_D \sqrt{a^2-x^2-y^2} dx dy$
 $= 8 \int_0^{2\pi} \int_0^{\frac{\sqrt{2}}{2}a} \sqrt{a^2-r^2} r dr d\theta$
 $= \frac{16}{3} a^3$

三重积分计算之 Fubini 定理

i) Ω 为柱状区域 $f(x,y,z)$ 在 Ω 上连续.
 $z = h_2(x,y)$
 $z = h_1(x,y)$
 D 为 xy -平面上的区域 Jordan 可测. $\Rightarrow f(x,y,z)$
 $\iiint_{\Omega} f(x,y,z) dx dy dz = \iint_D (\int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z) dz) dx dy$

eg. 求 $\Omega = \{x^2+y^2 \leq z^2 \leq 8-x^2-y^2\}$ 所围区域之体积
 $D: x^2+y^2 = 8-x^2-y^2 \Rightarrow x^2+y^2 = 4 \Rightarrow x=2\cos\theta, y=2\sin\theta$
 $V = \iint_D dV = \iint_D (8-2x^2-2y^2) dx dy = \int_0^{2\pi} \int_0^2 (8-8r^2) r dr d\theta = 8\pi$

ii) Ω 为所求体积区域 $\iint_{\Omega} f(x,y,z) dx dy dz = \int_0^b [\iint_{D(z)} f(x,y,z) dx dy] dz$


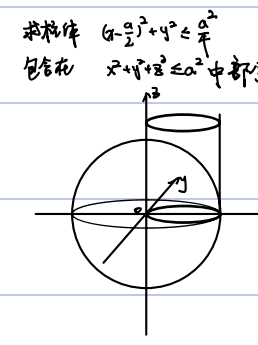
 $\iint_{\Omega} f(x,y,z) dx dy dz$
 $= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} f(x,y,z) dz dy dx$
 $= \int_0^1 [\int_0^{1-x} \int_0^{1-x-y} f(x,y,z) dy dz] dx$

eg. n 维情形
 $B_n(a) = \{x_1, \dots, x_n, x_1^2 + x_2^2 + \dots + x_n^2 \leq a^2\}$
 $V_n = V_n(B_n(a))$
 $\int_{\mathbb{R}^n} f(x_1, \dots, x_n, y_1, \dots, y_m) dx_1 \dots dx_n dy_1 \dots dy_m$
 $= \int_{\mathbb{R}^n} (\int_{\mathbb{R}^m} f(x_1, \dots, x_n, y_1, \dots, y_m) dy_1 \dots dy_m) dx_1 \dots dx_n$

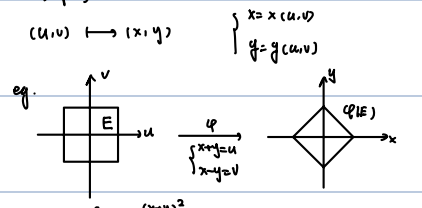
极坐标下重积分之计算

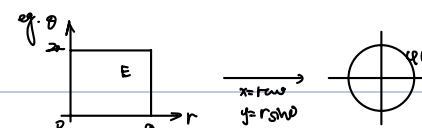
$\iint_D f(x,y) dx dy = ?$
 极坐标: $r = \text{Concl.}, \theta = \text{Concl.}$ 把 D 划分成小块
 $\Delta A = \frac{1}{2}(r_1 \Delta\theta + r_2 \Delta\theta) = r \Delta\theta + \frac{1}{2} \Delta r \Delta\theta \approx r \Delta r \Delta\theta$
 $g(r,\theta) = f(r \cos\theta, r \sin\theta)$
 $g(r,\theta) \Delta A \approx g(r,\theta) r \Delta r \Delta\theta$
 $\iint_D f(x,y) dx dy = \lim_{n \rightarrow \infty} \sum_{i=1}^n g(r_i^*, \theta_i^*) r_i^* \Delta r_i^* \Delta\theta_i^* = \int_E g(r,\theta) r dr d\theta$
 $= \int_{\theta_1}^{\theta_2} (\int_{r_1(\theta)}^{r_2(\theta)} f(r \cos\theta, r \sin\theta) \cdot r dr) d\theta$

Volume model

求体积 $x^2+y^2 \leq z^2 \leq \frac{a^2}{4}$
 包含在 $x^2+y^2 \leq a^2$ 中部分之体积 V

 $r = a \cos\theta$
 $z = \sqrt{x^2+y^2} = r$
 $V = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{a \cos\theta} \sqrt{r^2} \cdot r dr d\theta$
 $= \frac{4}{3} \int_0^{2\pi} (1 - \sin^2\theta) d\theta$
 $= \frac{4\pi}{3} \cdot [\frac{\pi}{2} + \int_0^{\pi/2} (-\cos^2\theta) d\theta]$
 $= \frac{4\pi^2}{3} \cdot (\frac{\pi}{2} - \frac{1}{2})$

二重积分之坐标变换公式

$(u,v) \mapsto (x,y) \begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$
 eg. 
 $\iint_{Q(E)} \frac{(x+y)^2}{1+(x-y)^2} dx dy$
 $\xrightarrow{u=x+y, v=x-y} \iint_E \frac{u^2}{1+v^2} du dv$
 $\xrightarrow{u=\sqrt{y}, v=\sqrt{x}} \iint_{Q(E)} \frac{x^2+y^2}{x^2-y^2} dx dy$

eg. θ

 $\iint_{Q(E)} e^{-x^2-y^2} dx dy$
 $\xrightarrow{u=r \cos\theta, v=r \sin\theta} \int_0^{2\pi} \int_0^1 e^{-r^2} r dr d\theta = \pi$

Thm. 设 $D \subset \mathbb{R}^2$ 为一开集, $\varphi: D \rightarrow \mathbb{R}^2, (u,v) \mapsto (x,y)$ 是 C^1 映射, $E \subset D$ 为 Jordan 可测集

若 i) $\det(D\varphi(u,v)) = \frac{\partial(x,y)}{\partial(u,v)} \neq 0, \varphi(u,v) \in E^0$
 ii) φ 在 E 上为单映射
 则 $\varphi(E)$ 仍为 Jordan 可测集, 且对 $\varphi(E)$ 上任意连续函数 f 有
 $\iint_{\varphi(E)} f(x,y) dx dy = \iint_E f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv$

$$\begin{cases} x = r \cos\theta \\ y = r \sin\theta \end{cases} \quad \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r \sin\theta \\ \sin\theta & r \cos\theta \end{vmatrix} = r$$

D φ : $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}^m$
 $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \xrightarrow{\varphi} \varphi(\vec{x}) = \begin{pmatrix} \varphi_1(x_1, \dots, x_n) \\ \vdots \\ \varphi_m(x_1, \dots, x_n) \end{pmatrix} \in \mathbb{R}^m$

φ 为不可微
 若 $\exists \vec{h} \in \mathbb{R}^n$ 使得 $A: \vec{h} \mapsto A\vec{h}$ 且 $\lim_{\|\vec{h}\| \rightarrow 0} \frac{\|\varphi(\vec{h}) - A\vec{h}\|}{\|\vec{h}\|} = 0$
 则 φ 在 \vec{h} 处可微
 注: φ 在 \vec{h} 处可微 $\Leftrightarrow \vec{x} \mapsto \varphi(\vec{x})$ 在 \vec{h} 处可微

$$A = D\varphi$$

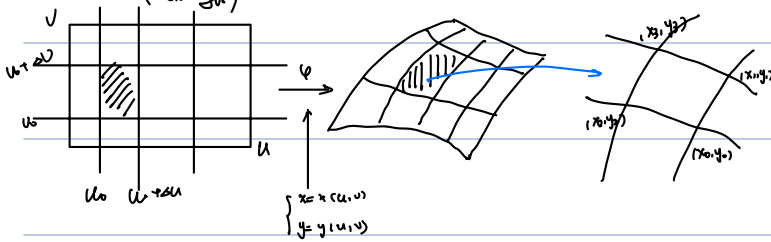
$$\vec{h} = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix} \mapsto A\vec{h} = \begin{vmatrix} \frac{\partial \varphi_1}{\partial x_1} & \frac{\partial \varphi_1}{\partial x_2} & \dots & \frac{\partial \varphi_1}{\partial x_n} \\ \frac{\partial \varphi_2}{\partial x_1} & \frac{\partial \varphi_2}{\partial x_2} & \dots & \frac{\partial \varphi_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \varphi_m}{\partial x_1} & \frac{\partial \varphi_m}{\partial x_2} & \dots & \frac{\partial \varphi_m}{\partial x_n} \end{vmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}$$

$$\varphi: (u, v) \mapsto (x, y)$$

$$x = x(u, v)$$

$$y = y(u, v)$$

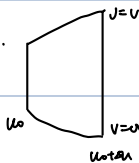
$$A = D\varphi = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$



$$(x_i - x_0, y_i - y_0) \approx \left(\frac{\partial x}{\partial u} \Big|_{(u_0, v_0)}, \frac{\partial y}{\partial u} \Big|_{(u_0, v_0)} \right) \Delta u$$

$$(x_i - x_0, y_i - y_0) \approx \left(\frac{\partial x}{\partial v} \Big|_{(u_0, v_0)}, \frac{\partial y}{\partial v} \Big|_{(u_0, v_0)} \right) \Delta v$$

$$|\vec{a} \times \vec{b}| = \left| \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right|$$

eg. 

$$A = \int_{u_0}^{u_0+\Delta u} \int_{v_0}^{v_0+\Delta v} [y(u, v) - y(u, v)] du dv$$

$$= \int_{u_0}^{u_0+\Delta u} \int_{v_0}^{v_0+\Delta v} \frac{\partial y}{\partial u} du dv$$

$$\frac{\partial y}{\partial u} = \left| \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right| = \frac{\partial y}{\partial u}$$

注: $\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \frac{1}{\left| \frac{\partial(u, v)}{\partial(x, y)} \right|}$ 面积可逆

Review

Thm: 设 $O \subset \mathbb{R}^2$ 为有界开集; $\varphi: O \rightarrow \mathbb{R}^2$ 为 C^1 映射; $E \subset O$ 为一闭 Jordan 可测集, 且有

i) det $D\varphi$ 在 E 内部每点不为 0

ii) $\varphi|_E$ 为单射

则 $\varphi(E)$ 为闭 Jordan 可测集, 且对 $\varphi(E)$ 上任意连续函数 f 有

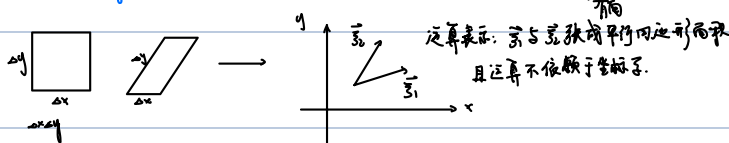
$$\iint_{\varphi(E)} f(x, y) dx dy = \iint_E f(x(u, v), y(u, v)) \left| \frac{D(x, y)}{D(u, v)} \right| du dv$$

i) $\frac{D(x, y)}{D(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

ii) $\frac{D(u, v)}{D(x, y)} \cdot \frac{D(x, y)}{D(u, v)} = 1$

证: $\iint_{\varphi(E)} f(x, y) dx dy = \iint_{\varphi(E)} f(x, y) d\sigma = \iint_{\varphi(E)} f(x, y) (dx dy)$

$x = r \cos \theta$
 $y = r \sin \theta$
 $dx dy = (dr d\theta - r \sin \theta dr + r \cos \theta d\theta)$



$$\vec{e}_1 \times \vec{e}_2$$

\mathbb{R}^n 中 $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ 张成平行多面体 (n 维) 有向体积。又乘而不加。

在 \mathbb{R}^2 中

$$\wedge: \vec{e}_1 \wedge (\vec{e}_1 + \vec{e}_2) = \vec{e}_1 \wedge \vec{e}_1 + \vec{e}_1 \wedge \vec{e}_2 = \vec{e}_1 \wedge \vec{e}_2$$

注意: $\vec{e}_1 \wedge \vec{e}_2 = -\vec{e}_2 \wedge \vec{e}_1$

$$\vec{e} = x \vec{e}_1 + y \vec{e}_2$$

$$\vec{e}_1 = x \vec{e}_1 + y \vec{e}_2$$

eg. \vec{e} 为基标准正交集

$$\begin{vmatrix} x & y \\ x & y \end{vmatrix} \rightarrow \vec{e}_1, \vec{e}_2 \text{ 张成平行四边形: 有向面积}$$

$$\Rightarrow \vec{e}_1 \wedge \vec{e}_1 = 0$$

$$\vec{e}_1 \wedge \vec{e}_2 = (x_1 \vec{e}_1 + y_1 \vec{e}_2) \wedge (x_2 \vec{e}_1 + y_2 \vec{e}_2)$$

$$= y_1 x_2 \vec{e}_1 \wedge \vec{e}_1 + x_1 y_2 \vec{e}_2 \wedge \vec{e}_2$$

$$= (x_1 y_2 - x_2 y_1) \vec{e}_1 \wedge \vec{e}_2$$

$$= \left| \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \right| \vec{e}_1 \wedge \vec{e}_2$$

外积与平行多面体的有向体积

$$\vec{e}_1 \wedge \vec{e}_2 \wedge \dots \wedge \vec{e}_n = \sum_{j_1 < j_2 < \dots < j_n} b_{j_1, j_2, \dots, j_n} \vec{e}_{j_1} \wedge \vec{e}_{j_2} \wedge \dots \wedge \vec{e}_{j_n}$$

$$E \subset \mathbb{R}^n$$

$$E^{\wedge k} := \{ \vec{e}_1 \wedge \vec{e}_2 \wedge \dots \wedge \vec{e}_k : \vec{e}_i \in E \}$$

E 的 k -外积空间

$$\wedge$$
 满足: i) $\vec{e}_1 \wedge \dots \wedge \vec{e}_i \wedge \vec{e}_i \wedge \dots \wedge \vec{e}_k = 0$

$$= (a_1 + a_2)(\vec{e}_1 \wedge \dots \wedge \vec{e}_k)$$

$$ii) \vec{e}_1 \wedge \dots \wedge \vec{e}_i \wedge \dots \wedge \vec{e}_j \wedge \dots \wedge \vec{e}_k = -\vec{e}_1 \wedge \dots \wedge \vec{e}_j \wedge \dots \wedge \vec{e}_i \wedge \dots \wedge \vec{e}_k$$

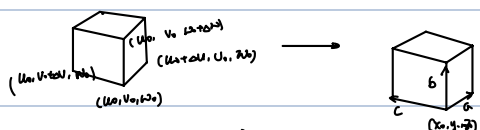
取 $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ 为 \mathbb{R}^n 的一组正交基

$$\vec{e}_i = a_{i1} \vec{e}_1 + a_{i2} \vec{e}_2 + \dots + a_{in} \vec{e}_n \quad i=1, \dots, n$$

$$n$$
 维 n -外积 $\vec{e}_1 \wedge \dots \wedge \vec{e}_n = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \vec{e}_1 \wedge \vec{e}_2 \wedge \dots \wedge \vec{e}_n$

$$\iint_E f(x, y, z) dx dy dz = \iint_E f(x(u, v, w), y(u, v, w), z(u, v, w)) \frac{D(x, y, z)}{D(u, v, w)} du dv dw$$

$$\text{其中 } \frac{D(x, y, z)}{D(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$



$$\text{证: } |\vec{a} \times \vec{b}|^2 = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}^2$$

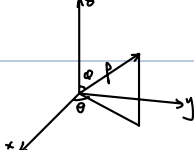
$$\vec{a} = (x(u, v), y(u, v))$$

$$= \left(\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v} \right) \Big|_{(u_0, v_0, w_0)} \cdot \Delta u$$

$$\vec{b} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right) \Big|_{(u_0, v_0, w_0)} \cdot \Delta v$$

$$\vec{c} = \left(\frac{\partial x}{\partial w}, \frac{\partial y}{\partial w}, \frac{\partial z}{\partial w} \right) \Big|_{(u_0, v_0, w_0)} \cdot \Delta w$$

eg. 球坐标变换



$$x = \rho \sin \phi \cos \theta$$

$$y = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

$$0 \leq \theta \leq 2\pi$$

$$0 \leq \phi \leq \pi$$

$$0 < \rho < +\infty$$

$$\left| \frac{D(x, y, z)}{D(\rho, \theta, \phi)} \right| = \begin{vmatrix} \sin \phi \cos \theta & \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ \cos \phi & -\rho \sin \phi & 0 \end{vmatrix} = \rho^2 \sin \phi$$

$$\text{证: } \iiint_{\mathbb{R}^3} \frac{1}{1 + \sqrt{x^2 + y^2 + z^2}} dx dy dz$$

$$= \int_0^{2\pi} \int_0^\pi \int_0^a \frac{1}{1+r} r^2 \sin \phi dr d\phi d\theta$$

$$= 4\pi \int_0^a \frac{r^2}{1+r} dr = 4\pi \left(\frac{1}{2} a^2 - a \ln(1+a) \right)$$

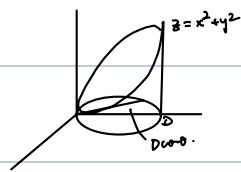
$$\iiint_{x^2+y^2+z^2 \leq a^2} 1 dx dy dz = \frac{4}{3} \pi a^3$$

$$\text{又球坐标: } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\begin{cases} x = ar \sin \phi \cos \theta \\ y = br \sin \phi \sin \theta \\ z = cr \cos \phi \end{cases}$$

eg. 柱坐标:

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \\ |J| = r \end{cases}$$

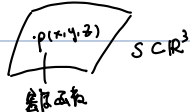


$$\begin{aligned} \iint_R |d\mathbf{u}| &= \iint_D \left(\sqrt{r^2 + dz^2} \right) dr d\theta \\ &= \int_{-\pi}^{\pi} \int_0^{\sqrt{2}} \sqrt{r^2 + 1} dr d\theta \\ &= \int_{-\pi}^{\pi} \left[\frac{1}{2} r^2 + \frac{1}{2} \ln(r^2 + 1) \right]_0^{\sqrt{2}} d\theta \end{aligned}$$

第一型曲面积分:

Q1: 曲面S的“面积”?

Q2: 面积为 $p(x, y, z)$, 对曲面求积?



本书: C^1 正则参数曲面

$S: D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$, D 为区域

$$(u, v) \in D \mapsto (x(u, v), y(u, v), z(u, v))$$

其中 $x = x(u, v)$
 $y = y(u, v)$
 $z = z(u, v)$
关于 (u, v) 连续.

eg. $(x, y, f(x, y))$

eg. $x^2 + y^2 + z^2 = a^2, z > 0$.

$$(x, y, \sqrt{a^2 - x^2 - y^2})$$

(a sine curve, a negative, a cone)

C^1 正则参数曲面 (开片, 不含边界)

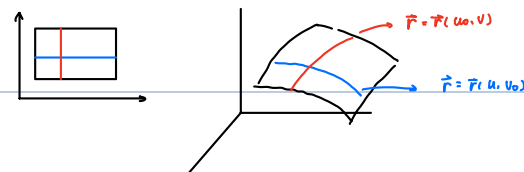
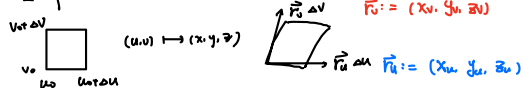
考虑: $S: D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ (D 为区域)

$$(u, v) \mapsto (x(u, v), y(u, v), z(u, v))$$

或记为 $F(u, v)$

其中 $x(u, v), y(u, v), z(u, v)$ 关于 u, v C^1

Q: $\{(x, y, z) : (u, v) \in D\}$ 何物称为“下”面?



若 $\vec{r}_u \times \vec{r}_v \neq \vec{0}$, 则 S 在 $p_0 = F(u_0, v_0) = (x(u_0, v_0), y(u_0, v_0), z(u_0, v_0))$ 正则

$$\begin{bmatrix} \vec{r}_u & \vec{r}_v & \vec{r} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{bmatrix} = \begin{pmatrix} D(x, y, z) \\ D(u, v) \end{pmatrix} = \begin{pmatrix} D(x, y, z) \\ D(u, v) \end{pmatrix}$$

eg. $\frac{D(x, y, z)}{D(u, v)} \neq 0 \rightarrow$ 隐函数存在定理

局部 u, v 用 x, y 表示
 $z = z(u, v) = z(x, y)$

正则曲面

S 上任一点 $p \in S$ 正则, 且映射 $(u, v) \mapsto (x, y, z)$ 为 D 到 S 的同胚映射
则称 S 为 D 上的正则参数曲面

参数曲面举例

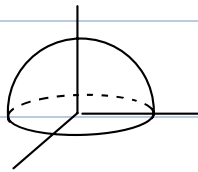
① $z = z(x, y), (x, y) \in D$

图像 $(x, y, z(x, y))$

$$x^2 + y^2 + z^2 = a^2, z > 0$$

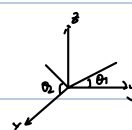
$$D = \{(x, y) : x^2 + y^2 < a^2\} \rightarrow \mathbb{R}^2$$

$$(x, y) \mapsto (x, y, \sqrt{a^2 - x^2 - y^2})$$

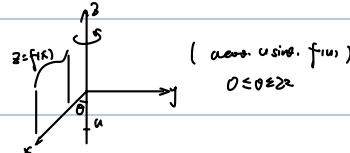


① 用球坐标参数化

$$\begin{cases} 0 \leq \phi < \frac{\pi}{2} \\ 0 \leq \theta < 2\pi \end{cases}$$



② 旋转面



正则参数曲面的面积

设 S 为 $D \subset \mathbb{R}^2$ 上 C^1 正则参数曲面

i) 若 $\lim_{D \rightarrow \infty} \sum_{D \rightarrow \infty} \|\vec{r}_u \times \vec{r}_v\| \Delta u \Delta v$ 存在

则 S 是有限面积之和的极限曲面

ii) 设 $p(x, y, z)$ 为 S 上连续函数 (通常可设 $p = 1 \rightarrow$ 面积)

若 $\lim_{D \rightarrow \infty} \sum_{D \rightarrow \infty} p(x, y, z) \|\vec{r}_u \times \vec{r}_v\| \Delta u \Delta v$ 存在, 则 p 在 S 上可积, 记为 $\iint_S p d\sigma$

$\iint_S 1 d\sigma = \iint_S p d\sigma$ 的面积

$\|\vec{r}_u \times \vec{r}_v\|$ 的计算

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = \begin{pmatrix} D(x, y, z) \\ D(u, v) \end{pmatrix} = \begin{pmatrix} D(x, y, z) \\ D(u, v) \end{pmatrix} = (A, B, C)$$

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{A^2 + B^2 + C^2}$$

A, B, C 含义?

A : \vec{r}_u 与 \vec{r}_v 平面法向量 (0, y_u, z_u) 与 \vec{r}_v 的叉积

$$\left(\frac{A}{\sqrt{A^2 + B^2 + C^2}} \cdot \frac{B}{\sqrt{A^2 + B^2 + C^2}} \cdot \frac{C}{\sqrt{A^2 + B^2 + C^2}} \right) = (n_1, n_2, n_3) = \vec{n} \quad |\vec{n}| = 1$$

面积与法向量

$$\|\vec{r}_u \times \vec{r}_v\| = \frac{A}{\sqrt{A^2 + B^2 + C^2}} = \frac{S_{xy}}{\cos \alpha}$$

叉积与外积

$$\vec{S} = x_u \vec{e}_1 + y_u \vec{e}_2 + z_u \vec{e}_3$$

$$\vec{S}_v = x_v \vec{e}_1 + y_v \vec{e}_2 + z_v \vec{e}_3$$

$$\vec{S}_u \wedge \vec{S}_v = \begin{vmatrix} \vec{e}_1 \wedge \vec{e}_2 & \vec{e}_2 \wedge \vec{e}_3 & \vec{e}_3 \wedge \vec{e}_1 \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}$$

$\|\vec{r}_u \times \vec{r}_v\|$ 其他计算方法

$$\iint_D \sqrt{A^2 + B^2 + C^2} du dv = \iint_D \sqrt{E, G, F} du dv$$

$$E = x_u^2 + y_u^2 + z_u^2 \quad G = x_v^2 + y_v^2 + z_v^2$$

$$F = x_u x_v + y_u y_v + z_u z_v$$

$$(\sqrt{A^2 + B^2 + C^2})^2 = (E, G, F)^2$$

$$(\vec{r}_u \times \vec{r}_v) \cdot (\vec{r}_u \times \vec{r}_v) = (\vec{r}_u \cdot \vec{r}_u)(\vec{r}_v \cdot \vec{r}_v) - (\vec{r}_u \cdot \vec{r}_v)^2$$

$$\|\vec{r}_u\| \|\vec{r}_v\| \sin \alpha = \|\vec{r}_u\| \|\vec{r}_v\| - \|\vec{r}_u\| \|\vec{r}_v\| \cos \alpha$$

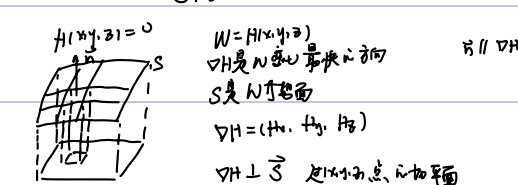
例:

$$\begin{aligned} \text{直角坐标系} \quad \begin{cases} x = x \\ y = y \\ z = f(x, y) \end{cases} \quad \begin{aligned} \vec{r}_u &= (1, 0, f_x) \\ \vec{r}_v &= (0, 1, f_y) \end{aligned} \quad \|\vec{r}_u \times \vec{r}_v\| = \left\| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} \right\| \\ &= \sqrt{1 + f_x^2 + f_y^2} \end{aligned}$$

$$z = x^2 + y^2 \quad \text{隐函数方程} \quad \text{决定曲面面积}$$

曲面 $S: \{(x, y, z) : (x, y, z) \text{ 满足方程 } H(x, y, z) = 0\}$

隐函数方程: ① $H(x, y, z)$ 关于 x, y, z C^1 (局部) 可微分方程可解: $z = z(x, y)$
② $H_z \neq 0$



关于梯度: $z = f(x, y)$ $\nabla f = (f_x, f_y)$

$$f(x, y) = C \quad (x(t), y(t)) \quad f(x(t), y(t)) = C$$

$$f_x x'(t) + f_y y'(t) = 0 \quad \vec{r} \perp (x'(t), y'(t))$$

$$\Rightarrow \vec{r} \cdot \nabla f = 0 \Rightarrow \vec{r} \perp \nabla f$$

$W = H(x, y, z)$ 为曲面 $H(x, y, z) = 0$ 为 $W = H(x, y, z)$ 的等值面

$$H(x(t), y(t), z(t)) = 0$$

$$\vec{r} = (x(t), y(t), z(t))$$

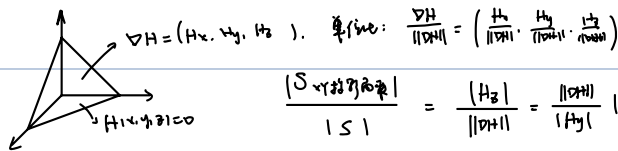
$$H_x x' + H_y y' + H_z z' = 0$$

$$\Rightarrow \nabla H \cdot \vec{r} = 0$$

$$\nabla f = (f_x, f_y) \Rightarrow \nabla f \cdot (x', y') = 0 = H(x, y, z)$$

$$\nabla H = (-f_x, -f_y, 1)$$

$$\Rightarrow |\nabla H| = \frac{|\nabla H|}{|H_z|}$$



$$\frac{|\nabla H|}{|H_z|} = \frac{|H_z|}{|\nabla H|} = \frac{|\nabla H|}{|H_z|} |S_y|$$

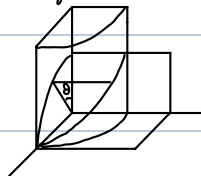
圆柱面: $x^2 + y^2 = R^2$ 在圆柱面 $x^2 + y^2 = R^2$ 上刻出的面积

$$z = \sqrt{R^2 - x^2 - y^2} \quad \begin{matrix} x = R \cos \theta \\ y = R \sin \theta \\ z = \sqrt{R^2 - x^2 - y^2} \end{matrix}$$

$$S = 8 \iint_D \sqrt{1 + \frac{x^2}{R^2} + \frac{y^2}{R^2}} dx dy$$

$$= 8 \int_0^R \int_0^{2\pi} \frac{R}{\sqrt{R^2 - x^2 - y^2}} dy dx = 8R^2$$

$$\text{或 } S = 8 \int_0^{\frac{\pi}{2}} R d\theta \cdot R \sin \theta = 8R^2$$



球坐标下S计算公式

$$\vec{r}(\theta, \phi) = (a \sin \phi \cos \theta, a \sin \phi \sin \theta, a \cos \phi)$$

$$\theta \in [0, 2\pi], \phi \in [0, \pi]$$

$$\|\vec{r}_\theta \times \vec{r}_\phi\| \quad \vec{r}_\theta = (-a \sin \phi \sin \theta, a \sin \phi \cos \theta, 0)$$

$$\vec{r}_\phi = (a \cos \phi \cos \theta, a \cos \phi \sin \theta, -a \sin \phi)$$

$$\|\vec{r}_\theta \times \vec{r}_\phi\| = a^2 \sin \phi$$

$$S_{\text{球}} = \int_0^\pi \int_0^{2\pi} a^2 \sin \phi d\theta d\phi = 4\pi a^2$$

例: 计算球面 $x^2 + y^2 + z^2 = a^2 (a > 0)$ 在圆柱面 $x^2 + y^2 = ax$ 之间的部分面积

$$(a \cos \phi \sin \theta, a \sin \phi \sin \theta, a \cos \phi)$$

$$a \sin \phi = a \cos \phi$$

$$\Rightarrow \theta + \phi = \frac{\pi}{2}$$

$$\Rightarrow \theta = \frac{\pi}{2} - \phi$$

$$S = 4 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} a^2 \sin \phi d\theta d\phi$$

$$= 4 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} a^2 \sin \phi d\theta d\phi$$

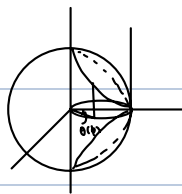
$$= 4 \int_0^{\frac{\pi}{2}} a^2 (1 - \sin \phi) d\phi$$

$$= 4a^2 (\frac{\pi}{2} - 1)$$

$$= 2a^2 - 4a^2$$

$$\text{另: } H(x, y, z) = x^2 + y^2 + z^2 - a^2 \quad H(x, y, z) = 0 \quad \nabla H = (2x, 2y, 2z), H_z = 2z$$

$$S = 4 \iint_D \frac{|\nabla H|}{|H_z|} dx dy = 4 \iint_D \frac{a}{\sqrt{a^2 - x^2 - y^2}} dx dy \quad \left(\frac{\nabla H}{H_z}\right) = \frac{a}{z}$$



其他参数曲面

$$\begin{cases} x = \cos u \cos v \\ y = \cos u \sin v \\ z = u \end{cases} \quad \begin{matrix} -\frac{\pi}{2} \leq u \leq \frac{\pi}{2} \\ 0 \leq v \leq 2\pi \end{matrix}$$

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = h \theta \end{cases} \quad \begin{matrix} 0 \leq r \leq a \\ 0 \leq \theta \leq 2\pi \end{matrix}$$

$$\sqrt{E G F^2} = \sqrt{r^2 + h^2}$$

$$S = \{ (x, y, z) \mid x^2 + y^2 + z^2 = 1 \}$$

$$\iint_S (a + by + cz) d\sigma = 2\pi \int_{-1}^1 f(\sqrt{1 - u^2}) du$$

Poisson 公式

$$\begin{bmatrix} u \\ v \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$$

$$A \text{ 的逆矩阵: } \left(\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \frac{c}{\sqrt{a^2 + b^2 + c^2}} \right)$$

$$u = \frac{ax + by + cz}{\sqrt{a^2 + b^2 + c^2}}$$

$$\text{取合适之 } a, b, c \text{ 使得 } S \text{ 上 } x^2 + y^2 + z^2 = 1 \Rightarrow \int_S (a + by + cz) d\sigma = \int_S (u + \sqrt{1 - u^2}) d\sigma$$

在球面上取元面积 dS 不变

有 $dS = dZ$

$$= \iint_S (a + by + cz) dS = \iint_S (u + \sqrt{1 - u^2}) dZ$$

$$= 2 \int_{-1}^1 (u + \sqrt{1 - u^2}) \frac{1}{\sqrt{1 - u^2}} du$$

$$= 2 \int_{-1}^1 \frac{1}{\sqrt{1 - u^2}} du$$

Review:

$$\iint_S p(F(u, v)) \|\vec{r}_u \times \vec{r}_v\| du dv = \iint_S p(x, y, z) d\sigma$$

① A, B, C E, F, G H

② 几何直观

第一型曲线积分与第二型曲线积分

线长, 质量

线密度 $\rho(x, y, z)$

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \quad \alpha \leq t \leq \beta$$

定向的 \vec{F}

面积与线积分关系
保守场与旋度

第一型曲线积分

① T 的曲线

② T 的质量

定义? 计算?

$\rho(x, y, z)$

$$T: \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \quad \alpha \leq t \leq \beta$$

① 若极限 $L(T) := \lim_{n \rightarrow \infty} \sum_{i=1}^n \|\vec{r}_i\|$ 存在, 称 T 为可求长曲线.

称 $L(T)$ 为 T 的弧长

$$\text{记: } \int_C p ds$$

若 T 为 C' 正则曲线

设 $T: [\alpha, \beta] \rightarrow \mathbb{R}^3$ 为连续映射, 若 $f|_{[\alpha, \beta]}$ 为单射, f' 在 (α, β) 上不为 0

$$L(T) = \int_\alpha^\beta p ds = \int_\alpha^\beta \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt$$

② 类似可求

$$\int_T p(x, y, z) ds = \int_\alpha^\beta p(x(t), y(t), z(t)) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt = \int_\alpha^\beta p(F(t)) \|F'(t)\| dt$$

若 T 为 C' 正则曲线

$\rho(F(t))$ 连续

$$\int_T p(x, y, z) ds = \int_\alpha^\beta p(x(t), y(t), z(t)) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt = \int_\alpha^\beta p(F(t)) \|F'(t)\| dt$$

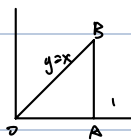
注: ① $\int_T k_1 p_1 + k_2 p_2 ds = k_1 \int_T p_1 ds + k_2 \int_T p_2 ds$

$$\text{② } \int_{T_1 \cup T_2} p ds = \int_{T_1} p ds + \int_{T_2} p ds$$

eg. 求 $\int_{\Gamma} \frac{z^2}{x^2+y^2} ds$, 其中 Γ 为圆柱螺线 $\begin{cases} x = a \cos t \\ y = a \sin t \\ z = bt \end{cases} \quad 0 \leq t \leq 2\pi$

$$= \int_0^{2\pi} \frac{b^2 t^2}{a^2} \sqrt{a^2 + b^2} dt = \frac{b^2 \sqrt{a^2 + b^2}}{3a^2} (2\pi)^3$$

eg.



$$\begin{aligned} \int_{\Gamma} xy^2 ds &= \int_0^1 xy^2 dy + \int_0^1 xy^2 \cdot \sqrt{2} dy \\ &= \frac{1}{3} + \frac{\sqrt{2}}{3} \end{aligned}$$

eg. $\int_{\Gamma} xy ds$ Γ 为球面 $x^2+y^2+z^2 = a^2$ 与平面 $x+y+z=0$ 相交曲线的一部分

第二型曲面积分

Q: 设 \vec{F} 在 \mathbb{R}^3 中为向量场, 以 Γ 为边界, 求 \vec{F} 沿 Γ 的线积分

如何求? 由什么?

考虑 Γ 的分段 $P = \{P_1, P_2, \dots, P_n\}$ $\|P\| = \max\{\|P_1\|, \|P_2\|, \dots, \|P_n\|\}$

若 $\lim_{\|P\| \rightarrow 0} \sum_{i=1}^n \vec{F}(\xi_i) \cdot d\vec{r}_i$ 存在, 则称 \vec{F} 沿 Γ 的第二型曲线积分可积

Q: 元组. 记号 $\vec{F} = (P, Q, R)$

$$\int_{\Gamma} \vec{F} \cdot d\vec{r} = \int_{\Gamma} P dx + Q dy + R dz$$



OT: $\vec{r}(t)$, $ds = \sqrt{x'^2 + y'^2 + z'^2} dt$, \vec{F} 为 Γ 上向量

$$\vec{r}(t+dt) - \vec{r}(t) = \frac{d\vec{r}}{dt} dt$$

$$N = \int_{\Gamma} \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} dt$$

eg. $\vec{r}(t) = t\vec{i} + t^2\vec{j} + t^3\vec{k}$ $t \in [0, 1]$

$$\vec{F}(x, y, z) = (y-x)\vec{i} + (z-y)\vec{j} + (x-z)\vec{k}$$

$$\oint_{\Gamma} \vec{F} \cdot d\vec{r} = \int_0^1 ((y-x)dt + (z-y)2tdt + (x-z)3t^2dt)$$

$$= \int_0^1 (t^2 - t^3)2t + (t - t^2)3t^2 dt$$

$$= \int_0^1 (3t^3 - 2t^4 - 3t^3 + 3t^2) dt$$

$$= \frac{3}{4} + \frac{2}{5} - \frac{3}{6} - \frac{3}{7} = \frac{29}{60}$$

$$\oint \vec{F} \cdot d\vec{r} = \int_{\Gamma} P dx + Q dy + R dz$$

格林公式与斯托克斯公式

$$\int P dx + Q dy + R dz = \int \left(\frac{P x'}{\sqrt{x'^2 + y'^2 + z'^2}} + \frac{Q y'}{\sqrt{x'^2 + y'^2 + z'^2}} + \frac{R z'}{\sqrt{x'^2 + y'^2 + z'^2}} \right) \sqrt{x'^2 + y'^2 + z'^2} dt$$

Thm. Green 公式

设 D 为 \mathbb{R}^2 中闭有界区域, 其边界 ∂D 有有限集 (分段) 光滑曲线组成, ∂D 的方向满足沿 ∂D 正向走使得区域在左手边, 设 $P(x, y)$, $Q(x, y)$ 在 D 上 C^1 , 则有

$$\oint_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

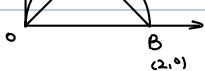
保守场的判定 (与 Green 公式, Stokes 公式...)

Def. 设 \vec{F} 为 \mathbb{R}^2 (或 \mathbb{R}^3) 中一道路连通开区域 Ω 上的向量场, 若 $\forall A, B \in \Omega$ 及连接 A, B (以 A 为起点, B 为终点) 的分段光滑曲线 Γ , $\int_{\Gamma} \vec{F} \cdot d\vec{r}$ 仅与 A, B 有关 (与路径 Γ 无关), 则称 \vec{F} 为 Ω 上的保守场

eg. $\vec{F} = (x^2+y^2, x^2-y^2)$

$$\int_0^2 \vec{F} \cdot d\vec{r} = \int_0^2 x^2 dx = \frac{8}{3}$$

$$\int_{\partial \Omega} \vec{F} \cdot d\vec{r} = \int_{OA} \vec{F} \cdot d\vec{r} + \int_{AB} \vec{F} \cdot d\vec{r}$$



$$= \int_{-1}^1 (x^2+y^2) dx + (x^2-y^2) dy$$

$$+ \int_{AB} (x^2+y^2) dx + (x^2-y^2) (-dy)$$

$$= \frac{4}{3}$$

一种判定方法 (用势函数)

Def. Ω 同上, 若 $f: \Omega \rightarrow \mathbb{R}$ 满足 $\nabla f = \vec{F}$, 则称 f 为向量场 \vec{F} 的一个势函数

Thm. 设向量场 \vec{F} 在一道路连通开区域 Ω 内连续, 则 \vec{F} 在 Ω 内是保守场 当且仅当 \vec{F} 有势函数 即 \vec{F} 为梯度场

证: $\vec{F} = (F_1, F_2, F_3)$ $\frac{d}{dt} = \frac{dx}{dt} \frac{\partial}{\partial x} + \frac{dy}{dt} \frac{\partial}{\partial y} + \frac{dz}{dt} \frac{\partial}{\partial z}$

$$\vec{F} \cdot d\vec{r} = (F_1 dx + F_2 dy + F_3 dz)$$

$$\int_{\Gamma} \vec{F} \cdot d\vec{r} = \int_a^b [F_1(t) x'(t) + F_2(t) y'(t) + F_3(t) z'(t)] dt$$

$$= \int_a^b \frac{d}{dt} f(\vec{r}(t)) dt = f(\vec{r}(b)) - f(\vec{r}(a))$$
 与路径无关

" \Rightarrow " 同是 $A \in \Omega$, 令 $f(A) = 0$, $\forall B \in \Omega$, 由 Ω 内道路连通

且包含在 Ω 内从 A 到 B 的分段光滑曲线 Γ_{AB}

$$\text{定义 } f(B) = \int_{\Gamma_{AB}} \vec{F} \cdot d\vec{r}$$

① f 的定义有效 (\vec{F} 保守场)

② $\nabla f = \vec{F}$: $\vec{F} = (P, Q, R)$

$$\frac{f(x+\Delta x, y, z) - f(x, y, z)}{\Delta x} = \frac{\nabla f(\Delta x, 0, 0)}{\Delta x} = \frac{\partial f}{\partial x} = P$$

Q: 如何判定有势函数

Q: 如何判定 $\vec{F} = \nabla f$? 如何找 f ?

假设 $(P, Q, R) = (f_x, f_y, f_z)$ P, Q, R 有连续偏导数

有 $\begin{cases} \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \\ \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \\ \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} \end{cases}$ 反之是否成立? ∂D 为 \mathbb{R}^2 , $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \Rightarrow (P, Q)$ 为保守场 + 简单连通区域假设

Q: ∂D , $\vec{F} = (P, Q)$ $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ 的物理意义?

$$\vec{F} \text{ 为保守场} \Leftrightarrow \oint_C \vec{F} \cdot d\vec{r} = 0$$

流进切 环流量

$$\frac{\oint_C \vec{F} \cdot d\vec{r}}{\text{Area}(C)}$$
 反映速度 M 点的 z 轴上旋转速度

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \text{ 旋转}$$



(x_0, y_0) $(x_0 + \Delta x, y_0 + \Delta y)$
 $P, Q \in C'$
 $\oint_{C'} \vec{F} \cdot d\vec{r} \xrightarrow{\Delta x, \Delta y \rightarrow 0} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ 格林公式

$$\begin{aligned}
 \oint_{C'} \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} P(x_0 + t, y_0) dt + \int_0^{2\pi} Q(x_0 + t, y_0 + t) ds \\
 &\quad - \int_0^{2\pi} P(x_0 + t, y_0 + t) dt - \int_0^{2\pi} Q(x_0, y_0 + t) ds \\
 &= \int_0^{2\pi} \left(\frac{\partial Q}{\partial x}(x_0 + t, y_0) - \frac{\partial P}{\partial y}(x_0 + t, y_0 + t) \right) dt \\
 &= \int_0^{2\pi} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)(x_0 + t, y_0 + t) ds dt
 \end{aligned}$$

Green 公式

设 $O \subset \mathbb{R}^2$ 为有界区域, 其边界 ∂D 由有限条光滑曲线组成, $P(x, y), Q(x, y)$ 在 $D \cup C'$ 上连续

则有 $\oint_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$

其中 ∂D^+ 为逆时针方向, 即“正向”或“逆时针”方向

注: $\vec{F} = (P, Q)$

$$\oint_{\partial D} \vec{F} \cdot d\vec{r} = \oint_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$$

$$\vec{F} = (P, Q)$$

$$\oint_{\partial D} \vec{F} \cdot d\vec{r} = \oint_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$$

考虑逆时针时, xy 轴位置并不对, 故会有符号出现

②: 再式上证明 (带上坐标信息, 有向面积元)

$$\oint_{\partial D} \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$$

$$\oint_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$$

$$dx \wedge dy = -dy \wedge dx$$

$$\oint_{\partial D} P dx = \iint_D \left(-\frac{\partial P}{\partial y} \right) dx \wedge dy$$

$$= \iint_D \frac{\partial P}{\partial y} dy \wedge dx$$

$$= \iint_D \frac{\partial P}{\partial y} dy \wedge dx$$

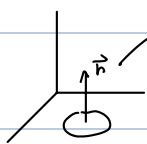
$$\oint_{\partial D} Q dy = \iint_D \frac{\partial Q}{\partial x} dx \wedge dy = \iint_D \frac{\partial Q}{\partial x} dx \wedge dy$$

$$\oint_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$$

$$= \iint_D \left(\frac{\partial Q}{\partial x} dx + \frac{\partial P}{\partial y} dy \right) \wedge dx \wedge dy = \iint_D \left(\frac{\partial Q}{\partial x} dx + \frac{\partial P}{\partial y} dy \right) \wedge dx \wedge dy$$

$$= \iint_D \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} \right) dx \wedge dy$$

③: 视觉结果



单位法向量 \vec{n} 与 ∂D^+ 选取一致

$$\oint_{\partial D} P dx + Q dy = \iint_D \begin{vmatrix} P & Q & R \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ dx & dy & dz \end{vmatrix} = \iint_D \left(P \frac{\partial R}{\partial x} + Q \frac{\partial R}{\partial y} + R \frac{\partial R}{\partial z} \right) dx \wedge dy \wedge dz$$

④: 证明 ②

⑤: 注意 Green 公式中 $\vec{F} = (P, Q)$ 在 C' 上连续

eg. $\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$ 是否为 $\mathbb{R}^2 \setminus \{(0,0)\}$ 上的保守场?

在 $(x,y) \neq (0,0)$

$$-\frac{\partial}{\partial y} \left(\frac{x}{x^2+y^2} \right) + \frac{\partial}{\partial x} \left(\frac{-y}{x^2+y^2} \right) = -\frac{(x^2+y^2) - 2y^2}{(x^2+y^2)^2} + \frac{x^2y^2 - 2x^2}{(x^2+y^2)^2} = 0$$

若环路包含 $(0,0)$, 不可用格林公式

$$\oint_{\Gamma^+} \vec{F} \cdot d\vec{r}$$

① Γ^+ 内部不含原点 $\Rightarrow \oint_{\Gamma^+} \vec{F} \cdot d\vec{r} = 0$

② Γ^+ 内部包含原点: $\oint_{\Gamma^+} \vec{F} \cdot d\vec{r} = \int_{\Gamma^+} \vec{F} \cdot d\vec{r} + \int_{\Gamma^-} \vec{F} \cdot d\vec{r} = \int_{\Gamma^+} \vec{F} \cdot d\vec{r}$

$$\Gamma^+ : \partial B(0,1)$$

$$= \int_0^{2\pi} \left(-\frac{\sin \theta}{1}, \frac{\cos \theta}{1} \right) \cdot (-\sin \theta d\theta, \cos \theta d\theta) = 2\pi$$

$$\vec{F} = (-cy, cx) \quad c > 0 \quad \vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

Green 公式简单应用

1. 用来计算平面二重曲线积分, 面积

eg. 求 $\oint_{\Gamma} (x + e^x \sin y) dx + (x + e^x \cos y) dy$

其中 Γ 是双纽线 $r^2 = \cos 2\theta$ 的右半支

$P, Q \in C'$

$$\frac{\partial Q}{\partial x} = 1 + e^x \cos y$$

$$\frac{\partial P}{\partial y} = e^x \sin y$$

$$\Rightarrow I = \iint_D dx dy = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\sqrt{\cos 2\theta}} r dr d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} \cos 2\theta d\theta = \frac{1}{2}$$

eg. $\int_{\Gamma} (e^x \sin y + \sin x - y) dx + (e^x \cos y - \sin y) dy$

其中 Γ 为上半圆 $0 \leq y \leq \sqrt{a^2 - x^2}$ ($0 \leq x \leq a$) 的边界 作题

1. 求面积

$$\iint_D dx dy = \oint_{\partial D} x dy - y dx = \frac{1}{2} \oint_{\partial D} x dy - y dx$$

2. 保守场判定

设 $\vec{F} = (P, Q)$ 为平面内单连通区域 D 内的 C' 向量场

则 \vec{F} 为 D 内保守场 \Leftrightarrow 在 D 上 $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$

单连通区域

① 简单闭曲线 $\Gamma: \phi: [a, b] \rightarrow \mathbb{R}^2$ 连续, 在 (a, b) 上为单射, 且有 $\phi(a) = \phi(b)$

② Jordan 定理 \mathbb{R}^2 上任一简单闭曲线 Γ 将平面分成 Γ 内部, Γ 外部, 一个有界, 一个无界

所有内部点称为 Γ 的内部

③ 单连通区域

设 D 为 \mathbb{R}^2 中任一区域, 若对 D 中任一简单闭曲线 Γ , Γ 的内部均包含在 D 中, 则称 D 为单连通区域

$$\oint_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy = \iint_D \begin{vmatrix} P & Q & R \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ dx & dy & dz \end{vmatrix} dx \wedge dy \wedge dz$$

流量与散度

$\vec{F}(x, y)$ 为向量场, Γ 为闭曲线, 流量: $\oint_{\Gamma} \vec{F} \cdot \vec{n} ds$ (例 6)

\vec{n} 为 Γ 的单位法向量, \vec{n} 与 Γ 成右手系

用弧长参数 s 表示 Γ 参数



$$\vec{F}(s) = (x(s), y(s))$$

$$\vec{r}(s) = (x(s), y(s))$$

$$\|\vec{r}'(s)\| = 1$$

$$\vec{n}(s) = \vec{r}'(s) \times \vec{F}(s) = (y'(s), -x'(s))$$

$$\vec{F} \cdot \vec{n} = P(x(s), y(s)) \frac{dy}{ds} - Q(x(s), y(s)) \frac{dx}{ds}$$

$$\oint_{\Gamma} \vec{F} \cdot \vec{n} ds = \oint_{\Gamma} \left(P \frac{dy}{ds} - Q \frac{dx}{ds} \right) ds = \oint_{\Gamma} (P dy - Q dx)$$

$$= \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx dy = \iint_D \nabla \cdot \vec{F} dx dy$$

eg. u 在 $\mathbb{R}^2 \subset C^2$, $\vec{F} = \nabla u = (u_x, u_y)$

$$\oint_{\partial D} \frac{\partial u}{\partial n} ds = \oint_{\partial D} \nabla u \cdot \vec{n} ds = \oint_{\partial D} \nabla u \cdot \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) dx dy = 0$$

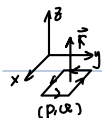
$$= \iint_D \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) dx dy = \iint_D \Delta u dx dy$$

$\Delta u = 0$ 调和函数

三维保守场的判定

2D: \vec{F} 保守场 $\Leftrightarrow \vec{F} = \nabla f$ 单连通 $\Leftrightarrow \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$

3D: $\vec{F} = (F_1, F_2, F_3) = (P, Q, R) \Leftrightarrow \begin{cases} \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \\ \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z} \\ \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z} \end{cases}$



($\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}$) \vec{i} 反映场绕 z 轴旋转的平面的

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}) \vec{i} \cdot d\vec{S}$$

$$\vec{r} = (x, y, z) \quad (\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}) \vec{i}$$

$$\nabla \times \vec{F} = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{pmatrix}$$

$$\oint_{\partial \Delta} \vec{F} \cdot d\vec{r} = \iint_{\Delta} \nabla \times \vec{F} \cdot \vec{n}$$

$$\vec{n} = (\cos \theta, \sin \theta, \cos \phi)$$

$$(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}) \cdot \Delta S \cdot \cos \theta + (\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}) \cdot \Delta S \cdot \sin \theta + (\frac{\partial P}{\partial z} - \frac{\partial Q}{\partial z}) \Delta S \cos \phi$$

① Stokes 公式 $\oint_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \vec{n} dS$

② Gauss 公式 $\oint_{\partial V} \vec{F} \cdot \vec{n} dS = \iiint_V \nabla \cdot \vec{F} \cdot dVol$

第二型曲面积分

可定向曲面 S: 不交叉边界上可取分向

Def: 可定向曲面 S

可定向法向量 \vec{n} 若 \vec{n} 随某边运动, 或任意从 P 出发沿路径 γ , \vec{n} 连续变化
应用时 P 时仍为 \vec{n}

参考曲面

① 单片 $(u, v) \mapsto (x, y, z)$ C^1 映射 $\vec{n} \times \vec{n} \neq 0$

$$\vec{n} = \pm \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$$

② M 块参考曲面

设 S 为可定向曲面, \vec{n} 为取定的单位法向量 $\vec{r} = (p, q, r) \in \mathbb{R}^3$

“流量”定义 $\iint_S \vec{F} \cdot \vec{n} d\sigma$

常用方法: 分片用参考曲面处理

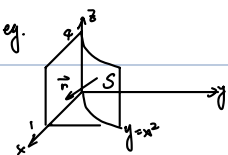
$$(u, v) \mapsto (x, y, z) \quad \textcircled{1} \quad \vec{r} = \iint_{Duv} \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) \cdot d\sigma$$

$$\textcircled{2} \quad \iint_S P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy = \iint \nabla \cdot \vec{F} dV$$

$\Delta u = \text{div } \nabla u = 0$ 调和 harmonic

$\text{div} \cdot \text{rot } \vec{F} \neq 0$

eg. $\vec{F} = y\vec{i} + x\vec{j} - z^2\vec{k}$
求 $\iint_S \vec{F} \cdot \vec{n} d\sigma$



曲面 S: $\vec{r}(x, z) = (x, x^2, z)$

$$\vec{r}_x \times \vec{r}_z = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2x\vec{i} - \vec{j}$$

$$\vec{n} = \frac{2x\vec{i} - \vec{j}}{\sqrt{4x^2 + 1}}$$

$$\iint_S \vec{F} \cdot \vec{n} d\sigma = \iint_D \frac{1}{\sqrt{4x^2 + 1}} (2x \cdot x^2 - x) dx dz$$

$$= \int_0^4 \int_0^1 (2x^3 - x) dx dz$$

eg. $\vec{r} = f(x, y)$

$$\vec{r}(x, y) = (x, y, f(x, y))$$

$$\vec{r}_x \times \vec{r}_y = -f_x \vec{i} - f_y \vec{j} + \vec{k}$$

$$\iint_S \vec{F} \cdot \vec{n} d\sigma = \pm \iint_{Dxy} (-f_x P - f_y Q + R) dx dy$$

$$\iint_S \vec{F} \cdot \vec{n} d\sigma = \iint_S (P, Q, R) \cdot \vec{n} d\sigma = \iint_S (0, 0, 0) \cdot \vec{n} d\sigma + \iint_S (0, 0, R) \cdot \vec{n} d\sigma$$

$$= \iint_S P dy \wedge dz + \iint_S Q dz \wedge dx + \iint_S R dx \wedge dy$$

$$= \pm \iint_{Dyz} P(x(y, z), y, z) dy dz \pm \iint_{Dxz} Q(x(y, z), y, z) dz dx \pm \iint_{Dxy} R(x(y, z), y, z) dx dy$$

eg. $I = \iint_{S^+} (y - z) dx dy + (x - y) dz dy$

$$\vec{F} = (x, y - z, 0, x - y)$$

$$\iint_{S^+} \vec{F} \cdot \vec{n} d\sigma = \iint_{S^+} (x - y) dx dy$$

$$\iint_{S^+} \vec{F} \cdot \vec{n} d\sigma = 0$$

$$\iint_{S^+} \vec{F} \cdot \vec{n} d\sigma$$

$$x = \cos \theta, y = \sin \theta, z = 0$$

$$A = \frac{\partial(x, y)}{\partial(\theta, z)} = \begin{pmatrix} \cos \theta & 0 \\ 0 & 1 \end{pmatrix} = \cos \theta$$

$$B = \frac{\partial(z, x)}{\partial(\theta, z)} = \begin{pmatrix} 0 & -\sin \theta \\ 1 & 0 \end{pmatrix} = \sin \theta$$

$$C = \frac{\partial(x, y)}{\partial(\theta, z)} = 0$$

$$\int_0^{2\pi} \cos \theta \sin \theta d\theta = -\frac{\pi}{2}$$

eg. $I = \iint_{S^+} xy z dx dy$ S^+ 为 $x^2 + y^2 + z^2 = 1, x, y > 0$, \vec{n} 指向外

$$z = \sqrt{1 - x^2 - y^2}$$

$$\vec{F} = (0, 0, xy z)$$

$$\vec{n} = (-x, -y, -z)$$

$$-\frac{1}{2} \int_0^1 \sqrt{1-u} u du$$

$$\iint_{S^+} \vec{F} \cdot \vec{n} d\sigma = - \iint_{S^+} xy z^2 d\sigma = -2 \iint_{Dxy} xy (1 - x^2 - y^2) \frac{1}{\sqrt{1-x^2-y^2}} dx dy = -\frac{2}{15}$$

$$x = \sin \theta \cos \phi, y = \sin \theta \sin \phi, z = \cos \theta$$

$$\vec{r}_\theta \times \vec{r}_\phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \theta \cos \phi & \sin \theta \sin \phi & 0 \end{vmatrix} = (A, B, C)$$

$$= (\sin^2 \theta \cos \phi, \sin^2 \theta \sin \phi, \sin \theta \cos \theta)$$

$$I = \int_0^{\pi/2} \int_0^{2\pi} \sin^2 \theta \cos \phi \sin \theta d\theta d\phi$$

$$I = \iint_{S^+} R dx dy = - \iint_{Dxy} xy \sqrt{1-x^2-y^2} dx dy + \iint_{Dxy} xy (1 - \sqrt{1-x^2-y^2}) dx dy$$

Gauss 公式: 设 $\Omega \subset \mathbb{R}^3$ 有界

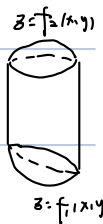
$S = \partial \Omega$ 封闭 由有限块可定向单片光滑曲面构成

$\vec{r} = (p, q, r)$ 在 $\Omega \cup \partial \Omega \subset \mathbb{R}^3$

$$\oint_{S^+} \vec{F} \cdot \vec{n} d\sigma = \iiint_{\Omega} (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}) dx dy dz$$

Proof: 对 $\vec{F} = (p, 0, 0), (0, q, 0), (0, 0, r)$ 分别证明

$$\iint_{S^+} R dx dy = \iint_D R(x, y, f(x, y)) dx dy - \iint_D R(x, y, f(x, y)) dx dy = \iint_D \frac{\partial R}{\partial z} dz dx dy$$



7. $\lim_{V \rightarrow 0} \frac{\oint \vec{F} \cdot \vec{n} d\sigma}{V(B)} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \Big|_{M_0} =: \text{div } \vec{F} \Big|_{M_0}$
 divergence

$\text{div } \vec{F} = 0$ 无源场, $\text{div } \vec{F} = 0$

$\vec{F} = \nabla f, \oint_{\partial V} \vec{F} \cdot \vec{n} d\sigma = \iint_V \nabla f \cdot \vec{n} d\sigma$

C' 假设

$E(x, y, z) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^3} \vec{r} = \frac{q}{4\pi\epsilon_0} \frac{(x, y, z)}{(x^2+y^2+z^2)^{3/2}}$
 $(x, y, z) \neq (0, 0, 0), \text{div } E = 0$

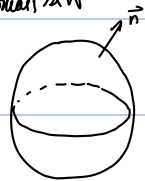
$\oint_{\partial V} \vec{E} \cdot \vec{n} d\sigma = 0 \quad \Omega \neq (0, 0, 0)$

$\iint_{\partial V} \vec{F} \cdot \vec{n} d\sigma = \sum_{i=1}^n \pm \iint_{D_{x_i}} \vec{F} \cdot (\vec{n}_i \times \vec{n}_j) d\sigma_{ij}$

$\iint_S P dy \wedge dz + \iint_S Q dz \wedge dx + \iint_S R dx \wedge dy$

$= \sum \pm \iint_{D_{y,z}} P(x(y,z), y, z) dy \wedge dz \pm \iint_{D_{z,x}} Q(x(y,z), y, z) dz \wedge dx \pm \iint_{D_{x,y}} R(x(y,z), y, z) dx \wedge dy$

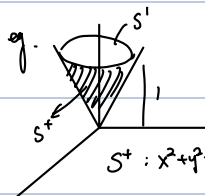
Green 公式



$\Omega \subset \mathbb{R}^3$ 开集
 $S = \partial\Omega$ 闭面
 分片光滑 可定向
 $\vec{F} = (P, Q, R)$ 在 $\Omega \cup \partial\Omega \subset C^1$

$\oint_{\partial V} \vec{F} \cdot \vec{n} d\sigma = \iint_{S^-} P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy$
 $= \iint_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx \wedge dy \wedge dz$

$\oint_{\partial V} P dy \wedge dz = \iint_{\Omega} dP \wedge dy \wedge dz = \iint_{\Omega} \left(\frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy + \frac{\partial P}{\partial z} dz \right) \wedge dy \wedge dz$
 $= \iint_{\Omega} \frac{\partial P}{\partial x} dx \wedge dy \wedge dz$



$I = \iint_{S'} (y-z) dy \wedge dz + (z-x) dz \wedge dx + (x-y) dx \wedge dy$

$I = \oint_{S'+S} (y-z) dy \wedge dz + (z-x) dz \wedge dx + (x-y) dx \wedge dy - \iint_S (y-z) dy \wedge dz + (z-x) dz \wedge dx + (x-y) dx \wedge dy$

$= \iint_{S'} 0 \cdot dV - \iint_{S'} (x-y) dx \wedge dy$

$= \iint_{S'} (y^2-x) dx \wedge dy = \int_0^1 \int_0^{\sqrt{1-r^2}} (r^2 \sin\theta - r \cos\theta) \cdot r dr d\theta = \frac{\pi}{4}$

Stokes 公式

设 S : 分片 C^1 可定向曲面
 $\Gamma = \partial S$: 分片 C^1 , 定向与 \vec{n} 相称

$\vec{F} = (P, Q, R) \in C^1(S \cup \partial S)$
 则: $\oint_{\Gamma} P dx + Q dy + R dz = \iint_S \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \vec{n} d\sigma = \iint_S \begin{vmatrix} dy \wedge dz & dz \wedge dx & dx \wedge dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$

Proof

$\oint_{\Gamma} P dx + Q dy + R dz = \iint_{S^+} \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz \wedge dx + \left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} \right) dx \wedge dy$

$\oint_{\Gamma} P dx = \iint_{S^+} dP \wedge dx = \iint_{S^+} \left(\frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy + \frac{\partial P}{\partial z} dz \right) \wedge dx = \iint_{S^+} \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial P}{\partial z} dz \wedge dx$

tips: ① $R=0, \Gamma: S \subset \mathbb{R}^2, \oint_{\Gamma} P dx + Q dy = \iint_{S^+} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$

② $\oint_{\Gamma} P dx + Q dy + R dz \rightarrow \text{rot } \vec{F} \cdot \vec{n}$

③ $\text{rot } \vec{F}$ 与大量切面垂直

④ S 至少 C^2 闭面

⑤ 证明: 分别对 $(P, 0, 0), (0, Q, 0), (0, 0, R)$ 证明

1. $z = f(x, y), \Gamma^+: \begin{cases} x = x(t) \\ y = y(t) \\ z = f(x(t), y(t)) \end{cases} \downarrow \oint_{\Gamma^+} P dx = \iint_{S^+} \frac{\partial P}{\partial z} dz \wedge dx - \frac{\partial P}{\partial y} dx \wedge dy$

$= \oint_{C^+} P(x(t), y(t), f(x(t), y(t))) x'(t) dt$
 $= \oint_{C^+} P(x, y, f(x, y)) dx = \oint_{C^+} \tilde{P}(x, y) dx$

$= \iint_{\Omega} - \frac{\partial}{\partial y} \tilde{P} dx \wedge dy$

$= \iint_{\Omega} \left(- \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) dx \wedge dy$

$= \iint_{S^+} \left(- \frac{\partial P}{\partial y} \right) dx \wedge dy - \iint_{\Omega} \frac{\partial}{\partial x} (x, y, f(x, y)) f_y dx \wedge dy$

$\vec{n} = \frac{1}{\sqrt{1+f_x^2+f_y^2}} (-f_x, -f_y, 1) = (\cos, \cos, \cos)$

$\frac{1}{\sqrt{1+f_x^2+f_y^2}} d\sigma = dx \wedge dy$

$\frac{-f_y}{\sqrt{1+f_x^2+f_y^2}} d\sigma = dz \wedge dx$

$= f_y dx \wedge dy = - dz \wedge dx$

2. $S: y = f(x, z)$

3. $S: x = f(y, z)$

$\oint_{\Gamma^+} P dx = \iint_{S^+} \frac{\partial P}{\partial z} dz \wedge dx - \frac{\partial P}{\partial y} dx \wedge dy$

$= \int_{\Gamma^+} P(f(y, z), y, z) (f_y y' + f_z z') dt$

$= \oint_{C^+} \frac{P(f(y, z), y, z) f_y dy + P(f(y, z), y, z) f_z dz}{\tilde{Q}} \Rightarrow \tilde{P} \cdot \tilde{\alpha} \cdot C^1 \Rightarrow \int_{\Gamma^+} \tilde{P} \cdot \tilde{\alpha} \cdot C^2$

$= \iint_{\Omega} \left[\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right] dy \wedge dz$

$= \iint_{\Omega} \left[\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} \right] f_z + P(f(y, z), y, z) f_y - \left[\left(\frac{\partial P}{\partial z} + \frac{\partial Q}{\partial x} \right) f_y + P(f(y, z), y, z) f_z \right] dy \wedge dz$

$= \iint_{\Omega} \left(\frac{\partial P}{\partial z} f_z - \frac{\partial Q}{\partial x} f_y \right) dy \wedge dz = \iint_{S^+} -P_y dx \wedge dy + P_z dz \wedge dx$

$\frac{1}{\sqrt{1+f_x^2+f_y^2}} d\sigma = dy \wedge dz$

$\frac{-f_y}{\sqrt{1+f_x^2+f_y^2}} d\sigma = dz \wedge dx$

$\frac{-f_x}{\sqrt{1+f_x^2+f_y^2}} d\sigma = dx \wedge dy$

4. $x^2+y^2+z^2=2\rho x, z \geq 0, x^2+y^2=2rx, r=R$
 求: $I = \oint_{\Gamma^+} (y^2+z^2) dx + (x^2+y^2) dy + (x^2+y^2) dz$
 $\vec{F} = (y^2+z^2, x^2+z^2, x^2+y^2)$

$\text{rot } \vec{F} = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{pmatrix} = (2(y-z), 2(z-x), 2(x-y))$

$(P, Q, R) = (y^2+z^2, x^2+z^2, x^2+y^2)$

$S: z = f(x, y) = \sqrt{R^2 - (x-R)^2 - y^2}$

$\vec{n} = \left(\frac{x-R}{R}, \frac{y}{R}, 1 \right) \frac{1}{\sqrt{1+\frac{(x-R)^2}{R^2} + \frac{y^2}{R^2}}}$

$I = \iint_{S^+} \text{rot } \vec{F} \cdot \vec{n} d\sigma$
 $= \iint_{\Omega} \left((y-z) \frac{x-R}{R} + (z-x) \frac{y}{R} + (x-y) \right) dx \wedge dy$

$= 2\pi R^2$

eg. 求 $I = \oint_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$
 其中 Γ 是平面 $x+y+z=1$ ($-1 < h < 1$) 与球面 $x^2+y^2+z^2=1$ 的交线
 从右起看是逆时针的.

$$\vec{n} = (1, 1, 1)$$

$$I = \iint_{\Sigma} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} \cdot \frac{1}{\sqrt{3}} (1, 1, 1) d\sigma$$

$$= \iint_{\Sigma} (-2y - 2z) \vec{i} + (-2x - 2z) \vec{j} + (-2x - 2y) \vec{k} \cdot \frac{1}{\sqrt{3}} (1, 1, 1) d\sigma$$

$$= -\frac{4}{\sqrt{3}} \cdot \pi \cdot (1 - \frac{1}{3})$$

eg. 求 $I = \oint_{\Gamma} (y-z) dx + (z-x) dy + (x-y) dz$
 $x^2+y^2 = a^2$ $\frac{x}{a} + \frac{y}{b} = 1$

$$I = \iint_{\Sigma} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z & z-x & x-y \end{vmatrix} \cdot \frac{1}{\sqrt{a^2+b^2}} (\frac{b}{a}, -\frac{a}{b}, 1) d\sigma$$

$$= \iint_{\Sigma} (-2\vec{i} - 2\vec{j} - 2\vec{k}) \cdot \frac{1}{\sqrt{a^2+b^2}} (\frac{b}{a}, -\frac{a}{b}, 1) d\sigma$$

$$= -2 \iint_{\Sigma} \frac{a+b}{\sqrt{a^2+b^2}} d\sigma$$

$$= -2(a+b)a^2$$

三维保场的判定.

\mathbb{R}^3 中简单(道路)连通区域 Ω (即曲面单连通):
 对 Ω 中任意简单闭曲线 Γ , $\exists \Omega$ 中片 C^2 连续曲面 S , st $\partial S = \Gamma$
 且 Γ 可以在 S 连续收缩到一点

Thm: 证 Ω 为 \mathbb{R}^3 简单连通区域. $\vec{F} = (P, Q, R)$ 为 Ω 中 C^1 向量场
 则 \vec{F} 是 Ω 上之保场 $\Leftrightarrow \nabla \times \vec{F} = 0$ 在 Ω 上

$$\Leftrightarrow \oint_{\Gamma} P dx + Q dy + R dz = \iint_{S} \nabla \times \vec{F} \cdot \vec{n} d\sigma = 0$$

证 \Rightarrow : 反证法:

取: $\oint_{\Gamma} \vec{F} \cdot d\vec{r} \neq 0$ $\vec{F}|_{M_0} \neq 0$

$$\oint_{\Gamma} P dx + Q dy + R dz = \iint_{S} \nabla \times \vec{F} \cdot \left(\frac{\nabla \vec{F}}{\|\nabla \vec{F}\|} \right) \Big|_{M_0} > a \cdot \pi r^2 > 0$$

$$\nabla(a\vec{F} \pm b\vec{G}) \quad \text{div}(a\vec{F} + b\vec{G}) \quad \text{rot}(a\vec{F} + b\vec{G})$$

$$\text{rot}(u\vec{F}) = u \text{rot}\vec{F} + \nabla u \times \vec{F}$$

$$\text{div}(u\vec{F}) = u \text{div}\vec{F} + \langle \nabla u, \vec{F} \rangle$$

$$\text{div}(\vec{F} \times \vec{G}) = \text{rot}\vec{F} \cdot \vec{G} - \vec{F} \cdot \text{rot}\vec{G}$$

$$\text{rot}(\nabla u) = 0$$

$$\text{div}(\text{rot}\vec{v}) = 0$$

常微分方程 ordinary differential equation ODE PDE

若一元函数 $y = y(x)$, $x \in I$ (区间), 满足方程 $F(x, y, y', \dots, y^{(n)}) = 0$, $n \geq 1$ (*)
 和 $y = y(x)$ 为 (*) 在 I 上的一个特解. F 称为 y 的 n 阶常微分方程

ODE 相关问题: (*) 的所有解
 即关于 $y = y(x)$ 关于 x 的变化规律

ODE 举例:

$$\textcircled{1} \frac{dR}{dt} = -\alpha R \quad \Rightarrow R = C_0 e^{-\alpha t}$$

$$\textcircled{2} u'(z) = u^2 + z^2 \quad \text{Riccati 方程} \quad \text{无初值函数表示}$$

$$\textcircled{3} \text{ 抛体 } m x''(t) = mg - k x'(t)^2 \quad \text{牛顿定律} \quad k \text{ 阻尼系数}$$

$$\frac{dx'}{dt} = g - \frac{k}{m} x'^2 \quad k = k_1 \text{ 不开伞} \quad k = k_2 \gg k_1 \text{ 开伞}$$

$$\frac{dx'}{g - \frac{k}{m} x'^2} = dt$$

$$\frac{1}{2g} \left(\frac{1}{\sqrt{g} - \sqrt{\frac{k}{m}} x'} + \frac{1}{\sqrt{g} + \sqrt{\frac{k}{m}} x'} \right) dx' = dt$$

$$\frac{1}{2g\sqrt{E}} \ln \frac{\sqrt{g} + \sqrt{\frac{k}{m}} x'}{\sqrt{g} - \sqrt{\frac{k}{m}} x'} = t + C$$

$$\textcircled{4} m \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = - \frac{k m M}{(x^2 + y^2)^{3/2}} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\textcircled{5} m x'' = -kx - y_1(x) x' + y_2(x) m g + f_0(t)$$

$$(*) \begin{cases} x'(t) + p(t)x'(t) + q(t)x(t) = f(t) \\ x(t_0) = x_0 \quad x'(t_0) = x'_0 \end{cases} \quad \text{线性 ODE} \quad \begin{matrix} \nearrow v \text{ 为常数} \end{matrix}$$

eg. Bessel 方程 $d^2 w(z)/dz^2 + \frac{1}{z} dw(z)/dz + (1 - \frac{v^2}{z^2}) w(z) = 0$

注: 令 $\vec{Y}(t) = \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix}$, 则 (**)

$$\vec{Y}'(t) = \begin{pmatrix} x'(t) \\ -p(t)x'(t) - q(t)x(t) + f(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} \vec{Y} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}$$

$M(t)$

$$\Rightarrow \vec{Y}'(t) = M(t) \vec{Y}(t) + \vec{b}(t)$$

③ 线性微分方程组

$$\vec{Y}'(t) = M(t) \vec{Y}(t) + \vec{b}(t)$$

$$\vec{Y} \in \mathbb{R}^n \quad M(t) \in M_n(\mathbb{R})$$

④ 以 $M_{2 \times 2}$ 矩阵为例

$$\frac{dR(t)}{dt} = -R(t)$$

$$\frac{dR(t)}{dt} = -R(t) + A(t) \quad \text{常数变易法} \quad R(t) = C(t) e^{-t}$$

$$\begin{cases} \dot{R}(t) = -R(t) + \sum_{i=1}^n \rho_i^R A_i^S(t) \\ \dot{S}(t) = -S(t) + \sum_{i=1}^n \rho_i^S A_i^R(t) \end{cases} \quad \begin{cases} \dot{R}(t) = -R(t) + \rho^R A^S \\ \dot{S}(t) = -S(t) + \rho^S A^R \end{cases}$$

Gone with the wind

S: Scarlett O'Hara
 R: Rhett Butler

$$\begin{cases} \dot{R}(t) = -R(t) + \rho^R A^S + \gamma_S(t) \\ \dot{S}(t) = -S(t) + \rho^S A^R + \gamma_R(t) \end{cases} \quad \begin{matrix} \nearrow k_R S(t) e^{-\rho_R S(t)} \\ \nearrow k_S R(t) e^{-\rho_S R(t)} \end{matrix}$$

$$\vec{Y} = \begin{pmatrix} R \\ S \end{pmatrix} \quad \vec{Y}'(t) = \vec{f}(\vec{Y}(t))$$

$$\oint \vec{Y}' \cdot \vec{n} dS = \oint \vec{f}(\vec{Y}) \cdot \vec{n} dS = \iint \text{div} \vec{f}(\vec{Y}) d\sigma = -2 \iint d\sigma \neq 0$$

不会形成周期

第 1 章 ODE

$$\begin{cases} y'(x) = f(x, y) \\ y(x_0) = y_0 \end{cases} \quad (*)$$

- i) 对特解 $f(x, y)$, 求解 y
ii) 对一般 f , $(*)$ 何时有解, 解结构

$$\textcircled{1} \begin{cases} \frac{dy}{dx} = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

② n 阶线性 ODE

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \dots + p_1(x)y' + p_0(x)y = g(x)$$

$$\vec{Y}(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{pmatrix}, \quad \vec{F}(x, \vec{Y}) = \begin{pmatrix} f_1(x, \vec{Y}) \\ f_2(x, \vec{Y}) \\ \vdots \\ f_n(x, \vec{Y}) \end{pmatrix}$$

$$\vec{Y}'(x) = M(x)\vec{Y}(x) + \vec{b}(x)$$

$$\begin{cases} y_1(x) = y(x) \\ y_2(x) = y'(x) \\ \vdots \\ y_n(x) = y^{(n-1)}(x) \end{cases} \quad \vec{Y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \quad \vec{Y}'(x) = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ -p_0 & -p_1 & \dots & -p_{n-1} \end{pmatrix} \vec{Y}(x) + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ g(x) \end{pmatrix}$$

③ 分离变量法 (一) 齐次方程

$$\frac{dy}{dx} = f(x)g(y)$$

$$1^\circ g(y) = 0 \Rightarrow y = y_0 \text{ 是一个特解}$$

$$2^\circ g(y) \neq 0 \quad \frac{1}{g(y)} \frac{dy}{dx} = f(x)$$

$$\text{若 } G(y) \text{ 为 } \frac{1}{g(y)} \text{ 的一个原函数} \quad G'(y) = \frac{1}{g(y)} \\ G(y(x)) = \int f(x) dx$$

④ 一阶线性 ODE

$$\frac{dy}{dx} + p(x)y = q(x)$$

Bernoulli 方程:

$$\frac{dy}{dx} + p(x)y = q(x)y^\alpha$$

$$\vec{Y}'(x) = M(x)\vec{Y}(x) + \vec{b}(x)$$

$$\text{Cauchy 问题} \begin{cases} y'(x) = f(x, y) \\ y(x_0) = y_0 \end{cases} \rightarrow \begin{cases} \frac{d\vec{Y}}{dx} = \vec{F}(x, \vec{Y}) \\ \vec{Y}(x_0) = \vec{Y}_0 \end{cases}$$

$$\text{分离变量法} \quad \frac{dy}{dx} = f(x)g(y) \quad \text{原方程分离}$$

$$\frac{1}{g(y)} \frac{dy}{dx} = f(x)$$

$$\text{eg. } \frac{dy}{dx} = f(ax+by+c)$$

一阶线性 ODE

$$\frac{dy}{dx} + p(x)y = q(x), \quad p(x), q(x) \in C(I)$$

① $q(x) \equiv 0$

$$\frac{dy}{dx} = -p(x)y(x)$$

$$\frac{1}{y(x)} \frac{dy}{dx} = -p(x)$$

$$\Rightarrow y = C e^{-\int p(x) dx}$$

② $q(x) \neq 0$ 常数变易法

$$\text{令 } y(x) = C(x) e^{-\int p(x) dx}$$

$$\frac{dy}{dx} = C'(x) e^{-\int p(x) dx} - p(x) C(x) e^{-\int p(x) dx}$$

$$= C'(x) e^{-\int p(x) dx} - p(x) y(x)$$

$$\Rightarrow y'(x) + p(x)y(x) = C'(x) e^{-\int p(x) dx}$$

$$\Rightarrow C'(x) = Q(x) e^{\int p(x) dx}$$

$$\Rightarrow C(x) = \int Q(x) e^{\int p(x) dx} dx$$

eg. Bernoulli 方程

$$\frac{dy}{dx} + p(x)y = Q(x)y^\alpha$$

$$\frac{1}{y^\alpha} \frac{dy}{dx} + y^{1-\alpha} p(x) = Q(x)$$

齐次方程

$$\frac{dy}{dx} = -\frac{P(x, y)}{Q(x, y)} \quad (*)$$

其中 $P(x, y), Q(x, y)$ 是 $DCIR^2$ 上的连续函数且 $Q(x, y) \neq 0$

$$\Rightarrow P(x, y) + Q(x, y) \frac{dy}{dx} = 0$$

$$\text{若 } \exists f(x, y) \text{ s.t. } \begin{cases} \frac{\partial f}{\partial x} = P(x, y) \\ \frac{\partial f}{\partial y} = Q(x, y) \end{cases}$$

$$\text{则 } (*) \text{ 的解为 } f(x, y) = C$$

$$\nabla f = (P, Q)$$

给定 $DCIR^2$ 上连续 $P(x, y), Q(x, y) \in C^1(D)$, 则 $\exists f$ s.t. $\nabla f = (P, Q)$

$$\Leftrightarrow \frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$$

$$\text{eg. } (x \cos y + 2xy^2) dx + (-\frac{1}{2} x^2 \sin y + 2x^2 y + y^3) dy = 0$$

$$d(\frac{1}{2} x^2 \cos y + x^2 y^2 + \frac{1}{3} y^3) = 0$$

$$\frac{1}{2} x^2 \cos y + x^2 y^2 + \frac{1}{3} y^3 = C$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

积分因子法

$$M(x, y) dx + N(x, y) dy = 0$$

$$\frac{\partial N}{\partial x} \neq \frac{\partial M}{\partial y}$$

$$u(x, y) M(x, y) dx + u(x, y) N(x, y) dy = 0 \quad \text{若 } \mu + \nu dy = 0$$

$$\exists u, \text{ s.t. } \frac{\partial(uM)}{\partial x} = \frac{\partial(uN)}{\partial y}$$

$$\text{eg. } (x - y\sqrt{1+x^2}) dx - x\sqrt{1+x^2} dy = 0$$

$$(\frac{x}{\sqrt{1+x^2}} - y) dx - x dy = 0$$

$$\Rightarrow d(\sqrt{1+x^2} - xy) = 0$$

$$N \frac{\partial u}{\partial x} + u \frac{\partial N}{\partial x} = M \frac{\partial u}{\partial y} + u \frac{\partial M}{\partial y}$$

$$\textcircled{1} u(x, y) \equiv u(x)$$

$$\textcircled{2} u(x, y) \equiv u(y)$$

$$\text{eg. } (2xy^2 - y) dx + (x + 3y^2) dy = 0$$

$$\frac{\partial}{\partial y} (u(2xy^2 - y)) = \frac{\partial u}{\partial y} (2xy^2 - y) + (4xy - 1)u$$

$$\frac{\partial}{\partial x} (u(x + 3y^2)) = \frac{\partial u}{\partial x} (x + 3y^2) + u$$

$$u = (4xy - 1)u + \frac{\partial u}{\partial y} (2xy^2 - y)$$

$$\frac{\partial u}{\partial y} = -\frac{1}{y}$$

$$u = \frac{1}{y}$$

$$(2x - \frac{1}{y}) dx + (\frac{x}{y} + 3) dy = 0$$

$$x^2 - \frac{1}{y} + 3y + C = 0$$

可降阶 $y'' = f(y)$ 方程

① $f(x, y', y'') = 0$

令 $z(x) = y'(x)$
 $f(x, z, z') = 0$

② $F(y, y', y'') = 0$

$p = y'$

$y'' = \frac{dp}{dy} \cdot y' = \frac{dp}{dy} \cdot p$

$\Rightarrow F(y, p, p \frac{dp}{dy}) = 0$

eg. $y'' = \sqrt{1+y'^2} = \frac{dy'}{dx}$

$\Rightarrow x = \operatorname{arsh} y' + C$

$\Rightarrow y' = \operatorname{sh}(x+C)$

$\Rightarrow y = \operatorname{ch}(x+C) + C'$

eg. $\begin{cases} 1+y'^2 = 2y \cdot y'' \\ y(1)=1, y'(1)=1 \end{cases}$

$1+y'^2 = 2y \cdot y' \frac{dy'}{dy}$

$1+p^2 = 2y p \frac{dp}{dy}$

$\frac{dy}{y} = \frac{2p dp}{1+p^2}$

$\ln y = \ln(1+p^2)$

$ay = 1 + y'^2$

$2y = 1 + y'^2$

$y' = -\sqrt{2y-1}$

$-x = \sqrt{2y-1} - 2$

Q: $\begin{cases} y'(x) = f(x, y) \\ y(x_0) = y_0 \end{cases} \quad (*)$ 何时有解, 何解唯一?

答: $\begin{cases} \frac{dy}{dx} = \tilde{f}(x, \tilde{y}) \\ \tilde{y}(x_0) = \tilde{y}_0 \end{cases}$

eg. $\begin{cases} \frac{dy}{dx} = \operatorname{Sign}(x) \\ y(0)=0 \end{cases}$ 无解

eg. $y' = \sqrt[4]{y^2}$
 $y=0$
 $y = \frac{1}{25}(x+C)^5$

eg. Bernoulli 方程
 $y' = x^2 + y^2$

Def. 设函数 $f(x, y)$ 在区域 D 内满足不等式 $|f(x, y_1) - f(x, y_2)| \leq L_0 |y_1 - y_2|$ 对 $(x, y_1), (x, y_2) \in D$ 成立, 其中 L_0 为常数.

则称 f 在 D 内对 y 满足 Lipschitz 条件

Thm. 设 f 在 $R = \{(x, y) : |x-x_0| \leq a, |y-y_0| \leq b\}$ 上连续, 且对 y 满足 Lipschitz 条件, 则 Cauchy 初值问题 $\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$ 在 $I = [x_0-h, x_0+h]$ 上有唯一解

其中 $h = \min\{a, \frac{b}{M}\}$, $M = \max\{|f(x, y)|\}$, $(x, y) \in R$

Proof. Picard 迭代

$y' = f(x, y)$

$y(x) - y_0 = \int_{x_0}^x y'(s) ds = \int_{x_0}^x f(s, y(s)) ds$

$y(x) = y_0 + \int_{x_0}^x f(s, y(s)) ds$

令 $y_0(x) = y_0$

$y_n(x) = y_0 + \int_{x_0}^x f(s, y_{n-1}(s)) ds$

若 $y_{n-1}(x) \rightarrow y(x)$

则 $y(x)$ 满足 $y(x) = y_0 + \int_{x_0}^x f(s, y(s)) ds$

Amazing

eg. $\begin{cases} y'(x) = y \\ y(0) = y_0 = 1 \end{cases} \Rightarrow y = e^x$

$y_n(x) = y_0 + \int_0^x y_{n-1}(s) ds$

$y_0(x)=1, y_1(x)=1+x, y_2(x)=1+x+\frac{1}{2}x^2, y_n(x)=1+x+\frac{1}{2}x^2+\dots+\frac{1}{n!}x^n$

$n \rightarrow \infty, y_n(x) \rightarrow e^x$

证明: 方程组情形

$\begin{cases} \frac{d\tilde{y}}{dx} = \tilde{f}(x, \tilde{y}) \\ \tilde{y}(x_0) = \tilde{y}_0 \end{cases}$

\tilde{f} 关于 \tilde{y} 满足 Lipschitz 条件

$\|\tilde{f}(x, \tilde{y}_1) - \tilde{f}(x, \tilde{y}_2)\| \leq L \|\tilde{y}_1 - \tilde{y}_2\|$

n 阶线性 ODE

$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \dots + p_0(x)y = g(x)$

$\tilde{y} = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}$

$\tilde{y}'(x) = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \\ -p_0 & \dots & -p_{n-1} \end{pmatrix} \tilde{y}(x) + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ g(x) \end{pmatrix}$

$\frac{d\tilde{y}}{dx} = M(x)\tilde{y} + \tilde{g}$

$\|\tilde{f}(x, \tilde{y}_1) - \tilde{f}(x, \tilde{y}_2)\| = \|M(x)\tilde{y}_1 - M(x)\tilde{y}_2\| = \|M(x)(\tilde{y}_1 - \tilde{y}_2)\| \leq \|M(x)\| \|\tilde{y}_1 - \tilde{y}_2\|$

$\|M(x)\| = \sup_{\|\tilde{y}\|=1} \frac{\|M\tilde{y}\|}{\|\tilde{y}\|}$

norm 范数

① 分离变量法 $\frac{dy}{dx} = f(x)g(y)$

② 常系数齐次 $y'(x) + p(x)y(x) = 0$

③ 全微分方程

④ 特殊方程

eg. $\begin{cases} x'(t) = \sin x \\ x(0) = x_0 \end{cases}$

$\frac{dx}{\sin x} = dt = \frac{\sin x}{\sin^2 x} dx = -\frac{d \cos x}{1 - \cos^2 x} = -\frac{1}{2} \left(\frac{1}{1 - \cos x} + \frac{1}{1 + \cos x} \right) d \cos x$
 $= \frac{1}{2} \ln |1 - \cos x| - \frac{1}{2} \ln |1 + \cos x| = \frac{1}{2} \ln \left| \frac{1 - \cos x}{1 + \cos x} \right| = t + C$

Thm. 设 f 在 $R = \{(x, y) : |x-x_0| \leq a, |y-y_0| \leq b\}$ 上连续, 且对 y 满足 Lipschitz 条件

即 $\exists L_R$ s.t. $|f(x, y_1) - f(x, y_2)| \leq L_R |y_1 - y_2| \quad \forall (x, y_1), (x, y_2) \in D$

则初值问题 $\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases} \quad (x)$ 在 $I = [x_0-h, x_0+h]$ 上有唯一解,

其中 $h = \min\{a, \frac{1}{L_R}\}$, $M = \max\{|f(x, y)|\}, (x, y) \in R$

① Picard 迭代

$y(x_0) = y_0$

$y_n(x) = y_0 + \int_{x_0}^x f(s, y_{n-1}(s)) ds, \quad x \in I$

$y(x) = y_0 + \int_{x_0}^x f(s, y(s)) ds$

② 解的存在性

③ 方程组情形

$\begin{cases} \frac{d\tilde{y}}{dx} = \tilde{f}(x, \tilde{y}) \\ \tilde{y}(x_0) = \tilde{y}_0 \end{cases} \quad \tilde{y} = \begin{pmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{pmatrix}, \quad \tilde{f}(x, \tilde{y}) = \begin{pmatrix} f_1(x, \tilde{y}) \\ f_2(x, \tilde{y}) \\ \vdots \\ f_n(x, \tilde{y}) \end{pmatrix}$

\tilde{f} 关于 \tilde{y} 满足 Lipschitz 条件

$\|\tilde{f}(x, \tilde{y}_1) - \tilde{f}(x, \tilde{y}_2)\| \leq L \|\tilde{y}_1 - \tilde{y}_2\|$

$\frac{d\tilde{y}}{dx} = A(x)\tilde{y} + \tilde{b}(x)$

$A_{n \times n} \in C([c, d], \mathbb{R}^{n \times n})$

$\tilde{b}_m \in C([c, d], \mathbb{R}^n)$

相关问题:

$\begin{cases} y'(x) + p(x)y'(x) + q(x)y(x) = g(x) \\ y(x_0) = y_0, y'(x_0) = y'_0 \end{cases} \quad p, q, g \in C([c, d])$
是否有解? 解的结构?

② $y^{(n)}(x) + p_{n-1}y^{(n-1)} + \dots + p_0y = g(x)$

③ $\tilde{y}'(x) = A(x)\tilde{y}(x) + \tilde{b}(x)$

Scap. 齐次

$y^{(n)}(x) + p(x)y'(x) + q(x)y(x) = 0 \quad (x) \quad p, q \in C([c, d])$

$S = \{y(x) \in C([c, d]) : y \text{ 在 } (c, d) \text{ 上满足 } (*)\}$

① S 是线性空间 $y_1, y_2 \in S \Rightarrow k_1 y_1 + k_2 y_2 \in S$

② $\dim S = ?$

因 $x_0 \in [c, d] \quad \tilde{y}'(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$

$\begin{cases} \tilde{y}'(x) = \begin{pmatrix} 0 & 1 \\ -q(x) & p(x) \end{pmatrix} \tilde{y} \\ \tilde{y}(x_0) = \tilde{y}_0 \end{cases}$

用解的存在唯一性定理

$\tilde{y}_0 \mapsto \tilde{y} \in S$ 线性同构 $\Rightarrow \dim$ 相同 $\Rightarrow \dim S = 2$

$\forall \tilde{y}_0 \in \mathbb{R}^2, \exists! \tilde{y} \in S, \text{ s.t. } \begin{pmatrix} y_1(x_0) \\ y_2(x_0) \end{pmatrix} = \tilde{y}_0$

$\tilde{y}_0 \in \mathbb{R}^2 \mapsto \tilde{y} \in S, \text{ s.t. } \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} \in S$ 线性同构

$\Rightarrow \dim S = 2$

Def. 线性相关. 设 m 个 $[c, d]$ 上的函数 u_1, u_2, \dots, u_m .

若 $\exists m$ 个不全为 0 的数 k_1, \dots, k_m , 使 $k_1 u_1 + k_2 u_2 + \dots + k_m u_m = 0, x \in [c, d]$

则称 u_1, \dots, u_m 在 $[c, d]$ 上线性相关, 否则称线性无关

Prop. 设 $\phi_1, \phi_2 \in S$, 则下列各式均行

① ϕ_1, ϕ_2 在 $[c, d]$ 上线性相关

Wronski

② $W(\phi_1, \phi_2)(x) = \begin{vmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{vmatrix} \equiv 0, x \in [c, d]$

③ 对某 $x_0 \in (c, d)$ 有 $W(\phi_1, \phi_2)(x_0) = 0$

$$C_1 \begin{pmatrix} \phi_1(x_0) \\ \phi_1'(x_0) \end{pmatrix} + C_2 \begin{pmatrix} \phi_2(x_0) \\ \phi_2'(x_0) \end{pmatrix} = 0$$

解存在唯一性定理: $\textcircled{1} \rightarrow \textcircled{3}$

证: $\textcircled{1} \Leftrightarrow \textcircled{2}$

$W(\phi_1, \phi_2)(x)$ 是 $\begin{pmatrix} \phi_1(x) \\ \phi_1'(x) \end{pmatrix}, \begin{pmatrix} \phi_2(x) \\ \phi_2'(x) \end{pmatrix}$ 张成的平行四边形的面积

$$W(x) = \frac{1}{dx} \begin{vmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{vmatrix} = \begin{vmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{vmatrix} = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = -p(x) W(x)$$

$$\Rightarrow \frac{d}{dx} W(x) = -p(x) W(x)$$

$$W(x) = W(x_0) e^{-\int_{x_0}^x p(x) dx}$$

$$\vec{y}'(x) = A(x) \vec{y}(x) \quad A(x) = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}$$

$$\frac{dW}{dx} = \text{trace } A(x) W(x)$$

$$R^2 \begin{pmatrix} x \\ z \end{pmatrix} \text{ 向量场 } \vec{V} \left(\begin{bmatrix} x \\ z \end{bmatrix} \right) = A \begin{pmatrix} x \\ z \end{pmatrix}$$

$$\text{div } \vec{V} = \text{div } \vec{V} = \text{trace } A$$

Scap 2. $y'(x) + p(x)y(x) + q(x)y(x) = f(x), p, q, f \in C[c, d]. (**)$

其解为 $y(x) = C_1 \phi_1(x) + C_2 \phi_2(x) + y^{**}$

其中 ϕ_1, ϕ_2 为 $y' + p(x)y + q(x)y = 0$ 的通解

y^{**} 为 $(**)$ 的一个特解

$$y(x) - y^{**} \in S \Rightarrow y(x) - y^{**} = k_1 \phi_1 + k_2 \phi_2$$

或 ϕ_1, ϕ_2 ?

或 y^{**} ?

y^{**}

① 常系数法

ϕ_1, ϕ_2 非线性组合

证 $\phi_1, \phi_2 \in S$, 假设 y^{**} 是 $(**)$ 的特解

且 y^{**} 满足: $\begin{pmatrix} y^{**}(x) \\ \frac{dy^{**}}{dx}(x) \end{pmatrix} = C_1(x) \begin{pmatrix} \phi_1 \\ \phi_1' \end{pmatrix} + C_2(x) \begin{pmatrix} \phi_2 \\ \phi_2' \end{pmatrix}, C_1(x), C_2(x)$ 为函数

$$\vec{y}'(x) = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \vec{y}(x) + \begin{pmatrix} 0 \\ f \end{pmatrix}$$

$$1) y^{**} = C_1(x) \phi_1 + C_2(x) \phi_2$$

$$2) (y^{**})' = C_1' \phi_1 + C_1 \phi_1' + C_2' \phi_2 + C_2 \phi_2' = C_1 \phi_1' + C_2 \phi_2'$$

$$\Rightarrow C_1' \phi_1 + C_2' \phi_2 = 0$$

$$3) \begin{pmatrix} y^{**} \\ (y^{**})' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \begin{pmatrix} y^{**} \\ (y^{**})' \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix}$$

$$\Rightarrow C_1(x) \begin{pmatrix} \phi_1 \\ \phi_1' \end{pmatrix} + C_2(x) \begin{pmatrix} \phi_2 \\ \phi_2' \end{pmatrix} + C_1'(x) \begin{pmatrix} \phi_1 \\ \phi_1' \end{pmatrix} + C_2'(x) \begin{pmatrix} \phi_2 \\ \phi_2' \end{pmatrix} = A(x) \begin{pmatrix} C_1(x) \begin{pmatrix} \phi_1 \\ \phi_1' \end{pmatrix} + C_2(x) \begin{pmatrix} \phi_2 \\ \phi_2' \end{pmatrix} \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix}$$

$$\Rightarrow \begin{cases} C_1'(x) \phi_1 + C_2'(x) \phi_2 = 0 \\ C_1'(x) \phi_1' + C_2'(x) \phi_2' = f \end{cases}$$

$$\Rightarrow C'(x) = \frac{\begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}}{W(\phi_1, \phi_2)(x)}$$

$$Q'(x) = \frac{\begin{pmatrix} \phi_1' & 0 \\ 0 & \phi_2' \end{pmatrix}}{W(\phi_1, \phi_2)(x)}$$

② 用机行事

$q(x)$

关于 ϕ_1, ϕ_2 的线性

常系数情形: $p(x) = p, q(x) = q, x \in [c, d]$

$$y'' + p y' + q y = 0 \quad (**)$$

找 $e^{\lambda x}$ 型特解

$$e^{\lambda x} (\lambda^2 + p\lambda + q) = 0$$

$$\lambda^2 + p\lambda + q = 0 \text{ 特征方程 由 (*)}$$

① $\exists \lambda_1 + \lambda_2 \in \mathbb{R}$ 或 λ_1, λ_2 为复数

$e^{\lambda_1 x}, e^{\lambda_2 x}$ 为 (*) 的通解

$$W(x) = \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} \end{vmatrix} = e^{(\lambda_1 + \lambda_2)x} \begin{vmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{vmatrix}$$

② $\lambda^2 + p\lambda + q = (\lambda - \lambda_1)^2 = 0$

$$\phi_1 = e^{\lambda_1 x} \quad \phi_2 = \lim_{\lambda \rightarrow \lambda_1} \frac{1}{\lambda - \lambda_1} (e^{\lambda x} - e^{\lambda_1 x}) = x e^{\lambda_1 x}$$

$$W(x) = \begin{vmatrix} e^{\lambda_1 x} & x e^{\lambda_1 x} \\ \lambda_1 e^{\lambda_1 x} & e^{\lambda_1 x} + \lambda_1 x e^{\lambda_1 x} \end{vmatrix} = e^{2\lambda_1 x} \begin{vmatrix} 1 & x \\ \lambda_1 & 1 + \lambda_1 x \end{vmatrix}$$

③ $\lambda^2 + p\lambda + q = 0$ 有复根

$$\lambda_1 = \alpha + i\beta$$

$$\lambda_2 = \alpha - i\beta$$

$$\phi_1(x) = e^{(\alpha + i\beta)x}$$

$$= e^{\alpha x} (\cos \beta x + i \sin \beta x)$$

$$W(\phi_1, \phi_2) \neq 0$$

$$\text{eq. } y'' + 4y' + 4y = \frac{3}{\sin x}$$

解: $\phi_1 = \cos 2x, \phi_2 = \sin 2x$ 是齐次方程的解

$$\text{设 } y_1 = C_1(x) \phi_1 + C_2(x) \phi_2$$

$$\begin{pmatrix} y_1 \\ y_1' \end{pmatrix} = C_1(x) \begin{pmatrix} \phi_1 \\ \phi_1' \end{pmatrix} + C_2(x) \begin{pmatrix} \phi_2 \\ \phi_2' \end{pmatrix}$$

$$C_1(x) = -3 \sin x$$

$$C_2(x) = \frac{3}{2} \ln \left| \frac{1}{\sin x} - \frac{\cos x}{\sin x} \right| + 3 \cos x$$

$$\text{eq. } y'' + 4y' + 4y = \cos 2x$$

$$\lambda^2 + 4\lambda + 4 = 0$$

$$\Rightarrow \lambda = -2$$

$$\phi_1 = e^{-2x} \quad \phi_2 = x e^{-2x}$$

$$y_1 = A \cos 2x + B \sin 2x$$

$$y_1' = -2A \sin 2x + 2B \cos 2x$$

$$y_1'' = -4A \cos 2x - 4B \sin 2x$$

eq. Euler 方程

$$x^2 y'' + \frac{3}{2} x y' - y = 0$$

令 $t = \ln x$

$$y^{(n)} + p_{n-1}(x) y^{(n-1)} + \dots + p_1(x) y' + p_0(x) y = f(x)$$

$$\begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}' = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & p_{n-1} \end{pmatrix} \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f \end{pmatrix}$$

$$\vec{y}'(x) = A(x) \vec{y}(x) + \vec{b}(x)$$

$$k_1 \phi_1 + \dots + k_n \phi_n + y^{**}$$

$\{\phi_i\}$ 为齐次方程的通解, y^{**} 为特解

$$p \text{ 取 } \lambda^n + p_{n-1} \lambda^{n-1} + \dots + p_0 = 0 \quad \phi_i = e^{\lambda_i x}$$

$$\begin{cases} \frac{dy}{dx} = x - y \\ \frac{dy}{dx} = 2x - y \end{cases}$$

Chapter 10 无穷级数

① 数项级数

② 函数项级数

$$\sum_{n=1}^{\infty} a_n \quad \sum_{n=1}^{\infty} (a_n + b_n) \quad \lim_{x \rightarrow x_0} \frac{1}{x} \int_0^x f(x) dx$$

① 数项级数

②. 给定一列实数 $\{a_n\}_{n=1}^{\infty}$ 如何定义求和?

逐项求和

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_n = a_1 + a_2 + \dots + a_n \quad \text{部分和}$$

若 $\lim_{n \rightarrow \infty} S_n = S$ 存在, 则称 $\{a_n\}_{n=1}^{\infty}$ 可求和. 和为 S , 或称级数 $\sum_{n=1}^{\infty} a_n$ 收敛

$$\text{记作: } S := \sum_{n=1}^{\infty} a_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k$$

若 $\lim_{n \rightarrow \infty} S_n$ 不存在, 则称 $\{a_n\}_{n=1}^{\infty}$ 不可求和, 或称级数 $\sum_{n=1}^{\infty} a_n$ 发散

③ 绝对收敛: $\sum_{n=1}^{\infty} |a_n|$ 收敛 ($\Rightarrow \sum_{n=1}^{\infty} a_n$ 收敛)

④ 条件收敛: $\sum_{n=1}^{\infty} a_n$ 收敛, 但 $\sum_{n=1}^{\infty} |a_n|$ 发散

注: 绝对收敛 \Rightarrow "+" 结合律, 交换律

⑤ 如何判断收敛

① 定义

② Cauchy 准则

③ 正项级数

④ 交错级数

⑤ 一般级数

⑥ 无穷乘积

⑦ ⑧ ⑨ 基础例子

$$\sum_{n=1}^{\infty} ar^{n-1}, r < 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, p > 1$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^p}, \sum_{n=1}^{\infty} \frac{\cos nx}{n^p}, p > 0, x \neq 2m\pi$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^8} = \frac{\pi^8}{75360}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{10}} = \frac{\pi^{10}}{9355680}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{12}} = \frac{\pi^{12}}{63551360}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{14}} = \frac{\pi^{14}}{135287680}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{16}} = \frac{\pi^{16}}{216476800}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{18}} = \frac{\pi^{18}}{328359360}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{20}} = \frac{\pi^{20}}{530010560}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{22}} = \frac{\pi^{22}}{860352000}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{24}} = \frac{\pi^{24}}{1394240000}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{26}} = \frac{\pi^{26}}{2239718400}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{28}} = \frac{\pi^{28}}{3619276800}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{30}} = \frac{\pi^{30}}{5664025600}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{32}} = \frac{\pi^{32}}{8642304000}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{34}} = \frac{\pi^{34}}{13088768000}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{36}} = \frac{\pi^{36}}{19775040000}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{38}} = \frac{\pi^{38}}{29246720000}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{40}} = \frac{\pi^{40}}{42429696000}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{42}} = \frac{\pi^{42}}{62297600000}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{44}} = \frac{\pi^{44}}{89977600000}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{46}} = \frac{\pi^{46}}{130887680000}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{48}} = \frac{\pi^{48}}{190927360000}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{50}} = \frac{\pi^{50}}{274400000000}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{52}} = \frac{\pi^{52}}{394240000000}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{54}} = \frac{\pi^{54}}{564240000000}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{56}} = \frac{\pi^{56}}{809240000000}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{58}} = \frac{\pi^{58}}{1159240000000}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{60}} = \frac{\pi^{60}}{1659240000000}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{62}} = \frac{\pi^{62}}{2359240000000}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{64}} = \frac{\pi^{64}}{3359240000000}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{66}} = \frac{\pi^{66}}{4759240000000}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{68}} = \frac{\pi^{68}}{6659240000000}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{70}} = \frac{\pi^{70}}{9359240000000}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{72}} = \frac{\pi^{72}}{12959240000000}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{74}} = \frac{\pi^{74}}{17959240000000}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{76}} = \frac{\pi^{76}}{24959240000000}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{78}} = \frac{\pi^{78}}{34459240000000}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{80}} = \frac{\pi^{80}}{47459240000000}$$

① 用定义判定 $\sum_{n=1}^{\infty} a_n$ 收敛性

$\lim_{n \rightarrow \infty} S_n$ 是否存在

$$\text{eg. } \sum_{n=1}^{\infty} 2^n \quad S_n = \dots \rightarrow \infty \text{ 发散}$$

$$\sum_{n=1}^{\infty} (-1)^n \quad S_{2n} = 0 \quad S_{2n+1} = -1 \text{ 发散}$$

$$\sum_{n=1}^{\infty} ar^{n-1} \quad a \neq 0$$

$$S_n = \sum_{k=1}^n ar^{k-1} = a \frac{1-r^n}{1-r}$$

$$|r| < 1 \text{ 收敛}$$

$$|r| > 1 \text{ 发散}$$

$$|r| = 1 \text{ 发散}$$

$$\sum_{n=1}^{\infty} ar^{n-1} \text{ 收敛}$$

$$\sum_{n=1}^{\infty} a(1-r)^{n-1} \text{ 收敛}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) \rightarrow 1$$

② 级数收敛的充分必要条件

1) 若级数 $\sum_{n=1}^{\infty} a_n$ 收敛, 则 $a_n \rightarrow 0 \quad n \rightarrow \infty$

$$a_{n+1} = S_{n+1} - S_n$$

$$\text{由 } \lim_{n \rightarrow \infty} S_{n+1} = \lim_{n \rightarrow \infty} S_n \Rightarrow a_{n+1} \rightarrow 0 \quad n \rightarrow \infty$$

$$\text{eg. } \sum_{n=1}^{\infty} \sin nd \text{ 收敛?}$$

$$\text{反证法: 若 } \sum_{n=1}^{\infty} \sin nd \text{ 收敛}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \sin nd = 0$$

$$\sin(n+1)d = \sin nd \cos d + \cos nd \sin d$$

$$\Rightarrow \sin nd \cos d \rightarrow 0$$

$$\Rightarrow \sin nd \rightarrow 0$$

$$\Rightarrow \sin nd \rightarrow 0$$

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③ 收敛级数满足结合律

对 $i_1 < i_2 < i_3 < \dots$

$$a_1 + \dots + a_{i_1} + a_{i_1+1} + \dots + a_{i_2} + \dots$$

$$= b_1 + b_2 + \dots$$

$$\sum_{n=1}^{\infty} a_n \text{ 收敛} \Rightarrow \sum_{n=1}^{\infty} b_n \text{ 收敛}$$

eg. $\sum_{n=1}^{\infty} \frac{1}{n}$ 发散

反证法: 若收敛

$$(1+\frac{1}{2}) + (\frac{1}{2}+\frac{1}{3}) + (\frac{1}{3}+\frac{1}{4}) + \dots + (\frac{1}{2^{n-1}} + \dots + \frac{1}{2^n})$$

$$> 1 + \frac{1}{2} + 2 \times \frac{1}{4} + 4 \times \frac{1}{8} + \dots + 2^n \times \frac{1}{2^n} + \dots$$

$$> \sum_{n=1}^{\infty} b_n = \infty \text{ 发散}$$

④ 充要条件 Cauchy 准则

$$\sum_{n=1}^{\infty} a_n \text{ 收敛} \Leftrightarrow \forall \epsilon > 0, \exists N > 0, \forall m > n, \sum_{k=n}^m a_k < \epsilon$$

$$\text{Corollary (推论): 若 } \sum_{k=1}^{\infty} a_k > 0, \text{ 则 } \forall N, \exists m > N, \sum_{k=N}^m a_k \geq \epsilon$$

$$\text{则 } \sum_{n=1}^{\infty} a_n \text{ 发散}$$

$$\text{eg. } \sum_{n=1}^{\infty} \frac{1}{n} \text{ 发散}$$

$$\text{若 } \sum_{k=1}^{\infty} a_k > 0, \text{ 则 } \forall N, \exists m > N, \sum_{k=N}^m a_k \geq \epsilon$$

$$\text{则 } \sum_{n=1}^{\infty} a_n \text{ 发散}$$

$$\text{eg. } \sum_{n=1}^{\infty} \frac{1}{n} \text{ 发散}$$

$$\text{若 } \sum_{k=1}^{\infty} a_k > 0, \text{ 则 } \forall N, \exists m > N, \sum_{k=N}^m a_k \geq \epsilon$$

$$\text{则 } \sum_{n=1}^{\infty} a_n \text{ 发散}$$

$$\text{eg. } \sum_{n=1}^{\infty} \frac{1}{n} \text{ 发散}$$

$$\text{若 } \sum_{k=1}^{\infty} a_k > 0, \text{ 则 } \forall N, \exists m > N, \sum_{k=N}^m a_k \geq \epsilon$$

$$\text{则 } \sum_{n=1}^{\infty} a_n \text{ 发散}$$

$$\text{eg. } \sum_{n=1}^{\infty} \frac{1}{n} \text{ 发散}$$

$$\text{若 } \sum_{k=1}^{\infty} a_k > 0, \text{ 则 } \forall N, \exists m > N, \sum_{k=N}^m a_k \geq \epsilon$$

$$\text{则 } \sum_{n=1}^{\infty} a_n \text{ 发散}$$

$$\text{eg. } \sum_{n=1}^{\infty} \frac{1}{n} \text{ 发散}$$

$$\text{若 } \sum_{k=1}^{\infty} a_k > 0, \text{ 则 } \forall N, \exists m > N, \sum_{k=N}^m a_k \geq \epsilon$$

$$\text{则 } \sum_{n=1}^{\infty} a_n \text{ 发散}$$

$$\text{eg. } \sum_{n=1}^{\infty} \frac{1}{n} \text{ 发散}$$

$$\text{若 } \sum_{k=1}^{\infty} a_k > 0, \text{ 则 } \forall N, \exists m > N, \sum_{k=N}^m a_k \geq \epsilon$$

$$\text{则 } \sum_{n=1}^{\infty} a_n \text{ 发散}$$

$$\text{eg. } \sum_{n=1}^{\infty} \frac{1}{n} \text{ 发散}$$

$$\text{若 } \sum_{k=1}^{\infty} a_k > 0, \text{ 则 } \forall N, \exists m > N, \sum_{k=N}^m a_k \geq \epsilon$$

$$\text{则 } \sum_{n=1}^{\infty} a_n \text{ 发散}$$

$$\text{eg. } \sum_{n=1}^{\infty} \frac{1}{n} \text{ 发散}$$

$$\text{若 } \sum_{k=1}^{\infty} a_k > 0, \text{ 则 } \forall N, \exists m > N, \sum_{k=N}^m a_k \geq \epsilon$$

$$\sum \frac{1}{n^p} \quad 1 + (\frac{1}{2^p} + \frac{1}{3^p}) + (\frac{1}{4^p} + \dots + \frac{1}{7^p}) + (\frac{1}{8^p} + \dots + \frac{1}{16^p}) + \dots$$

$$1 < p < 2 \quad \leq 1 + \frac{1}{2^{p-1}} + \dots + \frac{1}{2^{p-1}} < \dots < +\infty \quad \text{收敛}$$

Prop 2. 设 $\{a_n\}, \{b_n\}$ 为正数列, 且 $\exists N_0 > 0$, 当 $n > N_0$ 时 $a_n \leq b_n$

则有 i) $\sum_{n=1}^{\infty} b_n$ 收敛 $\Rightarrow \sum_{n=1}^{\infty} a_n$ 收敛

ii) $\sum_{n=1}^{\infty} a_n$ 发散 $\Rightarrow \sum_{n=1}^{\infty} b_n$ 发散

$$\sum \frac{1}{n^p} \quad 0 < p < 1, \quad \frac{1}{n^p} > \frac{1}{n}, \quad \sum \frac{1}{n} \text{ 发散} \Rightarrow \sum \frac{1}{n^p} \text{ 发散}$$

$$p > 2, \quad \frac{1}{n^p} < \frac{1}{n^2}, \quad \sum \frac{1}{n^2} \text{ 收敛} \Rightarrow \sum \frac{1}{n^p} \text{ 收敛}$$

eg. $\sum \frac{4n}{(n+1)(n+2)}, \sum \frac{1}{(n+1)(n+2)(n+3)}, \sum \frac{1}{\sqrt{n}} \sin \frac{1}{n}$

$$\sum \frac{n^2}{3^n}, \sum 2^n \sin \frac{\pi}{3^n}, \sum (1 - \cos \frac{\pi}{n}), x \in (0, \frac{\pi}{2}), \sum \cos \frac{\pi}{n}$$

Prop 2'. 设 $\{a_n\}, \{b_n\}$ 为正数列, 且有 $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = l$

则有 i) 若 $0 < l < +\infty$, 则 $\sum_{n=1}^{\infty} a_n$ 收敛 $\Leftrightarrow \sum_{n=1}^{\infty} b_n$ 收敛

ii) $l = 0$, 则 $\sum_{n=1}^{\infty} b_n$ 收敛 $\Rightarrow \sum_{n=1}^{\infty} a_n$ 收敛

iii) $l = +\infty$, 则 $\sum_{n=1}^{\infty} b_n$ 发散 $\Rightarrow \sum_{n=1}^{\infty} a_n$ 发散

$a_n \in \mathbb{R}$

$\sum_{n=1}^{\infty} a_n$ 收敛:

$$\lim_{k \rightarrow \infty} S_n = \lim_{k \rightarrow \infty} \sum_{n=1}^k a_n \text{ 收敛}$$

$$\sum_{n=1}^{\infty} a_n \text{ 收敛} \Leftrightarrow \forall \epsilon > 0, \exists N, \text{ 当 } n > N, p \geq 1, \left| \sum_{k=n+1}^{n+p} a_k \right| < \epsilon$$

必要充分条件 \Downarrow 满足结合律

$a_n \rightarrow 0, n \rightarrow \infty$

对正项级数: ① $\sum_{n=1}^{\infty} a_n$ 收敛 $\Leftrightarrow \exists C$ 使 $S_n \leq C, \forall n \in \mathbb{N}$

② 比较法

i) $a_n \leq b_n$

ii) $\frac{a_n}{b_n} \rightarrow l$

正项级数 $\sum a_n^r, \sum \frac{1}{n^p}$

① $\sum \frac{1}{n^p}$ 收敛 $\left\{ \begin{array}{l} \text{Cauchy} \quad \sqrt[n]{a_n} \\ \text{D'Alembert} \quad \frac{a_{n+1}}{a_n} \end{array} \right.$

② $\sum \frac{1}{n(\ln n)^a}$ 积分判别法

③ Gauss 判别法

$$\frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} + \frac{p}{n^2 \ln n} + o(\frac{1}{n \ln n})$$

一般项级数 ① 交错 $\sum (-1)^n a_n$

② Abel

Dirichlet

Prop 1 (D'Alembert 判别法) 设 $a_n > 0, n \in \mathbb{N}$

i) 若 $\exists r_0, 0 < r_0 < 1$, 使得对充分大 n 有 $\frac{a_{n+1}}{a_n} \leq r_0$, 则 $\sum_{n=1}^{\infty} a_n$ 收敛

ii) 若对充分大 n (即 $\exists N, \forall n > N$), 有 $\frac{a_{n+1}}{a_n} \geq 1$, 则 $\sum_{n=1}^{\infty} a_n$ 发散

Proof

i) $a_{n+m} = \frac{a_{n+m}}{a_{n+m-1}} \dots \frac{a_{n+1}}{a_n} \cdot a_n \leq r_0^m a_n$

$$S_{n+m} = \sum_{k=1}^n a_k + \sum_{k=n+1}^{n+m} a_k < C$$

ii) $a_n \not\rightarrow 0, n \rightarrow \infty$

Prop 1'. 设 $a_n > 0, n \in \mathbb{N}$

i) 若 $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = r_0 < 1$, 则 $\sum_{n=1}^{\infty} a_n$ 收敛

ii) 若 $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = r_0 > 1$, 则 $\sum_{n=1}^{\infty} a_n$ 发散

注: $r_0 = 1$, 无法判定

eg. $\sum \frac{1}{n^p}$

eg. $\sum_{n=1}^{\infty} \frac{x^n \cdot n!}{n^n}$ 收敛性, $x \geq 0$

固定 $x \geq 0$

$$\frac{a_{n+1}}{a_n} = \frac{x(n+1)}{(n+1)^{n+1}} \cdot n^n = \frac{x}{(1+\frac{1}{n})^n} \rightarrow \frac{x}{e}, n \rightarrow \infty$$

① $x > e, \sum_{n=1}^{\infty} a_n$ 发散

② $x < e, \sum_{n=1}^{\infty} a_n$ 收敛

③ $x = e, a_n = \frac{e^n \cdot n!}{n^n} \rightarrow 0$

Stirling 公式: $n! \sim n^n \cdot e^{-n} \cdot \sqrt{2\pi n} (1 + o(\frac{1}{n}))$

Prop 2. Cauchy 判别法: 设 $a_n > 0, n \in \mathbb{N}, \lim_{n \rightarrow \infty} \sqrt[n]{a_n} = r$

i) 若 $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = r_0 < 1$, 则 $\sum_{n=1}^{\infty} a_n$ 收敛

ii) 若 $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = r_0 > 1$, 则 $\sum_{n=1}^{\infty} a_n$ 发散

$r_0 = 1$?

eg. $\sum_{n=1}^{\infty} \frac{1}{n^2} (1 + \frac{1}{n})^n$

$$\sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \frac{(1+\frac{1}{n})^n}{n^2} = \frac{e}{2} > 1 \text{ 发散}$$

eg. $\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \dots$

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{1}{2^n + 3^n}}$$

Raabe 判别法 ($\sum \frac{1}{n^p}$ 收敛)

Lemma. 设 $\{a_n\}, \{b_n\}$ 为正数列, 若 $\exists N_0$ 使得 $\forall n \geq N_0$ 有 $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$

则有 i) 若 $\sum b_n$ 收敛, 则 $\sum a_n$ 收敛

ii) 若 $\sum a_n$ 收敛, 则 $\sum b_n$ 收敛

取 $b_n = \frac{1}{n^p}$

Prop 3. Raabe 判别法 设 $a_n > 0, n \in \mathbb{N}$

i) 若 $\exists r_0 > 1, N_0 \in \mathbb{N}$, 使得 $\forall n > N_0$ 有

$$n(\frac{a_n}{a_{n+1}} - 1) \geq r_0$$

则 $\sum_{n=1}^{\infty} a_n$ 收敛

ii) 若 $\exists N_0 \in \mathbb{N}$, 使得 $\forall n > N_0$ 有

$$n(\frac{a_n}{a_{n+1}} - 1) \leq 1$$

则 $\sum_{n=1}^{\infty} a_n$ 发散

Proof.

i) $\frac{a_n}{a_{n+1}} \geq 1 + \frac{r_0}{n}$ 充分大 n

设 $1 < r_0 < r$, 对充分大 n 有 $(1 + \frac{r_0}{n})^\sigma < 1 + \frac{r}{n}$

$$\lim_{n \rightarrow \infty} \frac{(1+\frac{r_0}{n})^\sigma}{1+\frac{r}{n}} = r < r_0$$

$$\frac{a_n}{a_{n+1}} \geq 1 + \frac{r_0}{n} > (1+\frac{r_0}{n})^\sigma = (\frac{n+1}{n})^\sigma = \frac{1}{(\frac{n}{n+1})^\sigma}$$

ii) 略.

Prop 3'. 设 $a_n > 0, n \in \mathbb{N}$

i) 若 $\lim_{n \rightarrow \infty} n(\frac{a_n}{a_{n+1}} - 1) = r_0 > 1$, 则 $\sum_{n=1}^{\infty} a_n$ 收敛

ii) 若 $\lim_{n \rightarrow \infty} n(\frac{a_n}{a_{n+1}} - 1) = r_0 < 1$, 则 $\sum_{n=1}^{\infty} a_n$ 发散

eg. $\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!}$ 收敛性

$$\left[\frac{(2n-1)!!}{(2n)!!} \right]^2 = \frac{1}{2n+1} \rightarrow \frac{1}{2}$$

$$\Rightarrow n(\frac{a_n}{a_{n+1}} - 1) = \frac{n}{2n+1} \rightarrow \frac{1}{2} \text{ 收敛}$$

eg. $\sum_{n=0}^{\infty} \frac{1}{n!}$ 收敛

D'Alembert 判别法

Raabe 判别法

注: Raabe 判别法

$$\sum \frac{1}{n(\ln n)^a}, a > 0$$

收敛性

$$\lim_{n \rightarrow \infty} n(\frac{a_n}{a_{n+1}} - 1)$$